Fuzzy finite element analysis based on the transformation between fuzzy and random variables

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Abstract:

In this paper, a fuzzy finite element method (FFEM) for determining the responses of structures is proposed by using the transformation between fuzzy and random variables. Firstly, the formulae for establishing normal random variables equivalent to symmetric triangular fuzzy numbers are presented based on the combination of the principle of insufficient reason and that of maximum specificity. As a result, fuzzy finite element analysis is transferred into stochastic finite element analysis. To solve this problem, the response surface method with the aid of standard normal random variables is utilized to approximate the real responses of structures. Then, the errors between the training and test sets are estimated to select a suitable response surface model amongst the regression models. Lastly, the formulae for determining the mean and the standard deviation values of the responses of structures are established. The accuracy and effectiveness of the proposed method are verified via an illustrative example.

Keywords: **fuzzy finite element, fuzzy sets theory, possibility-probability transformations, response surface method, stochastic finite element, surrogate model.**

Classification number: **2.3**

Introduction

FFEM is the combination of finite element method (FEM) and fuzzy sets theory [1], which is used to define the responses of structures in cases where the input quantities contain incomplete information such as loads, material and geometric properties, stiffness of supports, etc., and is described in the form of fuzzy numbers. In recent years, a large number of FFEMs have been proposed beginning with static analysis, then extending into the dynamic analysis of structures. Fundamental strategies of FFEM can be categorized into main groups as follows: the interval arithmetic approach for static analysis of structures [2-8], the optimization strategy for static and dynamic analysis of structures [9-20], the combination of interval arithmetic and optimization strategy for dynamic analysis of structures [21-23], and applying stochastic finite element methods (SFEM) [24] for fuzzy analysis of structures [25-27]. Amongst these approaches, the α-optimization method [9] has been gradually acknowledged as the standard procedure for FFEM. In this method, the search process is performed in the input domain to seek the exact bounds on the objective function by iteratively evaluating the objective function at designated points. Therefore, this method is very time-consuming because it must be accomplished with a large amount of finite element analysis.

To overcome this limitation, this study focuses on a novel approach for analysing fuzzy finite elements by using the transformation between fuzzy and random variables. Firstly, based on the combination of the principle of insufficient reason and that of maximum specificity, normal random variables equivalent to symmetric triangular fuzzy numbers of the input data are presented and explored in detail. As a result, these fuzzy numbers are replaced by normal random variables so SFEM such as Monte Carlo simulations, the perturbation method, the spectral stochastic finite element method, and so on can be used for the analysis of structures. In this study, the response surface method (RSM) is utilized to approximate the real responses of structures. Then, the least-squares error criterion between the training and the test sets is proposed to select the suitable response surface model amongst the regression models. Lastly, the mean and the standard deviation values of the responses of structures are directly calculated from the response surface model. The proposed method is verified via a two-bar truss structure with spring supports.

Methods

The overview of transformation principles and the formulae for determining the deviation of the equivalent normal variables

The transformation from fuzzy numbers into random quantities and its converse should be taken into account in any problem where heterogeneous uncertain and imprecise data appear together (e.g., information deficit, linguistic variables, statistical data). Representative transformation principles such as insufficient reason, maximum specificity, and uncertainty invariance are proposed by D. Dubois, et al. (1993) [28], D. Dubois, et al. (2004) [29], D. Dubois (2006) [30], and G.J. Klir (2005) [31], respectively. The advantage of Klir's principle is the information preservation

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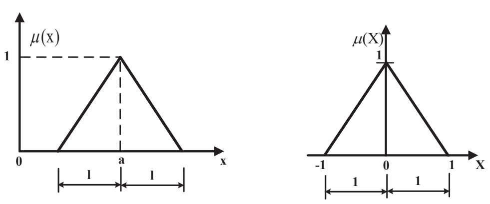
of fuzzy measure and equivalent probability. However, in Ref. [1], the results of using one may conflict with the probability/possibility consistency principle. Besides, three assumptions must be used in Klir's approach while Dubois's approach does not make any assumptions. Nevertheless, an initial fuzzy number will differ from the final fuzzy number after a transform forward to probability measure by the principle of insufficient reason and then a transform back to fuzzy measure by the principle of maximum specificity. Hence, transformations (from possibility to probability and converse) based on these two principles make non-conservative information. To overcome this drawback, T.H. Nguyen and H.X. Le (2019) [32] proposed an innovation transformation to calculate the deviation of the equivalent normal random variable. This approach is presented and detailed explored below.

Consider a symmetric triangular fuzzy number $\tilde{X}_i = (a, l)_{LR}$ (Fig. 1A) where a is the belief value (at the membership level μ=1) of the fuzzy number; l is the spread of the fuzzy number; and the standardized fuzzy variable $\tilde{x}_i = (0,1)_{LR}$ is defined as follows [19]:

$$\tilde{x}_i = \frac{\tilde{X}_i - a}{l} \quad (1)$$



(A) Original fuzzy variable (B) Standardized fuzzy variable

Fig. 1. Transformation to a standardized fuzzy variable.

