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# Robust Fixed Time Adaptive Terminal Sliding Mode Control of a QUAV Under Time Varying Disturbances and Payload

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**Abstract:** This paper proposes a fixed time adaptive fast terminal sliding mode controller (FTAFTSMC) to stabilize a quadrotor UAV system under time varying disturbances and variable payload. The first adaptation law is designed to mitigate the effect of abrupt disturbances acting on inner and outer loops, while the second one estimates the mass variation by adapting a coefficient. Moments of inertia and drag coefficients are considered to be time varying due to the introduced load shakes in the system and the uncertain nature of the environment respectively. The controller is formulated based on a fixed time terminal sliding surface, which guarantee fast convergence rate and insensitivity to initial conditions. Finally, a stability analysis is studied for each subsystem using Lyapunov theory. In order to demonstrate the performance of the proposed controller, two simulation scenarios are carried out in Matlab\Simulink and compared to two algorithms: A non-singular terminal super-twisting controller, and an adaptive non-singular fast terminal sliding-mode controller. An ISE performance index (Lower values are better) is considered to quantify the tracking precision performance of each algorithm for the first scenario, our proposed controller outperforms both controllers in term of accuracy (for z position, FTAFTSMC has an ISE index lower than both controllers at least by 37.65%). The simulation results for the second scenario show a clear lead for FTAFTSMC in term of response time and precision tracking due to the mass estimation law and robustness of the fixed time sliding manifold.

**Keywords:** Fixed time SMC, Adaptive control, Quadrotor UAV, Stability analysis, Trajectory tracking.

## 1. Introduction

### 1.1 Contexte et motivation

Finite time control is regarded as one of the most promising approaches to enhance tracking performance. Compared to infinite-time sliding manifold (asymptotic convergence), terminal sliding mode control (TSMC) offers greater control performance [1-3]. This technique has been used to control quadrotor system in several research studies. Systems that rely on finite-time stability, generally perform better than their infinite-time counterpart. However, there is an inevitable disadvantage of finite time control. Initial conditions that are far from equilibrium, result in a long stabilization time and a slow rate of convergence. Additionally, it might often be challenging if not impossible to get the initial conditions of dynamical systems in practice. The introduction of stabilization in fixed time has been introduced in order to address the issue of the slow rate of convergence in finite time sliding mode controllers [5-7]. In [8], the concept of fixed-time stability was brought out as an extension of finite-time stabilization, and it was formally described in [9]. The stabilization time is "uniformly" adjusted to initial conditions using fixed-time stability. The word "uniformly" in this context means that the stabilization time is bounded irrespective of the initial conditions [10-12].

### 1.2 Contribution

The problem of trajectory tracking of a quad-rotor system in the presence of disturbances and additive payload has been studied in [13-15]. The design of the controller can be challenging, especially for simple control structures (PID, LQR), since the

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Table 1. Model parameters

| Parameter | Meaning |
|---|---|
| $b, K_{f1i}, d$ | Aerodynamic coefficients |
| l | Quadrotor arm legnth |
| $I_i$ | Moment of inertia for axe $i$ |
| $m$ | Quadrotor mass |
| $D_{j=x,y,z,\phi,\theta,\psi}(t)$ | Disturbances acting on the QUAV |

mathematical model of the payload is taken into consideration. Our proposed method considers the payload as unmodeled dynamics. The fixed time adaptive fast terminal sliding mode controller (FTAFTSMC) guarantees fixed time convergence of the state variables, compensate for mass variation using adaptive law, and reject time varying external disturbances. Moments of inertia and drag coefficients are considered to be time varying due to the load shakes induced in the system and the uncertain nature of the environment respectively. In contrast to the work in [16] which uses a PID-based sliding surface (asymptotic convergence), our proposed controller is formulated based on a fixed-time terminal sliding surface, which guarantees a fast convergence rate and initial state insensitivity. The reaching phase converges also in fixed time, which means that the sliding surfaces are bounded in a fixed stabilization time. Finally, stability analysis is studied for each subsystem using Luyaponov's theory. In order to evaluate the performance of the proposed controller, the simulation results produced using Matlab\Simulink are compared to two controllers:

1. Position and attitude tracking of a quadrotor unmanned aerial vehicle based on non-singular terminal super-twisting algorithm [2].
2. Robust adaptive nonsingular fast terminal sliding-mode tracking control for an uncertain quadrotor UAV subjected to disturbances [19].

### 1.3 Paper organisation

This paper treats trajectory tracking problem for a quad-rotor system under time varying disturbances and variable additive payload. The organisation of the paper is as follow: Section 2 treats bravely the mathematical model of the system. Section 3 highlights design methodology of FTAFTSMC for translating a rational subsytems, and compensate the payload effect using adaptive estimation law. Section 4 study simulation results for two case scenarios. Section 5 presents a brief conclusion of simulation results compared to the controllers in [2, 19].

## 2. Mathematical model

The mathematical model can be formulated using the same approach in [17, 18]

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} -\frac{K_{f1x}}{m}\dot{x} + \frac{1}{m}u_x + D_x \\ -\frac{K_{f1y}}{m}\dot{y} + \frac{1}{m}u_y + D_y \\ -\frac{K_{f1z}}{m}\dot{z} + \frac{c(\phi)c(\theta)}{m}u_1 + D_z \\ \frac{I_2 - I_3}{I_1}\dot{\theta}\dot{\psi} - \frac{J_r\omega_r}{I_1}\dot{\theta} - \frac{K_{f2x}}{I_1}\dot{\phi}^2 + \frac{lu_2}{I_1} + D_\phi \\ \frac{I_3 - I_1}{I_2}\dot{\psi}\dot{\phi} - \frac{J_r\omega_r}{I_2}\dot{\phi} + \frac{K_{f2y}}{I_2}\dot{\theta}^2 + \frac{lu_3}{I_2} + D_\theta \\ \frac{I_1 - I_2}{I_3}\dot{\theta}\dot{\phi} - \frac{K_{f2z}}{I_3}\dot{\psi}^2 + \frac{l}{I_3}u_4 + D_\psi \end{bmatrix} \quad (1)$$

Control inputs and virtual control laws are defined in Eq. (2). Model parameters are defined in Table 1.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_1 = \sum_{i=1}^{4} b\omega_i^2 \\ (\omega_4^2 - \omega_2^2)bl \\ (\omega_3^2 - \omega_1^2)bl \\ (\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)d \\ c(\phi)c(\psi)s(\theta) + s(\phi)s(\psi) \\ c(\phi)s(\psi)s(\theta) - s(\phi)c(\psi) \end{bmatrix} \quad (2)$$

## 3. Control design

### 3.1 Translational subsystem controller using FTAFTSMC

In this part, FTAFTSMC will be presented for translational subsystem subjected to external disturbances. No additional payload is considered in this subsection, therefore only the nominal part of the mass is affecting the system.

For $i = x, y, z$, adding the terms $\frac{K_{f1i}}{m}\dot{x}$, $\frac{K_{f1i}}{m}\dot{y}$, $\frac{K_{f1i}}{m}\dot{z}$ to the disturbances $D_x$,$D_y$, $D_z$ respectively, the overall disturbances acting on the translational system are:

$$\overline{D_k}(t) = -\frac{K_{f1k}}{m}\dot{k} + D_k\,,\ \mathrm{k} = \{\mathrm{x,y,z}\} \quad (3)$$

**Assumption 1**: We assume that the disturbances are bounded, with $\left|\overline{D_i}(t)\right| \leq \Lambda$, where $\Lambda$ is a Lipschitz constant.

