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Anonymous and Separable Hedonic Coalition Formation Games: Nash Stability Under Different Membership Rights

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Abstract: *We consider hedonic coalition formation games. A hedonic coalition formation game is a pair which consists of a finite set of agents and a list of agents' preferences such that each agent has preferences over all coalitions containing her. We study the existence of a Nash stable partition under different membership rights for anonymous and separable hedonic coalition formation games. We prove that for anonymous and separable hedonic games, the existence of a Nash stable partition is always guaranteed when the membership rights are Free Exit-Approved Entry or Approved Exit-Free Entry, but the existence of a Nash stable partition is not guaranteed when the membership rights are Free Exit-Free Entry. We also analyze the relation of the anonymity and separability with the other sufficient conditions which guarantee the existence of a Nash stable partition under different membership rights.*

Keywords: Coalition Formation, Hedonic Games, Nash Stability, Membership Rights

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1. Introduction

In many social, economic and political instances, agents prefer to act in groups (coalitions) rather than staying alone. Some simple examples of coalition formation from daily life are peer groups, hobby groups, research teams in a faculty, project teams at a workplace, homework groups in a course, trade unions, or coalitions of political parties. Other important examples of coalition formation are agreements or cooperation between countries, such as the European Union, NATO, or the Kyoto protocol. Most of these coalition formation instances can be modeled as a hedonic coalition formation game.

A hedonic coalition formation game (briefly, a hedonic game) consists of a finite number of agents and a list of agents' preferences such that each agent has preferences over all coalitions which contain her. That is, each agent only cares about which agents are in her coalition, and does not care how other agents group among themselves. This is called the *hedonic aspect of preferences* and was introduced by Dreze and Greenberg (1980). The formal model of hedonic coalition formation games was first introduced by Banerjee, Konishi, and Sönmez (2001) and Bogomolnaia and Jackson (2002). An outcome of a hedonic game is a partition, that is, a collection of mutually disjoint coalitions such that their union is the set of agents.

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Examples of hedonic games are the well known marriage problems and roommate problems (Gale & Shapley, 1962; Roth & Sotomayor, 1990) at which either singleton or doubleton coalitions form and each agent only cares about his/her mate or roommate. Bogomolnaia and Jackson (2002, Section 2) provided an example of a local public good production model for a comparison of hedonic and non-hedonic settings. Banerjee et al. (2001) provided examples of economic problems that can be considered as hedonic coalition formation models, such as Benassy's (1982) uniform reallocation, Shapley and Scarf's (1974) housing market, Moulin and Shenker's (1994) average cost sharing problem, Farrell and Scotchmer's (1988) partnership game, etc. For futher motivation and examples of hedonic games, we refer to Bogomolnaia and Jackson (2002) and Banerjee et al. (2001).

The existence of stable partitions (for various stability concepts) for a given hedonic game is one of the main focuses in the literature on hedonic coalition formation. Stability concepts consider the deviation of individuals as well as coalitions from a given partition[1]. One of the most studied stability concepts for hedonic games is Nash stability[2]. A partition in a hedonic game is Nash stable if no agent has the incentive to leave her coalition in the partition and join another existing coalition in the partition or stay alone (in this case we say that she joins the empty set). Considering Nash stability, when an agent leaves her coalition in the partition and joins another existing coalition of that partition, she does not care about how the remaining agents in her coalition or the agents in the coalition she joined are affected (positive or negative), that is the deviating agent does not ask the permission of the agents affected by her deviation.

Karakaya (2011) introduced the membership rights approach of Sertel (1992) into hedonic coalition formation games while analyzing stability notions of a partition. That is, when considering a stability concept, the allowed movements for those who deviate should also be considered, along with the agents who deviate. Movements allowed for the deviating agent(s) are determined by specifying membership rights, that is, membership rights define whose approval is required for a particular deviation. Sertel (1992) introduced four membership rights, namely, Free Exit and Free Entry (FX-FE) membership rights, Free Exit and Approved Entry (FX-AE) membership rights, Approved Exit and Free Entry (AX-FE) membership rights, and Approved Exit and Approved Entry (AX-AE) membership rights.

Free Exit and Free Entry (FX-FE) membership rights describe situations in which an agent does not need the permission of anyone while leaving a coalition she is in and joining another coalition. *Free Exit and Approved Entry (FX-AE) membership rights* describe situations in which, while leaving a coalition and joining another one, an agent does not need the permission of any agent of the coalition she leaves, but she needs the permission of every agent of the coalition she wants to join. *Approved Exit and Free Entry (AX-FE) membership rights* describe situations in which, an agent needs the permission of each agent of the coalition she leaves, but she does not need the permission of agents of the coalition she wants to join. *Approved Exit and Approved Entry (AX-AE) membership rights* describes situations in which an agent needs the permission of every agent of coalitions that she leaves and joins.

We say that an agent Nash blocks a partition under the given membership rights if she gets strictly better off by moving to another coalition of the partition and all agents whose permissions are needed under the given membership rights for this movement do not get worse off. A partition is Nash stable under the given membership rights if there is no agent who Nash blocks it under the given membership rights.

We consider anonymous and separable hedonic games and study the existence of a Nash stable partition under different membership rights. An agent's preferences are anonymous if she is indifferent between coalitions of the same size. A hedonic game is anonymous if every agent has anonymous preferences. An agent has separable preferences if, when considering two coalitions that differ by an agent, she prefers coalitions with that agent to coalitions without that agent if and only if that agent is a good agent, she prefers coalitions without that agent to coalitions with that agent if and only if that agent is a bad agent, and she is indifferent between the two coalitions if and only if that agent is a neutral agent. A hedonic game is separable if every agent has separable preferences.

If an agent has anonymous and separable preferences, then she views all other agents as the same. In other words, agents other than herself are either all good, all bad, or all neutral agents according to herself.

Banerjee et al. (2001) showed that if preferences of an agent are anonymous and separable, then her preferences are additively separable, and she assigns the same cardinal utility to every other agent[3].

When we consider an anonymous and separable hedonic game, if an agent in this hedonic game assigns positive utility to other agents (i.e., she considers all other agents as good agents), then she prefers coalitions with higher sizes to coalitions with fewer sizes. If an agent in this hedonic game assigns negative utility to other agents (i.e., she considers all other agents as bad agents), then she prefers coalitions with fewer sizes to coalitions with higher sizes. If an agent in this hedonic game assigns zero utility to other agents (i.e., she considers all other agents as neutral agents), then she is indifferent among all coalitions that contain her. Therefore, for an anonymous and separable hedonic game, it is expected that all agents who assign positive utility to other agents be together in a coalition (possibly with other agents) and an agent who assigns negative utility to other agents forms a a singleton coalition by herself.

In this paper, we investigate the existence of a Nash stable partition under different membership rights for anonymous and separable hedonic games. We prove that for anonymous and separable hedonic games, the existence of an FX-FE Nash stable partition is not guaranteed (Proposition 1). We show that there always exist an FX-AE Nash stable partition (Proposition 2) and an AX-FE Nash stable partition (Proposition 3) for anonymous and separable hedonic games. Moreover, we show the relationship between hedonic games that satisfy the anonymity and separability properties and other hedonic games in the literature in which the existence of an FX-AE Nash stable or an AX-FE Nash stable partition is always guaranteed.

2. Related Literature

The concepts of FX-FE Nash stability, FX-AE Nash stability, and AX-AE Nash stability was introduced by Bogomolnaia and Jackson (2002) with the names of Nash stability, individual stability, and contractual individual stability without considering the membership rights. Bogomolnaia and Jackson (2002) showed that if a hedonic game is additively separable and symmetric, then it has an FX-FE Nash stable partition. Burani and Zwicker (2003) studied hedonic games when agents have *descending separable preferences*, and showed that such a hedonic game always has a partition, which is called the *top segment partition*, that is both FX-FE Nash stable and core stable. Alcalde and Revilla (2004) introduced the *top responsiveness property* as a condition on agents' preferences that illustrates how each agent believes that other agents could complement her in the formation of research teams. Dimitrov and Sung (2006) showed that when agents preferences satisfy the top responsiveness property and *mutuality*, then there exists an FX-FE Nash stable partition. Dimitrov and Sung (2004) and Dimitrov, Borm, Hendrickx, and Sung (2006) introduced the *appreciation of friends* and the *aversion to enemies* properties. These properties are based on the cardinality of friends and the cardinality of enemies in each coalition. Dimitrov and Sung (2004) proved that for each hedonic game satisfying the appreciation of friends property and mutuality, there exists an FX-FE Nash stable partition, and similarly, for each hedonic game satisfying the aversion to enemies property and mutuality, there exists an FX-FE Nash stable partition. Suzuki and Sung (2010) and Aziz and Brandl (2012) introduced the *bottom responsiveness property* as a condition on agents' preferences that each agent pays more attention to the worst aspect of each coalition and has a tendency to choose among options a safer one. Aziz and Brandl (2012) showed that when agents preferences satisfy the bottom responsiveness property and the mutuality, then there exists an FX-FE Nash stable partition. Suksompong (2015) introduced the conditions called *subset neutrality* and *neutral anonymity*. He showed that if a hedonic games satisfies the subset neutrality property or the neutral anonymity property, then there always exists an FX-FE Nash stable partition.

Bogomolnaia and Jackson (2002) proved that if a hedonic game satisfies the *ordered characteristics property*, then there exists an FX-AE Nash stable partition. Suksompong (2015) showed that when a hedonic game satisfies the *common ranking property* of Farrell and Scotchmer (1988), there exists an FX-AE Nash stable partition.

The concept of AX-FE Nash stability was introduced by Sung and Dimitrov (2007) with the name of contractual Nash stability without considering the concepts of membership rights. They showed that if a hedonic game is *separable* and satisfies *weak mutuality*, then it has an AX-FE Nash stable partition.

Bogomolnaia and Jackson (2002) and Ballester (2004) proved that every hedonic game has an AX-AE Nash stable partition.

For a survey of Nash stability under different membreship rights we refer to Karakaya and Özbilen (2023). Computer scientists also work on Nash stability and Nash stability under different membership rights. They study the computational complexity analysis of the problem of finding stable partitions under various conditions. For computational complexity analysis studies, we refer the reader to Ballester (2004), Olsen (2009), Sung and Dimitrov (2010), Aziz, Harrenstein, and Pyrga (2011), Bilo, Fanelli, Flammini, Monaco, and Moscardelli (2018), and Kerkmann and Rothe (2019).

3. Hedonic Coalition Formation

Let $N = \{1,2,\ldots,n\}$ be a nonempty finite set of agents with $n \geq 2$. A nonempty subset S of N is called a *coalition of* N. For any agent $i \in N$, let $\mathcal{C}_i = \{S \subseteq N | i \in S\}$ denote the set of all coalitions of N containing agent i. Each agent $i \in N$ has complete and transitive preferences $\succeq_i$ over $\mathcal{C}_i$.[4] For any two coalitions $S, T \in \mathcal{C}_i$, $S \succeq_i T$ means that agent i (weakly) prefers coalition S to coalition T.

We denote the asymmetric part of $\succeq_i$ by $\succ_i$ which is agent i's strict preferences over coalitions in $\mathcal{C}_i$. For each $S, T \in \mathcal{C}_i$ with $S \neq T$, $S \succ_i T$ means that agent i strictly prefers coalition S to coalition T, and defined as $S \succ_i T$ if and only if [$S \succeq_i T$ but not $T \succeq_i S$]. The indifference relation of agent i will be denoted by $\sim_i$. For each $S, T \in \mathcal{C}_i$, $S \sim_i T$ means that agent i is indifferent between coalitions S and T, and defined as $S \sim_i T$ if and only if [$S \succeq_i T$ and $T \succeq_i S$].

For each $i \in N$, let $\mathcal{R}_i$ denote the set of all preferences of agent i over $\mathcal{C}_i$, and let $\mathcal{R} = \prod_{i \in N} \mathcal{R}_i$ denote the set of all preference profiles of agents in N.

A ***hedonic coalition formation game***, or simply a ***hedonic game***, consists of a finite set of agents N and their preferences $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n) \in \mathcal{R}$ and is denoted by $(N, \succeq)$.

A ***partition*** π for hedonic game $(N, \succeq)$ is a set $\pi = \{S_1, S_2, \ldots, S_K\}$ ($K \leq |N|$ is a positive integer) such that (i) for any $k \in \{1, \ldots, K\}$, $S_k \neq \emptyset$, (ii) $\bigcup_{k=1}^{K} S_k = N$, and (iii) for any $k, l \in \{1, \ldots, K\}$ with $k \neq l$, $S_k \cap S_l = \emptyset$. That is, a partition is a collection of pairwise disjoint coalitions such that their union is equal to the set of agents.

Given any partition π and any $i \in N$, we let $\pi(i)$ denote the unique coalition in π containing agent i. We denote the set of all partitions by Π. Since agents only care about their own coalitions, we have that for any agent $i \in N$ and partitions $\pi, \pi' \in \Pi$, $\pi \succeq_i \pi'$ if and only if $\pi(i) \succeq_i \pi'(i)$.

Next, we introduce the classical concept of voluntary participation, individual rationality, and the classical concept of efficiency, Pareto optimality. A partition π is ***individually rational*** for hedonic game $(N, \succeq)$ if for each $i \in N$, $\pi(i) \succeq_i \{i\}$, where $\{i\}$ denotes the coalition that contains only agent i. A partition π is ***Pareto optimal*** for hedonic game $(N, \succeq)$ if there exists no other partition $\pi' \in (\Pi \backslash \{\pi\})$ such that for each $i \in N$, $\pi'(i) \succeq_i \pi(i)$ and for some $j \in N$, $\pi'(j) \succ_j \pi(j)$.

4. Nash Stability Under Different Membership Rights

We define stability concepts based on individual deviations (Nash stability) under different membership rights[5].

When we have Free Exit and Free Entry (FX-FE) membership rights, a partition is Nash stable (FX-FE Nash stable) if there do not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition of the partition or to the empty set.