Defining the tracking errors and its derivatives as:

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$$\begin{bmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{bmatrix} = \begin{bmatrix} x - x_d \\ y - y_d \\ z - z_d \end{bmatrix}, \begin{bmatrix} \dot{e}_x(t) \\ \dot{e}_y(t) \\ \dot{e}_z(t) \end{bmatrix} = \begin{bmatrix} \dot{x} - \dot{x}_d \\ \dot{y} - \dot{y}_d \\ \dot{z} - \dot{z}_d \end{bmatrix} \quad (4)$$

The sliding surfaces for the position are as follow:

$$S_i = \dot{e}_i + \Upsilon(e_i)\left(\lambda_i sig^{q_i}(e_i) + \gamma_i S_{\Delta_i} + \eta_i e_i\right) \quad (5)$$

with $i = x, y, z$ . $\Upsilon(e_i)$ and $S_{\Delta_i}$ are calculated using the following equations:

$$\Upsilon(e_i) = \frac{1}{\aleph_{1i} + (1-\aleph_{2i})e^{-\aleph_{2i}|e_i|^{\aleph_{3i}}}} \quad (6)$$

$$S_{\Delta_i} = \begin{cases} sig^{p_i}(e_i) & if\ \sigma = 0\ or\ \sigma \neq 0\ |e_i| > \epsilon \\ \kappa_{1i}e_i + \kappa_{2i}sig^2(e_i) & if\ \sigma \neq 0\ |e_i| < \epsilon \end{cases} \quad (7)$$

$$\sigma = \dot{e}_i + \Upsilon(e_i)(\lambda_i sig^{q_i}(e_i) + \gamma_i sig^{p_i}(e_i) + \eta_i e_i) \quad (8)$$

$$\kappa_{1i} = (2 - p_i)\epsilon^{p_i - 1}\ \kappa_{2i} = (p_i - 1)\epsilon^{p_i - 2} \quad (9)$$

With: $0 < p_i < 1$ ; $q_i > 1$; $\lambda_i > 0$; $\eta_i > 0$
$0 < \aleph_{1i} < 1$ ; $\aleph_{2i} > 0$; $\aleph_{3i} > 0$ ; $\epsilon > 0$

The derivatives of Eq. (4), with $i = x, y, z$ are:

$$\dot{S}_i = \ddot{e}_i + \dot{\Upsilon}(e_i)\left[\lambda_i sig^{q_i}(e_i) + \gamma_i S_{\Delta_i} + \eta_i e_i\right] + \Upsilon(e_i)\left[\lambda_i q_i |e_i|^{q_i - 1}\dot{e}_i + \gamma_i \dot{S}_{\Delta_i} + \eta_i \dot{e}_i\right] \quad (10)$$

Where $\dot{\Upsilon}(e_i)$ and $\dot{S}_{\Delta_i}$ are calculated as:

$$\dot{\Upsilon}(e_i) = \frac{(1-\aleph_{1i})e^{-\aleph_{2i}|e_i|^{\aleph_{3i}}}\aleph_{2i}\aleph_{3i}|e_i|^{\aleph_{3i}-1}\dot{e}_i sign(e_i)}{\left(\aleph_{1i} + (1-\aleph_{2i})e^{-\aleph_{2i}|e_i|^{\aleph_{3i}}}\right)^2} \quad (11)$$

$$\dot{S}_{\Delta_i} = \begin{cases} p_i \dot{e}_i |e_i|^{p_i - 1} & if\ \sigma = 0\ or\ \sigma \neq 0\ \ |e_i| > \epsilon \\ \kappa_1 \dot{e}_i + 2\kappa_2 |e_i| \dot{e}_i & if\ \sigma \neq 0\ |e_i| < \epsilon \end{cases} \quad (12)$$

Forcing $\dot{S}_i = 0$ (where $i = x, y, z$ ), translational subsystem control signals are formulated as:

$$\text{Let } \varpi_{i=\{x,y,z\}} = -\dot{\Upsilon}(e_i)\left[\lambda_i sig^{q_i}(e_i) + \gamma_i S_{\Delta_i} + \eta_i e_i\right] - \Upsilon(e_x)\left[\lambda_i q_i |e_i|^{q_i - 1}\dot{e}_i + \gamma_i \dot{S}_{\Delta_i} + \eta_i \dot{e}_i\right] \quad (13)$$

$$u_{eq}x = \frac{m}{u_1}(\ddot{x}_d + \varpi_x),\ u_{eq}y = \frac{m}{u_1}(\ddot{y}_d + \varpi_y) \quad (14)$$
$$u_{eq}z = \frac{m}{c(\theta)c(\phi)}(\ddot{z}_d + \varpi_z)$$

The switching control laws are added to the equivalent control laws to enhance robustness in a disturbed and uncertain environment:

$$\text{Let } \varpi_{2i=\{x,y,z\}} = -\left[\lambda_{2i} sig^{q_{2i}}(S_i) + \gamma_{2i} sig^{p_{2i}}(S_i) + \eta_{2i} S_i + \frac{\hat{\delta}_i}{2\xi_i^2} S_i\right] \quad (15)$$

$$u_{sw}x = \frac{m}{u_1}\varpi_{2x}\ ;\ u_{sw}y = \frac{m}{u_1}\varpi_{2y};$$
$$u_{sw}z = \frac{m}{c(\theta)c(\phi)}\varpi_{2z} \quad (16)$$

Where $i = x, y, z$ . $\lambda_{2i}, \gamma_{2i}, \eta_{2i}$ are non zero positive coefficients, and $\hat{\delta}_i$ is an adaptive term to estimate the value of $\delta_i = \Lambda^2$.
With: $0 < p_{2i} < 1$; $q_{2i} > 1$; $\lambda_{2i}, \gamma_{2i}, \eta_{2i} > 0$

The adaptation law is formulated as follow:

$$\dot{\hat{\delta}} = \alpha_{1i}\left(\frac{S_i^2}{2\xi_i^2} - \alpha_{2i}\hat{\delta}_i\right),\ \ \alpha_{1i} > 0\ \ \alpha_{2i} > 0 \quad (17)$$

Consequently, the ultimate controllers for the position subsystem:

$$ux = \frac{m}{u_1}(\ddot{x}_d + \varpi_x + \varpi_{2x}) \quad (18)$$

$$uy = \frac{m}{u_1}(\ddot{y}_d + \varpi_y + \varpi_{2y}) \quad (19)$$

$$u_1 = \frac{m}{c(\theta)c(\phi)}(\ddot{z}_d + \varpi_z + \varpi_{2z}) \quad (20)$$

**Theorem 1**: Using robust control laws $ux$, $uy$, $u_1$, the translation subsystem is then considered practically fixed-time stable.
**Proof**: Choosing a Lyapunov function as:

$$V_z = \frac{1}{2}S_z^2 + \frac{1}{2\alpha_{1z}}\tilde{\delta}_z^2 \quad \tilde{\delta}_z = \delta_z - \hat{\delta}_z \quad (21)$$

The derivative of $V_z$ is given as:

$$\dot{V}_z = S_z\big(\ddot{e}_z + \dot{\Upsilon}(e_z)\left[\lambda_z sig^{q_z}(e_z) + \gamma_z S_{\Delta_z} + \eta_z e_z\right] + \Upsilon(e_z)\left[\lambda_z q_z |e_z|^{q_z - 1}\dot{e}_z + \gamma_z \dot{S}_{\Delta_z} + \eta_z \dot{e}_z\right]\big) - \frac{1}{\alpha_{1z}}\dot{\hat{\delta}}_z\tilde{\delta}_z \quad (22)$$

$$\dot{V}_z = S_z\big(u_1\frac{\cos(\theta)\cos(\phi)}{m} - \ddot{z}_d + \overline{D_z}(t) + \dot{\Upsilon}(e_z)\left[\lambda_z sig^{q_z}(e_z) + \gamma_z S_{\Delta_z} + \eta_z e_z\right] + \Upsilon(e_z)\left[\lambda_z q_z |e_z|^{q_z - 1}\dot{e}_z + \gamma_z \dot{S}_{\Delta_z} + \eta_z \dot{e}_z\right]\big) - \frac{1}{\alpha_{1z}}\dot{\hat{\delta}}_z\tilde{\delta}_z \quad (23)$$