Definition 1: ***FX-FE Nash Stability***

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi$ is ***FX-FE Nash stable*** *if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that $S \cup \{i\} \succ_i \pi(i)$. If such a pair (i, S) exists, we then say that agent i* ***FX-FE Nash blocks*** *π (by joining coalition S).*

When we have Free Exit and Approved Entry (FX-AE) membership rights, a partition is Nash stable (FX-AE Nash stable) if there do not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition of the partition (or to the empty set), and the members of this coalition do not become worse off when this agent joins their coalition.

Definition 2: ***FX-AE Nash Stability***

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi$ is ***FX-AE Nash stable*** *if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that*

- *$S \cup \{i\} \succ_i \pi(i)$ and*
- *for all $j \in S$, $S \cup \{i\} \succeq_j S$.*

If such a pair (i, S) exists, we then say that agent i ***FX-AE Nash blocks*** *π (by joining coalition S).*

When we have Approved Exit and Free Entry (AX-FE) membership rights, a partition is Nash stable (AX-FE Nash stable) if there do not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition of the partition (or to the empty set), and each agent of her coalition under the given partition she leaves does not worse off, i.e., each member of her coalition that she leaves approves her leaving.

Definition 3: ***AX-FE Nash Stability***

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi$ is ***AX-FE Nash stable*** *if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that*

- *$S \cup \{i\} \succ_i \pi(i)$ and*
- *for all $k \in (\pi(i) \backslash \{i\})$, $\pi(i) \backslash \{i\} \succeq_k \pi(i)$.*

If such a pair (i, S) exists, we then say that agent i ***AX-FE Nash blocks*** *π (by joining coalition S).*

When we have Approved Exit and Approved Entry (AX-AE) membership rights, a partition is Nash stable (AX-AE Nash stable) if there do not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition of the partition (or to the empty set), and every agent of the coalitions that she leaves and joins does not get worse off by this movement.

Definition 4: ***AX-AE Nash Stability***

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi$ is ***AX-AE Nash stable*** *if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that*

- *$S \cup \{i\} \succ_i \pi(i)$,*
- *for all $j \in S$, $S \cup \{i\} \succeq_j S$, and*
- *for all $k \in (\pi(i) \backslash \{i\})$, $\pi(i) \backslash \{i\} \succeq_k \pi(i)$.*

If such a pair (i, S) exists, we then say that agent i ***AX-AE Nash blocks*** *π (by joining coalition S).*

We give the following example to better understand the concepts of Nash stability, membership rights, and blocking under different membership rights.

Example 1:

Let $G = (N, \succeq)$ where $N = \{1,2,3\}$ and players' preferences are as follows:

$$\succeq_1 : \{1,2,3\} \succ_1 \{1,2\} \succ_1 \{1,3\} \succ_1 \{1\}$$

$$\succeq_2 : \{1,2\} \succ_2 \{2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\}$$

$$\succeq_3 : \{1,3\} \succ_3 \{2,3\} \succ_3 \{3\} \succ_3 \{1,2,3\}$$

There are five partitions, i.e., $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$, where $\pi_1 = \{\{1,2\},\{3\}\}$, $\pi_2 = \{\{1,3\},\{2\}\}$, $\pi_3 = \{\{1\},\{2,3\}\}$, $\pi_4 = \{\{1,2,3\}\}$, and $\pi_5 = \{\{1\},\{2\},\{3\}\}$.

- Consider the partition π_1. We show that the partition π_1 is FX-FE Nash stable.
- The movements among the coalitions of π_1 that agent 1 can do are as follows:
  - Agent 1 leaves her current coalition $\{1,2\}$ and joins the existing coalition $\{3\}$. However, she does not want to do this movement because $[\{1,2\} \succ_1 \{1,3\}]$.
  - Agent 1 leaves her current coalition $\{1,2\}$ and joins the empty set. However, she does not want to do this movement because $[\{1,2\} \succ_1 \{1\}]$.
  - So agent 1 cannot FX-FE Nash block the partition π_1.
- The movements among the coalitions of π_1 that agent 2 can do are as follows:
  - Agent 2 leaves her current coalition $\{1,2\}$ and joins the existing coalition $\{3\}$. However, she does not want to do this movement because $[\{1,2\} \succ_2 \{2,3\}]$.
  - Agent 2 leaves her current coalition $\{1,2\}$ and joins the empty set. However, she does not want to do this movement because $[\{1,2\} \succ_2 \{2\}]$.
  - So agent 2 cannot FX-FE Nash block the partition π_1.
- The movements among the coalitions of π_1 that agent 3 can do are as follows:
  - Agent 3 leaves her current coalition $\{3\}$ and joins the existing coalition $\{1,2\}$. However, she does not want to do this movement because $[\{3\} \succ_3 \{1,2,3\}]$.
  - So agent 3 cannot FX-FE Nash block the partition π_1.
- No agent can FX-FE Nash block the partition π_1. Hence, π_1 is FX-FE Nash stable. Since no agent has a benecial movement for her among the coalitions of π_1 without taking any permission (i.e., under FX-FE membership rights), no agent can have a beneficial movement for her if she is required to take permissions from agents of coalitions that she leaves and/or joins. So, π_1 is also FX-AE Nash stable, AX-FE Nash stable, and AX-AE Nash stable.
- π_1 is individually rational because $\{1,2\} \succ_1 \{1\}$ and $\{1,2\} \succ_2 \{2\}$.

- Consider the partition $\pi_2 = \{\{1,3\},\{2\}\}$. Agent 1 FX-FE Nash blocks the partition π_2, i.e., she leaves her current coalition $\{1,3\}$ and joins the existing joins coalition $\{2\}$ because $[\{1,2\} \succ_1 \{1,3\}]$. Therefore, π_2 is not FX-FE Nash stable. Moreover, π_2 is not FX-AE Nash stable, because agent 1 FX-AE Nash blocks the partition π_2. Agent 1 leaves her coalition $\{1,3\}$ and joins the existing joins coalition $\{2\}$, and agent 2 permits her joining since $[\{1,2\} \succ_2 \{2\}]$.
- We show that π_2 is AX-FE Nash stable.
  - Agent 1 wants to leave $\{1,3\}$ and join the existing coalition $\{2\}$. However, agent 3 does not permit agent 1 to leave $\{1,3\}$, because $[\{1,3\} \succ_3 \{3\}]$. Thus, agent 1 cannot AX-FE Nash block the partition π_2.

  - Agent 2 does not want to join coalition $\{1,3\}$ because, $[\{2\} \succ_2 \{1,2,3\}]$. Thus, agent 2 cannot AX-FE Nash block the partition π_2.
  - Agent 3 does not want to leave coalition $\{1,3\}$ because it is the best coalition for agent 3. Thus, agent 3 cannot AX-FE Nash block the partition π_2.
  - Therefore, π_2 is AX-FE Nash stable.
- π_2 is individually rational because $\{1,3\} \succ_1 \{1\}$ and $\{1,3\} \succ_3 \{3\}$.
- Consider the partition $\pi_3 = \{\{1\},\{2,3\}\}$. We show that π_3 is not AX-AE Nash stable.
  - Agent 3 AX-AE Nash blocks π_3 by leaving her coalition $\{2,3\}$ and joining the existing coalition $\{1\}$ because, $[\{1,3\} \succ_3 \{2,3\}]$, and agents 1 and 2 approve this movement since for agent 1 we have $[\{1,3\} \succ_1 \{1\}]$. and for agent 2 we have $[\{2\} \succ_2 \{2,3\}]$.
  - Since agent 3 AX-AE Nash blocks π_3 , agent 3 also AX-FE Nash blocks π_3, FX-AE Nash blocks π_3, and FX-FE Nash blocks π_3. Then, π_3 is also not AX-FE Nash stable, not FX-AE Nash stable, and not FX-FE Nash stable.
- π_3 is not individually rational because for agent 2 we have $\{2\} \succ_2 \{2,3\}$.

- Consider the partition $\pi_4 = \{\{1,2,3\}\}$. The partition π_4 is not FX-AE Nash stable since agent 2 FX-AE Nash blocks it by leaving $\{1,2,3\}$ and joining the empty set, since $[\{2\} \succ_2 \{1,2,3\}]$, and there is no agent to get permission for joining the empty set. So, π_4 is also not FX-FE Nash stable (since agent 2 also FX-FE Nash blocks π_4). The partition π_4 is AX-FE Nash stable since the coalition $\{1,2,3\}$ is the best coalition for agent 1 and she does not permit any one to leave $\{1,2,3\}$. Thus, π_4 is also AX-AE Nash stable.
- π_4 is not individually rational because for agents 2 and 3 we have $\{2\} \succ_2 \{1,2,3\}$ and $\{3\} \succ_3 \{1,2,3\}$.

- Consider the partition $\pi_5 = \{\{1\},\{2\},\{3\}\}$. We show that π_5 is not AX-AE Nash stable.
  - Agent 1 AX-AE Nash blocks π_5 by leaving $\{1\}$ and joining $\{2\}$ because, $[\{1,2\} \succ_1 \{1\}]$, and agent 2 approves this movement since $[\{1,2\} \succ_2 \{2\}]$. π_5 is not AX-AE Nash stable then, it is also not AX-FE Nash stable, not FX-AE Nash stable, and not FX-FE Nash stable. π_5 is individually rational.

When a partition is FX-FE Nash stable it means that no agent can FX-FE Nash block it without taking any permission from any one. Moreover, since no agent can FX-FE Nash block a partition without taking any permission from any one, then no agent can block that partition when she is required to take some permissions from some agents. Then, if a partition is FX-FE Nash stable, it is also Nash stable under other membership rights, i.e., it is also FX-AE Nash stable, AX-FE Nash stable, and AX-AE Nash stable. With the same argument, we say that if a partition is FX-AE Nash stable then it is also AX-AE Nash stable; and if a partition is AX-FE Nash stable then it is also AX-AE Nash stable. However, a partition which is FX-AE Nash stable might not be AX-FE Nash stable, and vice versa. An FX-FE Nash stable partition and an FX-AE Nash stable partition is individually rational. However, an AX-FE Nash stable partition or an AX-AE Nash stable partition might not be individually rational.

5. Anonymous and Separable Hedonic Games

In this section, we first give the definitions of anonymity, separability, and additive separability. We then show that if the preferences of agents are anonymous and separable, then the preference of each agent can be one of three different preference types. An agent with Type 1 preferences prefers larger coalitions to smaller ones, an agent with Type 2 preferences prefers smaller coalitions to larger ones, and an agent with Type 3 preferences is indifferent among all coalitions.

Definition 5: ***Anonymity***

*Let $(N, \succeq)$ be a hedonic game. Preference $\succeq_i$ of an agent $i \in N$ satisfies **anonymity** if for each $S, T \in \mathcal{C}_i$ with $|S| = |T|$, we have $S \sim_i T$. A hedonic game $(N, \succeq)$ is **anonymous** if preferences of each agent are anonymous.*

If a hedonic game is anonymous then each agent is indifferent among coalitions each of which has the same size.

For an agent $i \in N$, let $G_i = \{j \in N \backslash \{i\} \mid \{i, j\} \succ_i \{i\}\}$ denote the set of agents such that agent i strictly prefers to form a coalition with each of them rather than staying alone (good agents for i), $U_i = \{j \in N \backslash \{i\} \mid \{i, j\} \sim_i \{i\}\}$ denote the set of agents such that agent i is indifferent between to form a coalition with each of them and staying alone (neutral agents for i), and $B_i = \{j \in N \backslash \{i\} \mid \{i\} \succ_i \{i, j\}\}$ denote the set of agents such that agent i strictly prefers to stay alone rather than forming a coalition with each of them (bad agents for i).

Definition 6: ***Separability***

*Let $(N, \succeq)$ be a hedonic game. Preference $\succeq_i$ of an agent $i \in N$ satisfies **separability** if, for every $j \in N$ with $i \neq j$ and for each coalition $S \in \mathcal{C}_i$ and $j \notin S$,*

- *$j \in G_i \Leftrightarrow S \cup \{j\} \succ_i S$,*
- *$j \in U_i \Leftrightarrow S \cup \{j\} \sim_i S$, and*
- *$j \in B_i \Leftrightarrow S \succ_i S \cup \{j\}$.*

*A hedonic game is **separable** if preferences of each agent are separable.*

*A separable hedonic game $(N, \succeq)$ satisfies **mutuality** if for each $i, j \in N$, we have $[\{i, j\} \succeq_i \{i\} \Leftrightarrow \{i, j\} \succeq_j \{j\}]$ and $[\{i\} \succeq_i \{i, j\} \Leftrightarrow \{j\} \succeq_i \{i, j\}]$.*

*A separable hedonic game $(N, \succeq)$ satisfies **weak mutuality** if for all agents $i \in N$ such that there exists $j \in N \backslash \{i\}$ with $\{i, j\} \succeq_i \{i\}$, then there exists $k \in N \backslash \{i\}$ with $\{i, k\} \succeq_k \{k\}$.*

An agent i has separable preferences if for any agent j, the effect of agent j on the preferences of agent i is consistent with whether j is a good, neutral, or bad agent for i regardless of to which coalition agent j is added. That is, an agent i has separable preferences if it holds that [agent j is a good agent for i (that is, $j \in G_i$) if and only if for any coalition S, agent i strictly prefers $S \cup \{j\}$ to S], [agent j is a neutral agent for i (i.e., $j \in U_i$) if and only if for any coalition S, agent i is indifferent between $S \cup \{j\}$ and S], and [agent j is a bad agent for i (that is, $j \in B_i$) if and only if for any coalition S, agent i strictly prefers S to $S \cup \{j\}$].

A separable hedonic game satisfies mutuality if, for each pair of agents, one agent prefers to cooperate with the other to stay alone if and only if the other behaves in the same way, and one agent prefers to be alone to cooperate with the other if and only if the other behaves in the same way. A separable hedonic game satisfies weak mutuality if, for each agent, there exists an agent with whom she prefers to stay together rather than to be alone, then there exists another agent who prefers staying with her rather than being alone.