Substituting control and adaptive laws from Eq. (17) to Eq. (20), results:

$$\dot{V}_z = S_z\big(\overline{D_z}(t) - \lambda_{2z} sig^{q_{2z}}(S_z) - \gamma_{2z} sig^{p_{2z}}(S_z) -\eta_{2z}S_z - \frac{\hat{\delta}_z}{2\xi_z^2}S_z\big) - \left(\frac{S_z^2}{2\xi_z^2} - \alpha_{2z}\hat{\delta}_z\right)\tilde{\delta}_z \quad (24)$$

$$\leq |S_z|\left|\overline{D_z}(t)\right| - \lambda_{2z}|S_z|^{q_{2z}+1} - \gamma_{2z}|S_z|^{p_{2z}+1} -\eta_{2z}S_z^2 - \frac{\hat{\delta}_z}{2\xi_z^2}S_z^2 - \left(\frac{S_z^2}{2\xi_z^2} - \alpha_{2z}\hat{\delta}_z\right)\tilde{\delta}_z \quad (25)$$

For any $\xi_z$ , the following inequality holds:

$$\left(\frac{|S_z|\Lambda}{\sqrt{2}\xi_z} - \frac{\xi_z}{\sqrt{2}}\right)^2 \geq 0 \;\rightarrow\; |S_z|\Lambda \leq \frac{|S_z|^2\Lambda^2}{2\xi_z^2} + \frac{\xi_z^2}{2} \;\rightarrow\; |S_z|\Lambda \leq \frac{|S_z|^2\delta_z}{2\xi_z^2} + \frac{\xi_z^2}{2} \quad (26)$$

For any $\epsilon_{1z} > 0.5$ , the following inequality holds:

$$\left(\frac{\tilde{\delta}_z}{\sqrt{2\epsilon_{1z}}} - \frac{\sqrt{\epsilon_{1z}}\delta_z}{\sqrt{2}}\right)^2 \geq 0 \rightarrow \tilde{\delta}_z\,\delta_z \leq \frac{\tilde{\delta}_z^2}{2\epsilon_{1z}} + \frac{\epsilon_{1z}\delta_z^2}{2} \quad (27)$$

$$\rightarrow \tilde{\delta}_z\,\delta_z - \tilde{\delta}_z^2 \leq \tilde{\delta}_z^2\left(\frac{1}{2\epsilon_{1z}} - 1\right) + \frac{\epsilon_{1z}\delta_z^2}{2} \quad (28)$$

$$\rightarrow \tilde{\delta}_z\hat{\delta}_z \leq \tilde{\delta}_z^2\,\frac{1-2\epsilon_{1z}}{2\epsilon_{1z}} + \frac{\epsilon_{1z}\delta_z^2}{2} \quad (29)$$

Based on Eq. (26), Eq. (29), and assumption 1, Eq. (25) becomes:

$$\dot{V}_z \leq \frac{|S_z|^2\delta_z}{2\xi_z^2} + \frac{\xi_z^2}{2} - \lambda_{2z}|S_z|^{q_{2z}+1} - \gamma_{2z}|S_z|^{p_{2z}+1} -\eta_{2z}S_z^2 - \frac{\hat{\delta}_z}{2\xi_z^2}S_z^2 - \frac{\tilde{\delta}_z}{2\xi_z^2}S_z^2 + \alpha_{2z}\left(\frac{\tilde{\delta}_z^2}{2\epsilon_{1z}} + \frac{\epsilon_{1z}\delta_z^2}{2}\right)$$
$$\leq -\eta_{2z}S_z^2 - \alpha_{2z}\frac{2\epsilon_{1z}-1}{2\epsilon_{1z}}\,\tilde{\delta}_z^2 + \alpha_{2z}\frac{\epsilon_{1z}\delta_z^2}{2} + \frac{\xi_z^2}{2}$$
$$\leq -\mu_z V_z + \Theta_z \qquad ;\ \mu_z, \Theta_z > 0 \quad (30)$$

$$\mu_z = \min\left\{2\eta_{2z}, 2\alpha_{1z}\alpha_{2z}\frac{2\epsilon_{1z}-1}{2\epsilon_{1z}}\right\} \quad (31)$$

$$\Theta_z = \alpha_{2z}\frac{\epsilon_{1z}\delta_z^2}{2} + \frac{\xi_z^2}{2} \quad (32)$$

Based on the Eq. (23), $S_z$ and $\tilde{\delta}_z$ are uniformly ultimately bounded. Fixed time stability and settling time are detailed in **Annex A**.
Using the same methodology for $x, y$ subsystems:

$$\dot{V}_j \leq -\eta_{2j}S_j^2 - \alpha_{2j}\frac{2\epsilon_{1j}-1}{2\epsilon_{1j}}\,\tilde{\delta}_j^2 + \alpha_{2j}\frac{\epsilon_{1j}\delta_j^2}{2} + \frac{\xi_j^2}{2}$$
$$\leq -\mu_j V_j + \Theta_j\,,\ \ j = \{x, y\} \quad (33)$$

$$\mu_j = \min\left\{2\eta_{2j}, 2\alpha_{1j}\alpha_{2j}\frac{2\epsilon_{1j}-1}{2\epsilon_{1j}}\right\} \quad (34)$$

$$\Theta_j = \alpha_{2j}\frac{\epsilon_{1j}\delta_j^2}{2} + \frac{\xi_j^2}{2}\ ,\ \ j = \{x, y\} \quad (35)$$

**Remak 1:** In contrast to the finite time control strategy, fixed time controller guarantees insensitivity to initial conditions and impose a fixed upper bound convergence time.
The controller in [2] has a reaching time:

$$T_{r1} \leq 2\,\frac{\sqrt{\lambda_{\max}\{R\}}}{\lambda_{\min}\{Q\}}\sqrt{V(0)} \quad (36)$$

Where V represent a Lyapunov function. $\lambda_{\max}\{R\}$, and $\lambda_{\min}\{Q\}$ represent respectively the maximum and minimum eigenvalue of the matrices R and Q.
The controller in [19] has a reaching time:

$$T_{r2} \leq t_0 + \frac{2}{\hbar_1}\ln\left(\frac{\hbar_1 V(t_0)^{1/2} + \hbar_2}{\hbar_2}\right) \quad (37)$$

Where $V(t_0)$ represent a Lyapunov function. $\hbar_1$ and $\hbar_2$ are parameters depending on state variables.
In both cases, the reaching time depends on initial conditions and can impact the overall convergence time if the state variables are located far from the equilibrium. From the results of Annex A, our proposed controller has a reaching time:

$$T \leq \frac{1}{\overline{\lambda_{2z}}(q_{2z}-1)} + \frac{1}{\overline{\gamma_{2z}}(1-p_{2z})} \quad (38)$$

It can be seen that the controller is not sensitive to initial conditions. The same can be applied for sliding phase, where fixed time sliding manifold exhibits a faster convergence rate compared to the finite time sliding surfaces [4].

## 3.2 Altitude controller using FTAFTSMC in the presence of time varying payload

In this part, FTAFTSMC will be presented to stabilize altitude subsystem subjected to external disturbances and mass variation due to additive payload. In order to have robustness against the payload effect, a correcting factor $\alpha > 0$ is considered in the new control signal. Thus the real mass is estimated as $m_l = \alpha m$ with a new control law $U_1 = \hat{\alpha}u_1$, where $\hat{\alpha}$ is the estimate value of $\alpha$.
From the previous subsection Eq. (10), we have:

$$\dot{S}_z = \ddot{e}_z + \dot{\Upsilon}(e_z)\left[\lambda_z sig^{q_z}(e_z) + \gamma_z S_{\Delta_z} + \eta_z e_z\right] + \Upsilon(e_z)\left[\lambda_z q_z |e_z|^{q_z-1}\dot{e}_z + \gamma_z \dot{S}_{\Delta_z} + \eta_z \dot{e}_z\right] \quad (39)$$

Replacing $U_1 = \hat{\alpha}u_1$ in the Eq. (26) results:

$$\dot{S}_z = \frac{U_1}{m_l}\cos(\theta)\cos(\phi) - \ddot{z}_d + \overline{D_z}(t) + \dot{\Upsilon}(e_z)\left[\lambda_z sig^{q_z}(e_z) + \gamma_z S_{\Delta_z} + \eta_z e_z\right] + \Upsilon(e_z)\left[\lambda_z q_z |e_z|^{q_z-1}\dot{e}_z + \gamma_z \dot{S}_{\Delta_z} + \eta_z \dot{e_z}\right] \quad (40)$$

**Theorem 2**: Using the robust control law, $U_1 = \hat{\alpha}u_1$. Altitude subsystem is considered practically fixed-time stable.