Definition 7: ***Additive separability***

*Let $(N, \succeq)$ be a hedonic game. Preference $\succeq_i$ of an agent $i \in N$ is **additively separable** if there exists a function $v_i: N \to \mathbb{R}$ such that for any $S, T \in \mathcal{C}_i$,*

$$S \succeq_i T \Leftrightarrow \sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j), where\ v_i(j) = 0\ for\ i = j.$$

*A hedonic game $(N, \succeq)$ is **additively separable** if preferences of each agent are additively separable. An additively separable hedonic game $(N, \succeq)$ satisfies **symmetry** if for any $i, j \in N$, we have $v_i(j) = v_j(i)$, and satisfies **mutuality** if for any $i, j \in N$, we have* $v_i(j) \geq 0 \Leftrightarrow v_j(i) \geq 0$ and $v_i(j) > 0 \Leftrightarrow v_j(i) > 0$.

For each agent i, v_i denotes her utility function that assigns a cardinal utility for every agent in N, where she assigns zero value to herself. For any coalition $S \in \mathcal{C}_i$, the total payoff that agent i obtains from being a member of this coalition is the sum of the utilities that she assigns to each agent in S, that is, $\sum_{j \in S} v_i(j)$. Then, any two coalitions containing agent i are compared according to the total payoffs that she obtains from these coalitions, that is, for any $S, T \in \mathcal{C}_i$, agent i prefers S to T if and only if the total payoff that i obtains from S is as big as the total payoff that she obtains from T. Note that additive separability implies separability.

Additively separable preferences are symmetric if every two agents assign the same utilities to each other, that is, for every agents i and j, $v_i(j) = v_j(i)$. Additively separable preferences satisfy mutuality if it holds for every two agents i and j that i assigns a positive (negative or zero, respectively) value to j if and only if j assigns a positive (negative or zero, respectively) value to i. Note that symmetry implies mutuality.

Banerjee et al. (2001) noted that when agent i has anonymous and separable preferences, her preferences are also additively separable[6]. Moreover, an agent assigns the same value to every other agent, i.e., for each $i \in N$ and each $j, k \in (N \backslash \{i\})$, $v_i(j) = v_i(k)$. To see this, consider all doubleton coalitions containing agent i. Since agent i has anonymous preferences, she is indifferent among all such coalitions. This fact, together with additive separability of preferences, implies that agent i assigns the same value to every other agent. So, when preferences are anonymous and separable, there are three types of preferences that an agent can have depending on whether an agent assigns a positive, negative or zero value to all other agents. So, an agent i with separable and anonymous preferences has one of the following type of preferences.

Type 1 preferences: When an agent assigns a positive value to all other agents, then her preferences are as follows: The unique best coalition is $\{N\}$, then the second best coalitions are ones with $n-1$ sizes (our agent is indifferent among all of them), then the third best coalitions are with $n-2$ sizes, and so on, and the worst coalition is staying alone ($\{i\}$).

Type 2 preferences: When an agent assigns a negative value to all other agents, then her preferences are as follows: The unique best coalition is staying alone ($\{i\}$), then the second best coalitions are ones with sizes 2 (our agent is indifferent among all of them), then the third best coalitions are with sizes 3, and so on, and the worst coalition is the grand coalition ($\{N\}$).

Type 3 preferences. When an agent assigns zero value to all other agents, then her preferences are as follows: She is indifferent among all coalitions.

Example 2: ***An Anonymous and Separable Hedonic Game***

Let $G = (N, \succeq)$ where $N = \{1,2,3\}$ and players' preferences are as follows:

$\succeq_1 : \{1,2,3\} \succ_1 \{1,3\} \sim_1 \{1,2\} \succ_1 \{1\}$.

$\succeq_2 : \{2\} \succ_2 \{1,2\} \sim_2 \{2,3\} \succ_2 \{1,2,3\}$,

$\succeq_3 : \{3\} \sim_3 \{1,3\} \sim_3 \{2,3\} \sim_3 \{1,2,3\}$.

This hedonic game is anonymous and separable. So, preferences are representable by additively separable functions $v = (v_i)_{i \in N}$:

$$v_1(1) = 0,\ v_1(2) = v_1(3) = x,$$

$$v_2(1) = v_2(3) = y,\ v_2(2) = 0,$$

$$v_3(1) = v_3(2) = v_3(3) = 0,$$

where $x, y \in \mathbb{R}$ with $x > 0$, and $y < 0$.

Agent 1 has Type 1 preferences, agent 2 has Type 2 preferences, and agent 3 has Type 3 preferences.

6. Results

In this section, we provide and prove our results.

Proposition 1: *The existence of an FX-FE Nash stable partition is not guaranteed for any anonymous and separable hedonic game* $(N, \succeq)$.

We prove Proposition 1 by providing a counter-example. The following hedonic game satisfies the anonymity and separability properties, however it has no FX-FE Nash stable partition.

Example 3:

Let $G = (N, \succeq)$ where $N = \{1,2,3\}$ and players' preferences are as follows:

$\succeq_1: \{1,2,3\} \succ_1 \{1,3\} \sim_1 \{1,2\} \succ_1 \{1\}$.

$\succeq_2: \{2\} \succ_2 \{1,2\} \sim_2 \{2,3\} \succ_2 \{1,2,3\}$,

$\succeq_3: \{3\} \succ_3 \{1,3\} \sim_3 \{2,3\} \succ_3 \{1,2,3\}$,

This hedonic game is anonymous and separable. So, preferences are representable by additively separable functions $v = (v_i)_{i \in N}$:

$$v_1(1) = 0, v_1(2) = v_1(3) = x,$$

$$v_2(1) = v_2(3) = y,\ v_2(2) = 0,$$

$$v_3(1) = v_3(2) = z,\ v_3(3) = 0,$$

where $x, y, z \in \mathbb{R}$ with $x > 0$, $y < 0$, and $z < 0$.

There are five partitions, i.e., $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$, where $\pi_1 = \{\{1\},\{2\},\{3\}\}$, $\pi_2 = \{\{1,2\},\{3\}\}$, $\pi_3 = \{\{1,3\},\{2\}\}$, $\pi_4 = \{\{1\},\{2,3\}\}$, and $\pi_5 = \{\{1,2,3\}\}$.

- Agent 1 FX-FE Nash blocks the partition π_1 by joining any of the existing coalitions $\{2\}$ or $\{3\}$.
- Agent 2 FX-FE Nash blocks the partition π_2 by leaving the coalition $\{1,2\}$ and staying alone.
- Agent 3 FX-FE Nash blocks the partition π_3 by leaving the coalition $\{1,3\}$ and staying alone.
- Agent 2 or agent 3 FX-FE Nash blocks the partition π_4 by leaving the coalition $\{2,3\}$ and staying alone. Agent 1 FX-FE Nash blocks the partition π_4 by joining the coalition $\{2,3\}$.
- Agent 2 or agent 3 FX-FE Nash blocks the partition π_5 by leaving the coalition $\{1,2,3\}$ and staying alone. Therefore, none of the partitions are FX-FE Nash stable.

We note that if a partition is FX-AE Nash stable then it is also individually rational. We now state and prove our second result.

Proposition 2: *Let* $(N, \succeq)$ *be an anonymous and separable hedonic game. Then, there exists an FX-AE Nash stable partition.*

***Proof*.** Let $(N, \succeq)$ be an anonymous and separable hedonic game. We show that there exists an FX-AE Nash stable partition.

If the society consists of only agents with Type 1 preferences then the partition $\pi = \{\{N\}\}$ that consists of the grand coalition is FX-AE Nash stable partition. If the society consists of only agents with Type 2 preferences then the partition $\pi' = \{\{1\},\{2\},\dots,\{n\}\}$ that contains all singleton coalitions is FX-AE Nash stable. If the society consists of only agents with Type 3 preferences then all partitions are FX-AE Nash stable.

When there are agents only with Type 1 and Type 2 preferences in the society. Let $S \subsetneq N$ denote agents with Type 1 preferences. Consider the partition $\tilde{\pi} = \{S,\ (\{j\})_{j \in N \setminus S}\}$, where agents with Type 1 preferences form the coalition S and each agent j with Type 2 preferences forms a singleton coalition by herself. We will show that the partition $\tilde{\pi}$ is FX-AE Nash stable. If an agent with Type 1 preferences wants to leave coalition S and join a singleton coalition formed by an agent with Type 2 preferences, then, since the membership rights is FX-AE, no agent with Type 2 preferences approves a joining of any agent to herself. So, no agent with Type 1 preferences can FX-AE Nash blocks the partition $\tilde{\pi}$. Moreover, no Type 2 preferences agent wants to move to another coalition. Hence, the partition $\tilde{\pi}$ is FX-AE Nash stable.

When there are agents only with Type 1 and Type 3 preferences in the society. Then, it is clear that the partition $\pi = \{\{N\}\}$ that consists of the grand coalition is FX-AE Nash stable.

When there are agents only with Type 2 and Type 3 preferences in the society. Then, the partition $\pi' = \{\{1\}, \{2\}, \dots, \{n\}\}$ that contains all singleton coalitions is FX-AE Nash stable.

We consider that there are agents with Type 1, Type 2, and Type 3 preferences in the society. Let $S \subsetneq N$ denote agents with Type 1 preferences and $H \subsetneq N$ denote agents with Type 3 preferences. In this case, we consider the partition $\hat{\pi} = \{S \cup H,\ (\{j\})_{j \in N \setminus (S \cup H)}\}$ at which all agents with Type 1 and Type 3 preferences form a coalition and each agent with Type 2 preferences forms a singleton coalition by herself. Since $|S \cup H| > 1$, no agent with Type 1 preferences wants to leave her coalition $S \cup H$ and to join a singleton coalition formed by an agent with Type 2 preferences. An agent with Type 2 preferences is in her best coalition under partition $\hat{\pi}$, so a Type 2 preferences agent does not want to move to another coalition. Since an agent with Type 3 preferences is indifferent among all coalitions containing her, a Type 3 preferences agent cannot gain by moving to another coalition. Hence, the partition $\hat{\pi}$ is FX-AE Nash stable.

We revisit Example 3. The partition π_1 is the unique FX-AE Nash stable partition (which is also individually rational). None of the remaining partitions is FX-AE Nash stable partition.

We note that if a partition is AX-FE Nash stable then it might not be individually rational. So then, the question of whether there exists a partition which is AX-FE Nash stable and individually rational for an anonymous and separable hedonic games arises. Our following third result answers this question.

Proposition 3: *Let $(N, \succeq)$ be an anonymous and separable hedonic game. Then, we have following:*

(i) There exists an AX-FE Nash stable partition.

(ii) There exists an AX-FE Nash stable and individually rational partition unless the society contains a unique agent with Type 1 preferences and some agents with Type 2 preferences.

***Proof*.** Let $(N, \succeq)$ be an anonymous and separable hedonic game. In order to prove (i) and (ii) we show that there exists an AX-FE Nash stable and individually rational partition except for the case that the society contains a unique agent with Type 1 preferences and some agents with Type 2 preferences, and for that case we will show that there exists an AX-FE Nash stable partition which is not individually rational.

If there exists no agent with Type 1 preferences (i.e., the society consists of agents with Type 2 and/or Type 3 preferences), then the partition $\pi' = \{\{1\}, \{2\}, \dots, \{n\}\}$ is AX-FE Nash stable and individually rational.

If there exists no agent with Type 2 preferences (i.e., the society consists of agents with Type 1 and/or Type 3 preferences), then the partition $\pi = \{\{N\}\}$ is AX-FE Nash stable and individually rational.

We now consider the case that there are agents with Type 1, Type 2, and Type 3 preferences in the society. Let $\emptyset \neq S \subsetneq N$ denote agents with Type 1 preferences, $\emptyset \neq T \subsetneq N$ denote agents with Type 2 preferences, and $\emptyset \neq H \subsetneq N$ denote agents with Type 3 preferences. Let $\hat{\pi} = \{S \cup H, (\{j\})_{j \in T}\}$, i.e., agents with Type 1 and Type 3 preferences form the coalition $S \cup H$ and each agent with Type 2 preferences forms a singleton coalition. Since $|S \cup H| \geq 2$, an agent with Type 1 preferences does not want to leave the coalition $S \cup H$ and enter a singleton coalition $\{j\}$ which is formed by a Type 2 preferences agent. Since a Type 3 preferences agent is indifferent among all coalitions that contain her, an agent with Type 3 preferences cannot gain by moving to another coalition of the partition $\hat{\pi}$. Since staying single is the best option for each agent with Type 2 preferences, no agent with Type 2 preferences wants to move to another coalition of the partition. Hence, the partition $\hat{\pi}$ is AX-FE Nash stable. The partition $\hat{\pi}$ is individually rational since for each $i \in N$, $\hat{\pi}(i) \succeq_i \{i\}$.

We now consider that if there are agents with Type 1 and Type 2 preferences in the society. Let $\emptyset \neq S \subsetneq N$ denote agents with Type 1 preferences and $\emptyset \neq N \backslash S \subsetneq N$ denote agents with Type 2 preferences. We will consider two cases:

• $|S| \geq 2$, there are at least two agents with Type 1 preferences.

We consider the partition $\tilde{\pi} = \{S, (\{j\})_{j \in N \backslash S}\}$, i.e., agents with Type 1 preferences form the coalition S and each agent with Type 2 preferences forms a singleton coalition. Since there are at least two agents in coalition S, no agent in S can gain by moving to another coalition of the partition. Each agent with Type 2 preferences is singleton at $\tilde{\pi}$ and it is the best coalition for each Type 2 preferences agent. So, the partition $\tilde{\pi}$ is AX-FE Nash stable. Moreover, it is individually rational since $\tilde{\pi}(i) \succeq_i \{i\}$ for all $i \in N$.

• $|S| = 1$, there exists only one agent with Type 1 preferences.

Consider a set of partitions $\Pi(S) = \{T, (\{j\})_{j \in N \backslash T}\}$, where $T \supsetneq S$ and each agent who is not in T stays single.

We show that any partition $\pi \in \Pi(S)$ is AX-FE Nash stable. It is clear that every singleton coalition at any partition $\pi \in \Pi(S)$ is formed by an agent who has Type 2 preferences, so no singleton agent wants to join coalition $T \in \pi$. Since $T \supsetneq S$, i.e., coalition T containing the unique agent i with Type 1 preferences also contains some other agent at partition π, agent i cannot benefit by leaving T and then joining a singleton coalition. Moreover, since the right structure is AX-FE, if some agent other than i wants to leave coalition T, agent i does not approve her to leave coalition T. Hence, each $\pi \in \Pi(S)$ is AX-FE Nash stable. In fact, $\Pi(S)$ consists of all partitions that are AX-FE Nash stable.