**Proof**: Choosing a Lyapunov function as:

$$V_{2z} = \frac{1}{2}S_z^2 + \frac{1}{2\alpha_{1z}}\tilde{\delta}_z^2 + \frac{1}{2\alpha\beta_1}\tilde{\alpha}^2 \quad (41)$$

Where $\tilde{\delta}_z = \delta_z - \hat{\delta}_z$ , $\tilde{\alpha} = \alpha - \hat{\alpha}$

The derivative of $V_{2z}$ is given as:

$$\dot{V}_{2z} = S_z\left(\frac{\alpha u_1}{\alpha m}\cos(\theta)\cos(\phi) - \frac{\tilde{\alpha}u_1}{\alpha m}\cos(\theta)\cos(\phi) - \ddot{z}_d + \overline{D_z}(t) + \dot{\Upsilon}(e_z)\left[\lambda_z sig^{q_z}(e_z) + \gamma_z S_{\Delta_z} + \eta_z e_z\right] + \Upsilon(e_z)\left[\lambda_z q_z|e_z|^{q_z-1}\dot{e}_z + \gamma_z\dot{S}_{\Delta_z} + \eta_z\dot{e_z}\right]\right) + \frac{1}{\alpha_{1z}}\dot{\tilde{\delta}}_z\tilde{\delta}_z + \frac{1}{\alpha\beta_1}\dot{\tilde{\alpha}}\tilde{\alpha} \quad (42)$$

Adaptations laws to update $\hat{\delta}_z$ and $\hat{\alpha}$ are:

$$\dot{\hat{\delta}}_z = \alpha_{1z}\left(\frac{S_z^2}{2\xi_z^2} - \alpha_{2z}\hat{\delta}_z\right),\ \alpha_{1z} > 0\ \ \alpha_{2z} > 0 \quad (43)$$

$$\dot{\hat{\alpha}} = -\beta_1\left(\frac{S_z u_1\cos(\theta)\cos(\phi)}{m} + \beta_2\hat{\alpha}\right), \beta_1, \beta_2 > 0 \quad (44)$$

Based on Eq. (16), Eq. (31), and Eq. (32) we have:

$$\dot{V}_{2z} = S_z\left(\overline{D_z}(t) - \frac{\tilde{\alpha}u_1}{\alpha m}\mathrm{c}(\theta)\mathrm{c}(\phi) - \lambda_{2z}sig^{q_{2z}}(S_z) - \gamma_{2z}sig^{p_{2z}}(S_z) - \eta_{2z}S_z - \frac{\hat{\delta}_z}{2\xi_z^2}S_z\right) - \left(\frac{S_z^2}{2\xi_z^2} - \alpha_{2z}\hat{\delta}_z\right)\tilde{\delta}_z + \frac{1}{\alpha}\left(S_z u_1\frac{\mathrm{c}(\theta)\,\mathrm{c}(\phi)}{m} + \beta_2\hat{\alpha}\right)\tilde{\alpha} \quad (45)$$

$$\dot{V}_{2z} \le |S_z|\left|\overline{D_z}(t)\right| - \lambda_{2z}|S_z|^{q_{2z}+1} - \gamma_{2z}|S_z|^{p_{2z}+1} - \eta_{2z}S_z^2 - \frac{\hat{\delta}_z}{2\xi_z^2}S_z^2 - \frac{S_z^2}{2\xi_z^2}\tilde{\delta}_z + \alpha_{2z}\hat{\delta}_z\tilde{\delta}_z + \frac{\beta_2}{\alpha}\hat{\alpha}\tilde{\alpha} \quad (46)$$

Same approach as Eq. (26) to Eq. (29), we have:

$$\dot{V}_{2z} \le -\eta_{2z}S_z^2 - \alpha_{2z}\frac{2\epsilon_1 - 1}{2\epsilon_1}\tilde{\delta}_z^2 - \frac{(2\epsilon_2-1)\beta_2}{2\epsilon_2\alpha}\tilde{\alpha}^2 + \alpha_{2z}\frac{\epsilon_1\delta_z^2}{2} + \frac{\epsilon_2\alpha\beta_2}{2} + \frac{\xi_z^2}{2} \quad (47)$$

$$\dot{V}_{2z} \le -\mu_{2z}V_{2z} + \Theta_{2z} \quad (48)$$

Where $\mu_{2z}$ and $\Theta_{2z}$ are positives parameters:

$$\mu_{2z} = \min\left\{2\eta_{2z}, \alpha_{1z}\alpha_{2z}\frac{2\epsilon_1-1}{2\epsilon_1}, \beta_1\beta_2\frac{2\epsilon_2-1}{2\epsilon_2}\right\} \quad (49)$$

$$\Theta_{2z} = \alpha_{2z}\frac{\epsilon_1\delta_z^2}{2} + \frac{\epsilon_2\alpha\beta_2}{2} + \frac{\xi_z^2}{2} \quad (50)$$

Where $\epsilon_1 > \frac{1}{2}$ ; $\epsilon_2 > \frac{1}{2}$; $\beta_1 > 0$ ; $\beta_2 > 0$ .

Based on Eqs. (28-34), $S_z, \tilde{\delta}_z, \tilde{\alpha}$ are uniformly ultimately bounded. Fixed time stability and settling time can be porved using **Annex A**.

**Remark 2**: To compensate for load variation in x and y subsystems, we can use $U_1$ instead of $u_1$ in virtual control laws $u_x$ and $u_y$.

### 3.3 Attitude controller using FTAFTSMC

In order to stabilise UAV attitude subsystem, FTAFTSMC is considered. Defining attitude errors and its derivatives as follows:

$$\begin{bmatrix} e_\phi(t) \\ e_\theta(t) \\ e_\psi(t) \end{bmatrix} = \begin{bmatrix} \phi - \phi_d \\ \theta - \theta_d \\ \psi - \psi_d \end{bmatrix}, \begin{bmatrix} \dot{e}_\phi(t) \\ \dot{e}_\theta(t) \\ \dot{e}_\psi(t) \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\phi}_d \\ \dot{\theta} - \dot{\theta}_d \\ \dot{\psi} - \psi_d \end{bmatrix} \quad (51)$$

Desired angles references are given as:

$$\begin{cases} \phi_d = \sin^{-1}\left(u_x\sin(\psi_d) - u_y\cos(\psi_d)\right) \\ \theta_d = \sin^{-1}\left(\frac{u_x\cos(\psi_d) + u_y\sin(\psi_d)}{\cos(\psi_d)}\right) \end{cases} \quad (52)$$

The sliding surfaces for attitude subsystems are:

$$S_i = \dot{e}_i + \Upsilon(e_i)(\lambda_i sig^{q_i}(e_i) + \gamma_i \mathrm{S}_{\Delta_i} + \eta_i e_i) \quad (53)$$

With $i = \phi, \theta, \psi$ , where $\Upsilon(e_i)$ and $S_{\Delta_i}$ are calculated using Eq. (5), and Eq. (6).

Using the same methodology as Eq. (10) to Eq. (17), control laws $u_2, u_3, u_4$ can be formulated as:

$$\text{Let } \chi_{i=\{\phi,\theta,\theta\}} = -\dot{\Upsilon}(e_i)\left[\lambda_i sig^{q_i}(e_i) + \gamma_i S_{\Delta_i} + \eta_i e_i\right] - \Upsilon(e_x)\left[\lambda_i q_i|e_i|^{q_i-1}\dot{e}_i + \gamma_i\dot{S}_{\Delta_i} + \eta_i\dot{e}_i\right] \quad (54)$$

$$\text{And } \chi_{2i=\{\phi,\theta,\theta\}} = -\left[\lambda_{2i}sig^{q_{2i}}(S_i) + \gamma_{2i}sig^{p_{2i}}(S_i) + \eta_{2i}S_i + \frac{\hat{\delta}_i}{2\xi_i^2}S_i\right] \quad (55)$$

$$u_2 = I_1\left(\ddot{\phi}_d - \frac{I_2 - I_3}{I_1}\dot{\theta}\dot{\psi} + \frac{J_r\omega_r}{I_1}\dot{\theta} + \frac{K_{f2x}}{I_1}\dot{\phi}^2 + \chi_\phi + \chi_{2\phi}\right) \quad (56)$$

$$u_3 = I_2\left(\ddot{\theta}_d - \frac{I_3-I_1}{I_2}\dot{\phi}\dot{\psi} - \frac{J_r\omega_r}{I_2}\dot{\phi} + \frac{K_{f2y}}{I_2}\dot{\theta}^2 + \chi_\theta + \chi_{2\theta}\right) \quad (57)$$

$$u_4 = I_3\left(\ddot{\psi}_d - \frac{I_1-I_2}{I_3}\dot{\theta}\dot{\phi} + \frac{K_{f2z}}{I_3}\dot{\psi}^2 + \chi_\psi + \chi_{2\psi}\right) \quad (58)$$

**Theorem 3:** Using the robust control law, $u_2, u_3$, and $u_4$ The attitude subsystem is considered practically fixed-time stable.

**Proof** : Choosing a Lyapunov function as:

$$V_i = \frac{1}{2}S_i^2 + \frac{1}{2\alpha_{1i}}\tilde{\delta}_i^2 \quad \tilde{\delta}_i = \delta_i - \hat{\delta}_i \;\; i = \{\phi, \theta, \psi\} \quad (59)$$

Using the same approach as Eq. (22) to Eq. (30), we have:

$$\dot{V}_i \leq -\mu_i V_i + \Theta_i \;, \mu_i > 0, \Theta_i > 0 \quad (60)$$

$$\mu_i = \min\left\{2\eta_{2i}, 2\alpha_{1i}\alpha_{2i}\frac{2\epsilon_{1i}-1}{2\epsilon_{1i}}\right\} \quad (61)$$

$$\Theta_i = \alpha_{2i}\frac{\epsilon_{1i}\delta_i^2}{2} + \frac{\xi_i^2}{2} \quad i = \{\phi, \theta, \psi\} \quad (62)$$

Therefore $S_i$ and $\tilde{\delta}_i$ are uniformly ultimately bounded. Fixed time stability and settling time can be proved using the same approach from **Annex A.**

## 4. Simulation results

This part is consecrated to the evaluation of the proposed control strategy based on the FTAFTSMC method for the problem of path tracking under different types of disturbances, model uncertainty, and payload variation using numerical simulations via Matlab\Simulink. In order to prove the superiority of the control strategy, a comparative study with two other controllers is envisaged. The existing controllers presented in this work are the non-singular terminal sliding mode controller supertwisting [2], and the non-singular adaptive fast terminal sliding mode controller [19]. Table 2 summarizes the physical parameters of the quadcopter used in these simulations. FTAFTSMC parameters are presented in the Table 4 and Table 5.

### 4.1 Simulation of the first scenario

In this scenario, the speed of the proposed controller is tested using step signals as a reference trajectory without external disturbances. Desired trajectory is given as step signals. The initial conditions of the vehicle are zero. The results of the simulation illustrated in Figs. 1-4 represent

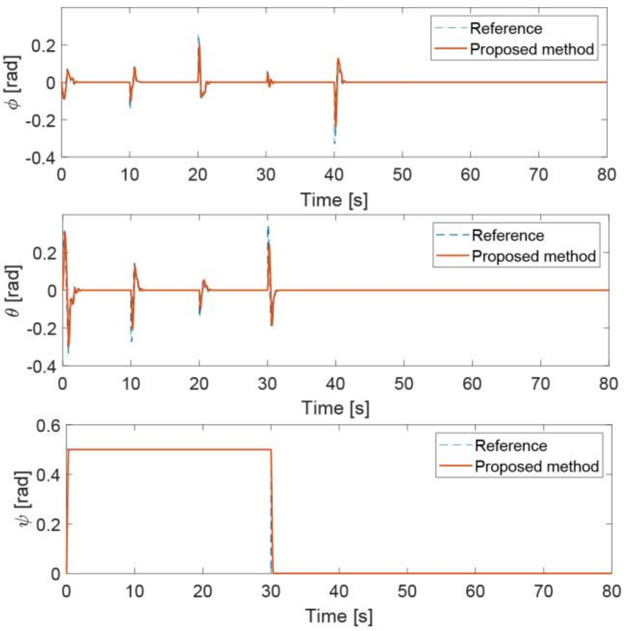


Figure. 1 Attitude tracking (scenario n°1)

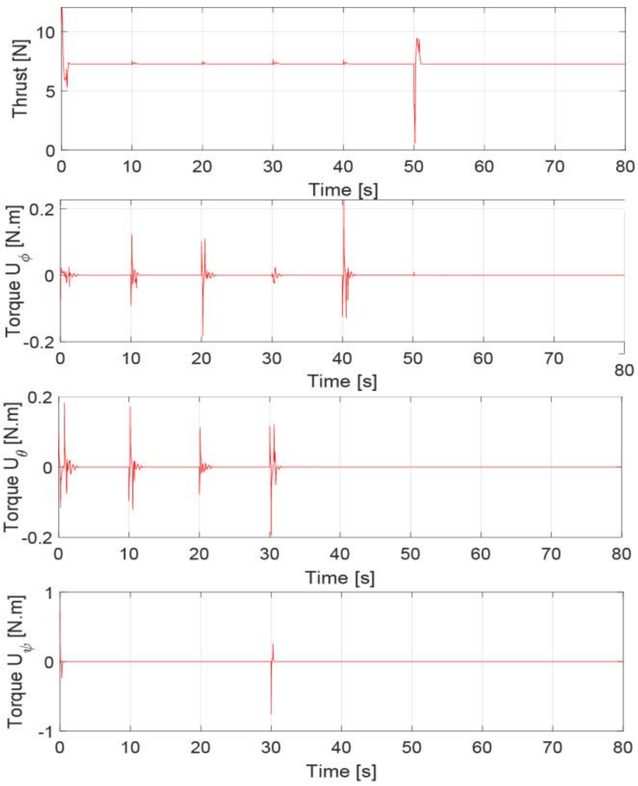


Figure. 2 Thrust and torques signals (scenario n°1)

quadcopter attitude and position tracking performance as well a comparison with RANFTSMC [19] and TSM-STA [2]. Fig. 1 and Fig. 4 represent the performance tracking for attitude and position of the quadcopter. It can be observed that the yaw angle and position variables are forced to their references quickly. Additionally, the roll and pitch angles converge to zero, indicating that the attitude

Table 2. Quadrotor parameters

| Param | Value | Param | Value |
|---|---|---|---|
| g | 9.81 | $K_{f1y}$ | 5.5670 e-4 |
| m | 0.74 | $K_{f1z}$ | 5.5670 e-4 |
| $I_1$ | 3.827 e-3 | $K_{f2x}$ | 5.5670 e-4 |
| $I_2$ | 3.827 e-3 | $K_{f2y}$ | 5.5670 e-4 |
| $I_3$ | 7.6566 e-3 | $K_{f2z}$ | 5.5670 e-4 |
| $J_r$ | 2.8385 e-5 | b | 2.9842 e-3 |
| $K_{f1x}$ | 5.5670 e-4 | K | 3.2320 e-2 |