We now show that each partition in $\Pi(S)$ is not individually rational. For each $\pi \in \Pi(S)$, coalition T contains some agent with Type 2 preferences, and the partition π is not individually rational for those agents. That is, for each agent j with Type 2 preferences (i.e., $j \in N \backslash S$) such that $j \in T$ we have for each $\pi \in \Pi(S)$ that $\{j\} \succ_j \pi(j)$. Hence, no partition in $\Pi(S)$ is individually rational. In fact, the unique individually rational partition is the one that consists of all singleton coalitions, and it is not AX-FE Nash stable (since the unique agent with Type 1 preferences AX-FE Nash blocks it by joining another singleton coalition).

We revisit Example 3. In the hedonic game there, the partition π_1 is not AX-FE Nash stable since agent 1 AX-FE Nash blocks it (e.g, by joining coalition $\{2\}$). So, in the proof of Proposition 3 we require that $T \supsetneq S$ when $|S| = 1$. Since there is only one agent with Type 1 preferences, i.e., $S = \{1\}$, we have $\Pi(S) = \{\pi_2, \pi_3, \pi_5\}$, and each partition in $\Pi(S)$ is AX-FE Nash stable. However, no partition in $\Pi(S)$ is individually rational. Hence, there is no partition that is both individually rational and AX-FE Nash stable. We note that the partition π_4 is not AX-FE Nash stable since agent 2 wants to leave coalition $\{2,3\}$ and stays alone, and agent 3 approves agent 2's leaving from $\{2,3\}$.

7. Relations Between Properties

In this section, we study the relationship between anonymity and separability properties of hedonic games and the properties of hedonic games in which the existence of an FX-AE Nash stable or an AX-FE Nash stable partition is guaranteed in the literature.

7.1. Ordered Characteristics Property

Bogomolnaia and Jackson (2002) introduced the ordered characteristics property and showed that it guarantees the existence of an FX-AE Nash stable partition. In this section, we show that if a hedonic game satisfies anonymity and separability, it also satisfies the ordered characteristics property. However, the converse is not true. A hedonic game which satisfies the ordered characteristics property may not be anonymous and separable.

We firstly give the definition of the ordered characteristics property. We will follow Bogomolnaia and Jackson (2002) to define the ordered characteristics property.

Let each coalition $S \subseteq N$ be described by a *characteristic* $c(S)$ that lies in $\{0,1,\ldots,|S|\}$. Let each agent $i \in N$ has single-peaked preferences on $\{0,1,\ldots,n\}$ with peaks denoted by p_i such that $p_i \geq 1$. Agents' preferences over coalitions correspond to the preference ranking of $c(S)$, that is, for all $i \in N$, all $S,T \in \mathcal{C}_i$, $S \succeq_i T$ if and only if $c(S) \succeq_i c(T)$.

Definition 8: ***Ordered characteristics***

A hedonic game $(N,\succeq)$ has ***ordered characteristics*** *if agents' preferences over coalitions depend on single-peaked preferences over characteristics $c(S)$ where:*

- *If $c(S) < |S|$ then $c(S) = p_j$ for some $j \in S$, and*
- *If $i \notin S$, $j \notin S$, and $p_i \geq p_j$, then $c(S \cup \{i\}) \geq c(S \cup \{j\})$. Moreover,*

if $c(S \cup \{i\}) > p_i$, then $c(S \cup \{i\}) = c(S \cup \{j\})$.

The first condition states that if a characteristic of a coalition is smaller than the size of that coalition, then the characteristic is the peak of some agent in that coalition. The first part of the second condition states that when comparing any two coalitions which differ by only one agent, the characteristics of these coalitions are ordered by the peaks of the agents who differ. The second part states that if the peak of the agent who has a higher peak than other agent is smaller than the characteristic of the coalition that contains her, then the characteristics of these two coalitions that differ by one agent are equal.

Proposition 4: *Let $(N,\succeq)$ be an anonymous and separable hedonic game. Then, $(N,\succeq)$ satisfies the ordered characteristics property.*

Bogomolnaia and Jackson (2002) noted that if in a hedonic game, agents' preferences are anonymous and single-peaked on the sizes of the coalitions to which they belong, then this hedonic game satisfies the ordered characteristics property. We prove Proposition 4 by showing that if in a hedonic game preferences of each agent are anonymous and separable, then their preferences are also single-peaked on the sizes of the coalitions to which they belong, hence the hedonic game satisfies the ordered characteristics property.

Proof. (Proposition 4)

Let $(N,\succeq)$ be an anonymous and separable hedonic game. Then, for each $i \in N$, for each $j,k \in N\backslash\{i\}$, $v_i(j) = v_i(k) = x$.

Let $i \in N$, and let $x > 0$. Then, the preference relation $\succeq_i$ of the agent i is a Type 1 preference. But then, $\succeq_i$ is anonymous and single-peaked on the sizes of the coalitions in $\mathcal{C}_i$ with the peak of agent i as $p_i = n$ and with ordering on sizes of coalitions as $[n \succ_i (n-1) \succ_i \ldots \succ_i 2 \succ_i 1]$.

Let $i \in N$, and let $x < 0$. Then, the preference relation $\succeq_i$ of the agent i is a Type 2 preference. But then, $\succeq_i$ is anonymous and single-peaked on the sizes of the coalitions in $\mathcal{C}_i$ with the peak of agent i as $p_i = 1$ and with ordering on sizes of coalitions as $[1 \succ_i 2 \succ_i \ldots \succ_i (n-1) \succ_i n]$.

Let $i \in N$, and let $x = 0$. Then, the preference relation $\succeq_i$ of the agent i is a Type 3 preference and she is indifferent between all coalitions in $\mathcal{C}_i$. But then, for each $S \in \mathcal{C}_i$, $c(S) = 1$, and $p_i = 1$, and for each $S,T \in \mathcal{C}_i$, since $c(S) = c(T)$, we have $S \sim_i T$.

Therefore, $(N,\succeq)$ satisfies the ordered characteristics property.

We note that a hedonic game satisfying the ordered characteristics property may not be separable. For such an example we refer to Example 11 of Bogomolnaia and Jackson (2002, page 219). The hedonic game, given there, is anonymous and agents' preferences are single-peaked over the sizes of coalition that they belong, and hence it satisfies the ordered characteristics property. However, that game is not separable.

7.2. Separability and Weak Mutuality

Sung and Dimitrov (2007) introduced the concept of AX-FE Nash stability (by naming it contractual Nash stability) and showed that if a hedonic game is separable and satisfies weak mutuality, then it has an AX-FE Nash stable partition.

In this section, we show that if a hedonic game is anonymous and separable, it may not be weakly mutual, and if a hedonic game is separable and weakly mutual, it may not be anonymous. In the following example, the hedonic game satisfies separability and weak mutuality but it does not satisfy anonymity.