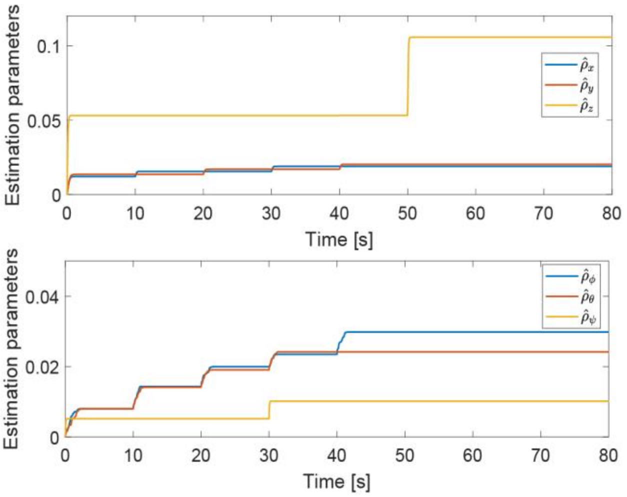

Figure. 3 Adaptation laws estimation (scenario nº1)

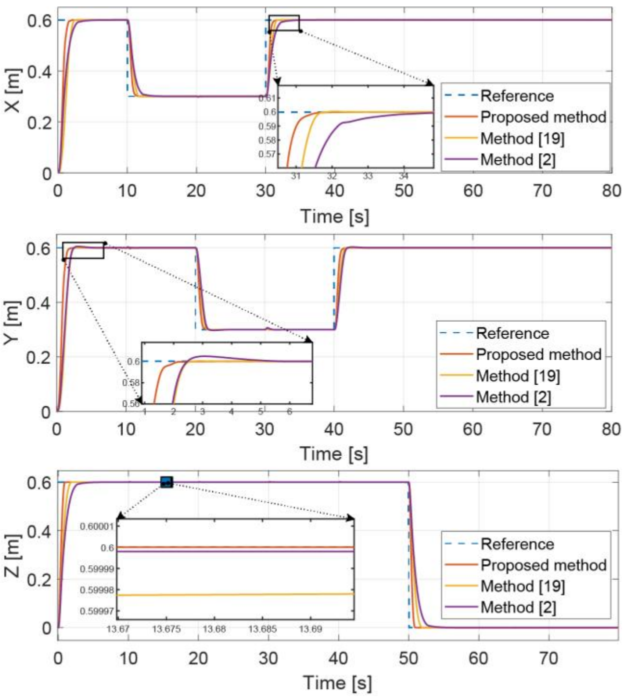

Figure. 4 Position tracking (scenario nº1)

subsystem is stable. Fig. 2 shows that control efforts $u_1$, $u_\phi$, $u_\theta, u_\psi$ are bounded and are considered within acceptable limits. These signals are also smooth and without chattering, the thrust converges towards $mg$ and the torques towards zero, which proves the effectiveness of the FTAFTSMC control strategy. The adaptation laws for the attitude and position subsystems are illustrated by Fig. 3. We can notice that $\hat{\delta}_i$ changes constantly with the variation of the setpoint to guarantee the stability of the system and improve the tracking of trajectory. Table 3 illustrates the "ISE" performance indices of the different command structures, which shows the superiority of the proposed command compared to RANFTSMC and TSM-STA in terms of the tracking speed and accuracy of in steady state phase.

Table 3. "ISE" performance indice (scenario nº1)

| State variable | Controller [19] | Controller [2] | Proposed controller |
|---|---|---|---|
| x | 0.308 | 0.354 | 0.2674 |
| y | 0.328 | 0.430 | 0.282 |
| z | 0.316 | 0.383 | 0.197 |
| $\psi$ | 0.059 | 0.068 | 0.049 |

### 4.2 Simulation of the second scenario

This part is dedicated to test the stability of the quadrotor in the presence of time-varying disturbances, uncertainty in the dynamic model, and payload variation. Disturbances are mainly caused by wind gusts when the quadrotor is flying outside. The effect of this varying load is simulated as static overload and exponential load decay. There are two possible ways in which this load might affect trajectory tracking performance. Since the thrust mathematical expression is proportional to the mass, The quadcopter altitude is directly impacted by the payload effect. On the other hand, when the quadcopter is moving, mass variation and vibration impact the momentum of inertia. In Fig. 5, sine waves at multiple frequencies in a period of time between 10 to 40 seconds are added to the moment of inertia to simulate the vibrations produced by displacements. Drag coefficients are considered uncertain by adding white noise. Initial conditions are [0.1, 0.1, 0.1] rad and [-0.2, -0.2, 0] m.

Fig. 6 shows trajectory tracking performance of the three controllers, we note for the x position, when t<10s the controllers FTAFTSMC, RANFTSMC have better tracking speed compared to TSM-STA. However, at t>10s the robustness of the RANFTSMC is clearly affected during the activation of disturbances and the effect of the payload variation, whereas the proposed controller FTAFTSMC rejects perfectly the disturbances and ensures a better stability against model uncertainty. The overall effect (mass variation and external disturbance) is less present in y position, which explain why the

RANFTSMC perform similar to scenario 1. For z position, the effect of the mass variation is more present, RANFTSMC controller at t>10s fails to reject the sudden combined effect (mass variation and disturbance), and manage only to track the trajectory after t>40s (phase of the exponential load decay). The TSM-STA controller at t>10s manages to follow the trajectory only after about five seconds, this robustness compared to the RANFTSMC is due to the super-twisting law in the reaching phase. The proposed controller FTAFTSMC, manages to completely reject the effect of mass variation, this is due to mass estimation from the adaptive control law, as shown in Fig. 7. Fig. 8 illustrates the tracking of roll, pitch, and yaw angle references. We can note that the controller track perfectly references angles, even in the presence of model uncertainty and external disturbances. Fig. 9 represents the tracking of the trajectory in 3D. It can be seen that for the same initial conditions compared to the other controllers, the FTAFTSMC ensures high tracking speed, better accuracy, and better robustness against model uncertainties (uncertain drag coefficients, mass variation), and external disturbances.

## 5. Conclusion

The trajectory tracking problem of a quadrotor system has been addressed using a robust adaptive sliding mode control technique in order to stabilize the vehicle in an uncertain environment (load variation and external disturbances). The simulation results are divided in two scenarios. Scenario 1 using step signals without disturbances. An ISE index has been used to quantify the superiority of the proposed methods, which shows a significant performance over RANFTSMC and TSM-STA controllers. In scenario 2, the problem of mass variation and abrupt time varying disturbances has been addressed using adaptation laws. The mass estimation adaptive law contributes the most to the tracking performance since other controllers could not cope with the combined effect of payload and disturbances. Fixed time stability of the control structure was detailed using Lyapunov theory.

## Conflicts of interest

The authors declare no conflict of interest.

## Author contributions

The paper conceptualization, methodology, software, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing, visualization, have been done by 1st author; Supervision, project administration, have been done by 2nd author; Resources, data curation have been done by the third author.

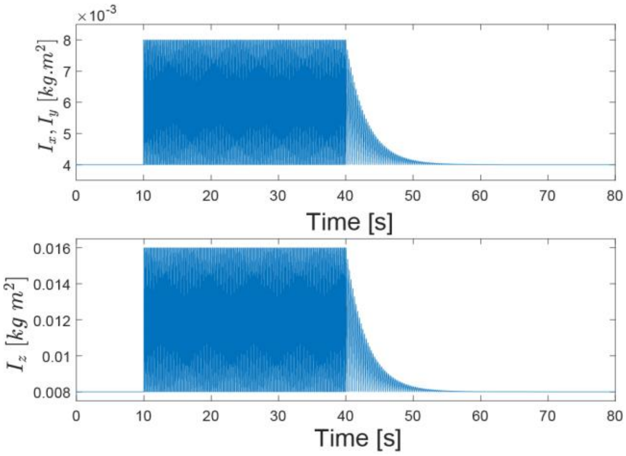


Figure. 5 Variations of moments of inertia (scenario nº2)

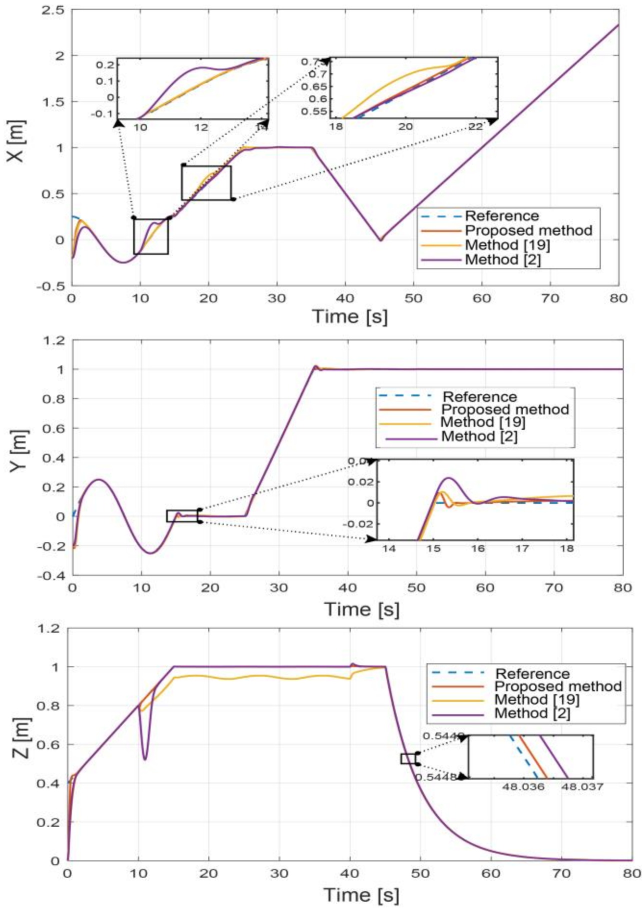


Fig. 6 Position tracking (scenario nº2)

## Annex A

**Lemma 1** [20]: System trajectory is practically fixed time stable, if there exist a Lyapunov function $V(x)$, and positive parameters $\lambda, \gamma, q, p$ s, $p < 1$, $q > 1$, $0 < \Gamma < \infty$, such that the following inequality holds

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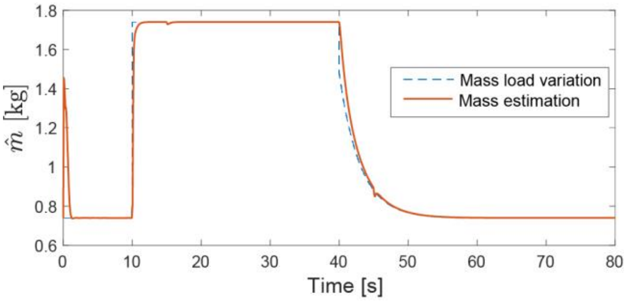


Figure. 7 Mass estimation (scenario nº2)

Table 4. FTAFTSMC parameters for inner loop

| Rotaionnal subsystem parameters | | | | | |
|---|---|---|---|---|---|
| Param | value | Param | value | Param | value |
| $q_{\phi,\theta}$ | 4 | $p_{\phi,\theta}$ | 0.1 | $\lambda_{\phi,\theta}$ | 2 |
| $\gamma_{\phi,\theta}$ | 2 | $\eta_{\phi,\theta}$ | 2 | $\alpha_{1\phi,1\theta}$ | 0.01 |
| $\alpha_{2\phi,2\theta}$ | 0.02 | $\xi_{x,y}$ | 0.08 | $\lambda_{2\phi,2\theta}$ | 1 |
| $\gamma_{2\phi,2\theta}$ | 2 | $\eta_{2\phi,2\theta}$ | 1 | $q_{2\phi,2\theta}$ | 2 |
| $p_{2\phi,2\theta}$ | 0.6 | $q_{\psi}$ | 2 | $p_{\psi}$ | 0.1 |
| $\lambda_{\phi}$ | 2 | $\gamma_{\psi}$ | 2 | $\eta_{\psi}$ | 4 |
| $\alpha_{1\phi}$ | 0.001 | $\alpha_{2\psi}$ | 0.005 | $\xi_{\psi}$ | 0.08 |
| $\aleph_{1\phi,1\theta,1\psi}$ | 1 | $\gamma_{2\psi}$ | 4 | $\eta_{2\psi}$ | 1 |
| $\aleph_{3\phi,3\theta,3\psi}$ | 2 | $p_{2\psi}$ | 0.4 | $\lambda_{2\phi}$ | 4 |
| $\aleph_{2\phi,2\theta,2\psi}$ | 30 | $q_{2\phi}$ | 2 | | |

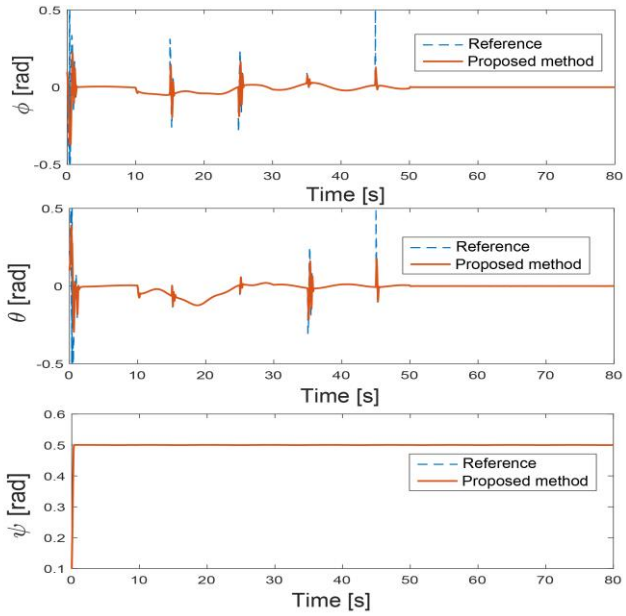


Fig. 8 Attitude tracking (scenario nº2)

$\dot{V}(x) \leq -\lambda V(x)^q - \gamma V(x)^p + \Gamma$ . The system states are confined in a small region, with a settling time

$$T \leq \frac{1}{\gamma(1-p)} + \frac{1}{\lambda(q-1)} . \tag{63}$$

**Lemma 2** [21]: let $\Omega_1,...,\Omega_n \geq 0$, $q > 1$, $0 < p \leq 1$,

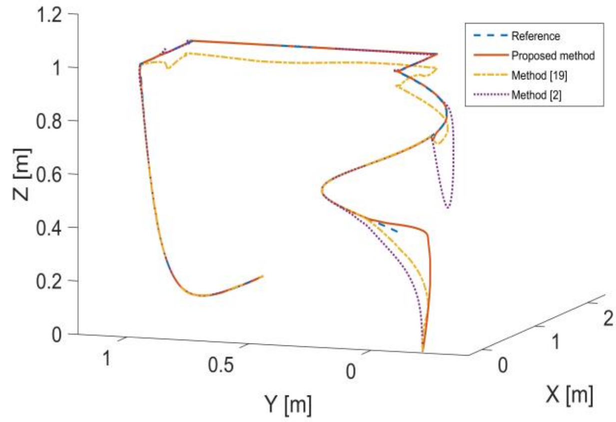


Figure. 9 Trajectory tracking in 3d (scenario nº2)