Example 4: ***Separable and weakly mutual hedonic game***

Let $(\mathrm{N}, \succeq)$ be a hedonic game with $\mathrm{N} = \{1,2,3\}$ and $\mathrm{G}_1 = \{2\}$, $\mathrm{G}_2 = \{3\}$, $\mathrm{G}_3 = \{1\}$, $\mathrm{U}_1 = \{3\}$, $\mathrm{U}_2 = \{1\}$, $\mathrm{U}_3 = \{2\}$, and for each $\mathrm{i} \in \mathrm{N}$, $\mathrm{B_i} = \emptyset$. Then, we have the following separable preference profile:

$\succeq_1: \{1,2\} \sim_1 \{1,2,3\} \succ_1 \{1\} \sim_1 \{1,3\}$,

$\succeq_2: \{2,3\} \sim_2 \{1,2,3\} \succ_2 \{2\} \sim_1 \{1,2\}$,

$\succeq_3: \{1,3\} \sim_3 \{1,2,3\} \succ_3 \{3\} \sim_3 \{2,3\}$.

This hedonic game satisfies weak mutuality, since for agent 1 we have $\{1,2\} \succ_1 \{1\}$, for agent 2 we have $\{2,3\} \succ_2 \{2\}$, and for agent 3 we have $\{1,3\} \succ_3 \{3\}$.

This hedonic game does not satisfy anonymity since no agent is indifferent between same-size coalitions.

We revisit Example 3. The hedonic game there satisfies anonymity and separability. We observe that $v_1(2) = v_1(3) = x > 0$, $v_2(1) = v_2(3) = y < 0$, and $v_3(1) = v_3(2) = z < 0$. The weak mutuality property is violated, i.e., for agent 1 we have $\{1,2\} \succ_1 \{1\}$ and $\{1,3\} \succ_1 \{1\}$, however, for agents 2 and 3 we have that $\{2\} \succ_2 \{1,2\}$ and $\{3\} \succ_3 \{1,3\}$.

7.3. Common Ranking Property

In this section, we will show that if a hedonic game is anonymous and separable, it may not satisfy the common ranking property, and if a hedonic game satisfies the common ranking property, it may not be anonymous and separable.

The *common ranking property* was introduced by Farrell and Scotchmer (1988). The common ranking property requires that there is a linear order on the set of all coalitions which coincides with any agent's preference ordering over her coalitions. Suksompong (2015) proved that when a hedonic game satisfies the common ranking property, there exists an FX-AE Nash stable partition.

Definition 9: ***Common ranking property***

A hedonic game $(N, \succeq)$ satisfies the ***common ranking property*** *if there exists an ordering $\trianglerighteq$ over $2^N \backslash \{\emptyset\}$ such that for each $i \in N$ and each $S, T \in \mathcal{C}_i$, we have*

$S \succeq_i T \;\; \Leftrightarrow \;\; S \trianglerighteq T.$

In the following example, the hedonic game satisfies the common ranking property but it is not anonymous and separable.

Example 5: ***A hedonic game satisfying the common ranking property***

Let $(N, \succeq)$ where $N = \{1,2,3\}$ and the preferences of agents are as follows:

$\succeq_1: \{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\}$,

$\succeq_2: \{1,2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{2\}$,

$\succeq_3: \{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}$.

This hedonic game satisfies the common ranking property with respect to the ordering $\succeq$, where $[\succeq : \{1,2\} \succ \{1,3\} \succ \{2,3\} \succ \{1,2,3\} \succ \{1\} \sim \{2\} \sim \{3\}]$.

However, this game is neither anonymous nor separable. It is clear that it is not anonymous since for coalitions $\{1,2\}$ and $\{1,3\}$ we have $|\{1,2\}| = |\{1,3\}| = 2$, but $\{1,2\} \succ_1 \{1,3\}$. This game is not separable. Consider agent 1's preferences that agent 2 is a good agent for agent 1, i.e., $2 \in G_1$, since $\{1,2\} \succ_1 \{1\}$. Then, separability requires that agent 1 prefers coalition $\{1,2,3\}$ to coalition $\{1,3\}$, however, we have that $\{1,3\} \succ_1 \{1,2,3\}$, So, agent 1's preferences are not separable.

The hedonic game in Example 3 satisfies anonymity and separability, however it does not satisfy the common ranking property. We have $\{1,2,3\} \succ_1 \{1,2\}$ but $\{1,2\} \succ_2 \{1,2,3\}$, so there is no linear order $\succeq$ on the set of all coalitions which coincides with any agent's preferences.

8. Conclusion

We consider anonymous and separable hedonic coalition formation games and study the existence of Nash stable partitions under different membership rights. First, we analyze anonymous and separable hedonic coalition formation games in detail and show that each agent's preferences can be one of three different types in these games. Next, we prove that the existence of an FX-FE Nash stable partition is not guaranteed in anonymous and separable hedonic coalition formation games, but there always exist an FX-AE Nash stable partition and an AX-FE Nash stable partition. We also study the relations of anonymity and separability with other conditions that guarantee the existence of an FX-AE Nash stable or an AX-FE Nash stable partition in the literature.

This paper is the first one in the literature on hedonic coalition formation that comprehensively handles anonymous and separable hedonic coalition formation games and analyzes the concept of Nash stability by associating it with the concepts of membership rights. We note that the anonymity and separability conditions imply the ordered characteristics property of Bogomolnaia and Jackson (2002), while other sufficient conditions in the literature (for the existence of an FX-AE Nash stable partition and an AX-FE Nash stable partition) and anonymity and separability are independent.

This paper raises some open research questions in the field of hedonic coalition formation games. Anonymity and separability properties are sufficient for the existence of a partition which is FX-AE Nash stable or AX-FE Nash stable (by Proposition 2 and Proposition 3), however they are not necessary. Besides, all the properties we mention in Section 7 are also sufficient conditions that guarantee the existence of a Nash stable partition under different membership rights. So, an open research area is to study the existence of both sufficient and necessary conditions that guarantee the existence of a Nash stable partition under different membership rights. Another open question is to study the conditions that gurantee the uniqueness of a Nash stable partition under different membership rights.

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Endnotes

1. *We refer the reader to Sung and Dimitrov (2007) for taxonomic analysis of some stability concepts and to Hajduková (2006) for a literature review of hedonic games.*
2. *The other one is core stability. A partition in a hedonic game is core stable if no group of agents has the incentive to come together and form a new coalition.*
3. *An agent's preferences are additively separable if there exists a utility function, which maps the set of agents to real numbers and assigns a cardinal utility to each agent.*
4. *A preference relation $\gtrsim$ over $\mathcal{C}_i$ satisfies completeness if for all $S, T \in \mathcal{C}_i$, $S \gtrsim T$ or $T \gtrsim S$, and it satisfies transitivity if for all $S, T, U \in \mathcal{C}_i$, if $S \gtrsim T$ and $T \gtrsim U$, then $S \gtrsim U$.*
5. *We note that Nash stability under different membership rights can also be defined by using the reachability notion (Karakaya, 2011), for such definitions we refer readers to Karakaya and Özbilen (2023).*
6. *Banerjee et al. (2001) showed that an anonymous and separable hedonic game has a core stable partition*

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