Table 5. FTAFTSMC parameters for outer loop

| Translationnal subsystem parameters | | | | | |
|---|---|---|---|---|---|
| Param | value | Param | value | Param | value |
| $\aleph_{1x,1y,1z}$ | 0.8 | $q_z$ | 1.5 | $\epsilon_i$ | 0.05 |
| $\aleph_{2x,2y,2z}$ | 30 | $p_z$ | 0.1 | $\lambda_z$ | 0.3 |
| $\aleph_{3x,3y,3z}$ | 2 | $\gamma_z$ | 1 | $\eta_z$ | 0.3 |
| $\alpha_{1z}$ | 0.01 | $\alpha_{2z}$ | 0.001 | $\xi_z$ | 0.08 |
| $\lambda_{2z}$ | 0.5 | $\gamma_{2z}$ | 1 | $\eta_{2z}$ | 0.5 |
| $q_{2z}$ | 1.1 | $p_{2z}$ | 0.4 | $\beta_1$ | 0.8 |
| $\beta_1$ | 0.8 | $\beta_2$ | 0.01 | $q_{x,y}$ | 1.5 |
| $p_{x,y}$ | 0.1 | $\lambda_{x,y}$ | 0.4 | $\gamma_{x,y}$ | 0.4 |
| $\eta_{x,y}$ | 0.4 | $\alpha_{1x,1y}$ | 0.001 | $\alpha_{2x,2y}$ | 0.005 |
| $\xi_{x,y}$ | 0.08 | $\lambda_{2x,2y}$ | 0.2 | $\gamma_{2x,2y}$ | 2 |
| $\eta_{2x,2y}$ | 0.2 | $q_{2x,2y}$ | 4 | $p_{2x,2y}$ | 0.2 |

$$\sum_{i=1}^{n} \Omega_i^q \geq n^{1-q}\left(\sum_{i=1}^{n} \Omega_i\right)^q;\ \ \sum_{i=1}^{n} \Omega_i^p \geq \left(\sum_{i=1}^{n} \Omega_i\right)^p \tag{64}$$

**Fixed time stability and settling time analysis:** Eq. (30), can be rewritten as:

$$\dot{V}_z \leq -\lambda_{2z}(|S_z|^2)^{\frac{q_{2z}+1}{2}} - \gamma_{2z}(|S_z|^2)^{\frac{p_{2z}+1}{2}} \\ -\alpha_{2z}\frac{2\epsilon_{1z}-1}{2\epsilon_{1z}}\tilde{\delta}_z^2 + \alpha_{2z}\frac{\epsilon_{1z}\delta_z^2}{2} + \frac{\xi_z^2}{2} \tag{65}$$

$$\text{Let } \rho = \alpha_{1z}\alpha_{2z}\left(\frac{2\epsilon_{1z}-1}{2\epsilon_{1z}}\right) \tag{66}$$

Adding and subtracting the terms:

$\left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}}, \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}}$ to $\dot{V}_z$, results :

$$\dot{V}_z \leq -\lambda_{2z}(|S_z|^2)^{\frac{q_{2z}+1}{2}} - \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}} \\ -\gamma_{2z}(|S_z|^2)^{\frac{p_{2z}+1}{2}} - \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}} + \Gamma \tag{67}$$

Where $\Gamma = \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}} + \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}} - \frac{\rho}{\alpha_{1z}}\tilde{\delta}_z^2 + \alpha_{2z}\frac{\epsilon_{1z}\delta_z^2}{2} + \frac{\xi_z^2}{2}$ (68)

If $\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2 < 1$ is valid, then we have:

$$\left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}} + \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}} - \frac{\rho}{\alpha_{1z}}\tilde{\delta}_z^2 < \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}} - \frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2 < 1 \quad (69)$$

If $\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2 \geq 1$ is valid, then we have:

$$\left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}} + \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}} - \frac{\rho}{\alpha_{1z}}\tilde{\delta}_z^2 < \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}} - \frac{\rho}{\alpha_{1z}}\tilde{\delta}_z^2 \quad (70)$$

Since $\hat{\delta}_z$ is bounded based on results from Eq. (30), we have a for a certain set $A = \{\tilde{\delta}_z \mid |\tilde{\delta}_z| \leq \delta_{max}\}$.

$$\left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{p_{2z}+1}{2}} + \left(\frac{\rho}{2\alpha_{1z}}\tilde{\delta}_z^2\right)^{\frac{q_{2z}+1}{2}} - \frac{\rho}{\alpha_{1z}}\tilde{\delta}_z^2 < \max\left\{1, \left(\frac{\rho}{2\alpha_{1z}}\delta_{max}^2\right)^{\frac{q_{2z}+1}{2}} - 1\right\} \quad (71)$$

Let $\Omega_1 = \frac{1}{2}|S_z|^2, \Omega_2 = \frac{1}{2\alpha_{1z}}\tilde{\delta}_z^2$ , (72)

$$\bar{\Gamma} = \max\left\{1, \left(\frac{\rho}{2\alpha_{1z}}\delta_{max}^2\right)^{\frac{q_{2z}+1}{2}} - 1\right\} + \alpha_{2z}\frac{\epsilon_{1z}\delta_z^2}{2} + \frac{\xi_z^2}{2} \quad (73)$$

Thus, the inequality (51) can be rewritten as:

$$\dot{V}_z \leq -\lambda_{2z}2^{\frac{q_{2z}+1}{2}}.\Omega_1^{\frac{q_{2z}+1}{2}} - \rho^{\frac{q_{2z}+1}{2}}.\Omega_2^{\frac{q_{2z}+1}{2}} -\gamma_{2z}2^{\frac{p_{2z}+1}{2}}.\Omega_1^{\frac{p_{2z}+1}{2}} - \rho^{\frac{p_{2z}+1}{2}}.\Omega_2^{\frac{p_{2z}+1}{2}} + \bar{\Gamma}$$

$$\dot{V}_z \leq -min\left\{\lambda_{2z}2^{\frac{q_{2z}+1}{2}}, \rho^{\frac{q_{2z}+1}{2}}\right\}.\sum_{i=1}^{2}\Omega_i^{\frac{q_{2z}+1}{2}} - min\left\{\gamma_{2z}2^{\frac{p_{2z}+1}{2}}, \rho^{\frac{p_{2z}+1}{2}}\right\}.\sum_{i=1}^{2}\Omega_i^{\frac{p_{2z}+1}{2}} + \bar{\Gamma} \quad (74)$$

Using lemma 2 the inequality becomes:

$$\dot{V}_z \leq -2^{\frac{1-q_{2z}}{2}} min\left\{\lambda_{2z}2^{\frac{q_{2z}+1}{2}}, \rho^{\frac{q_{2z}+1}{2}}\right\}\left(\sum_{i=1}^{2}\Omega_i\right)^{\frac{q_{2z}+1}{2}} - min\left\{\gamma_{2z}2^{\frac{p_{2z}+1}{2}}, \rho^{\frac{p_{2z}+1}{2}}\right\}.\left(\sum_{i=1}^{2}\Omega_i\right)^{\frac{p_{2z}+1}{2}} + \bar{\Gamma} \quad (75)$$

Let $\overline{\lambda_{2z}} = 2^{\frac{1-q_{2z}}{2}} min\left\{\lambda_{2z}2^{\frac{q_{2z}+1}{2}}, \rho^{\frac{q_{2z}+1}{2}}\right\}$,
$\overline{\gamma_{2z}} = min\left\{\gamma_{2z}2^{\frac{p_{2z}+1}{2}}, \rho^{\frac{p_{2z}+1}{2}}\right\}$ (76)

$$\dot{V}_z \leq -\overline{\lambda_{2z}}\,(V_z)^{\frac{q_{2z}+1}{2}} - \overline{\gamma_{2z}}\,(V_z)^{\frac{p_{2z}+1}{2}} + \bar{\Gamma} \quad (77)$$

According to lemma 1, sliding surface will converge to a small set, with a settling time $T$:

$$T \leq \frac{1}{\overline{\lambda_{2z}}(q_{2z}-1)} + \frac{1}{\overline{\gamma_{2z}}(1-p_{2z})} \quad (78)$$

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