For the transformation from a standardized symmetric triangular fuzzy number (Fig. 1B) into a random quantity, the error of probability measure between the equivalent probability density function $p(x)$ of the standardized symmetric triangular fuzzy number obtained by the principle of insufficient reason and the probability density function of the normal random variable $p_l(x)$ is expressed by the following formula:

$$(P(A) - P_1(A))^2 \to min, \ \forall x \in [-1,0) \quad (2)$$

where p(x), $p_1(x)$, P(A), and P1(A) are determined as follows:

$$p(x) = \begin{cases} -\frac{1}{2} ln(-x) \ ; x[\in -1,0) \\ -\frac{1}{2} ln(x) \ ; x \in 0,1] \end{cases} \quad (3)$$

$$p_1(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{x^2}{2\sigma^2}\right)} \quad (4)$$

$$P(A) = \frac{1}{2}[x - xln(-x) + 1] \quad (5)$$

$$P_1(A) = \int_{-1}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx \quad (6)$$

with σ being the deviation of the normal random variable, {A} is the event that has $1 \leq x_o \leq x$, and P(A) and $P_1(A)$ are probabilities of event A for the density distribution functions p(x) and $p_1(x)$.

From Eq. (2), we obtain:

$$F_1(\sigma) = \int_{-1}^{0} (P(A) - P_1(A))^2 \, dx \to min \quad (7)$$

The probability density function p(x) and normal random variable $p_1(x)$ have different domains. Therefore, to ensure that the probability of the density function $p_1(x)$ in (-∞,1) is insignificant, one needs

$$F_2(\sigma) = P_1[A*: x_o \in (-\infty, -1)] = \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx \to min \quad (8)$$

Combining Eqs. (7) and (8), we have:

$$F(\sigma) = F_1(\sigma) + F_2(\sigma) = \int_{-1}^{0} (P(A) - P_1(A))^2 \, dx + \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx \to min \quad (9)$$

For the reverse transformation, that is, from the normal random variable into an equivalent fuzzy number, the error of possibility measure between the equivalent fuzzy number of the normal random variable obtained by the principle of maximum specificity and the standardized fuzzy number is expressed by the following formula:

$$G(\sigma) = \int_{-1}^{0} (\pi_1(x) - \pi(x))^2 \, dx + \int_{-\infty}^{-1} \pi_1{}^2(x) \, dx = \int_{-1}^{0} (\pi_1(x) - 1 - x)^2 \, dx + \int_{-\infty}^{-1} \pi_1{}^2(x) \, dx \to min \quad (10)$$

where π(x) is the membership function of the standardized fuzzy number in [-1,0] as:

$$\pi(x) = 1 + x \quad (11)$$

Then, $\pi_1(x)$ is the equivalent fuzzy number of a normal random variable determined as follows (with limits –∞ and +∞ replaced by -6σ and 6σ, respectively):

$$\pi_1(x) = \pi_1(-x) = \int_{-6\sigma}^{x} p_1(y) dy + \int_{-x}^{6\sigma} p_1(y) dy \quad (12)$$

To solve the multi-objective optimization problem of Eqs. (9) and (10), one transforms multiple objectives into a scalar objective function by multiplying each objective function by a weighting factor and summing up all contributors as in:

$$H(\sigma) = \gamma F(\sigma) + (1 - \gamma) G(\sigma) \to \text{min} \quad (13)$$

where γ ∈[0,1].

The mathematical meaning of Eq. (13) is an extension that modifies the equivalent characteristic according to two principles: the principle of insufficient reason when going from a fuzzy number to random variable and the principle of maximum specificity when going from a random variable to fuzzy number. To solve Eq. (13), a genetic algorithm (GA) is applied using built-in functions in Matlab. The relation between the weighting factor γ and deviation σ is depicted in detail in Fig. 2.

One realizes that the proposed transformation encodes a family of normal random variables, including the result of Klir's method, from the standardized fuzzy number. Indeed, the deviation of a normal random variable using the uncertainty invariance principle is 0.3989 [25] and the same result is attained according to the proposed transformation when the weighting factor is 0.728. Hence, the constraints generated in the proposed transformation are more flexible than Klir's method.

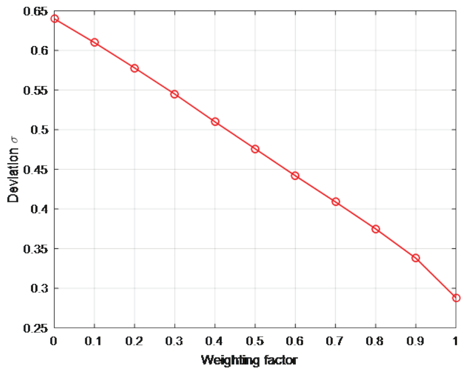


Fig. 2. Representation of the relation between deviation and weighting factor.

The calculation for the responses of structures

After a transformation from fuzzy to random variables, fuzzy finite element analysis becomes stochastic finite element analysis. Then, RSM is utilized to define the responses of the structures. The basic idea of RSM is to approximate an implicit function by an equivalent polynomial function. The calculation procedure is expressed in the next sub-sections.

Design of experiments: The design of the experiments is based on the sampling plan in design variable space (i.e., input data space). The design plays a very valuable role in determining the regression coefficients of surrogate models. The important problem is how we evaluate the goodness of such designs with a limited number of samples. In the present paper, the Box-Behnken design [33], which has good results in actual problems, is selected to be the design of the experiments. In the Box-Behnken design, sample points are chosen at all possible combinations of the mean values (μ_i=0) and $\mu_i \pm 3\sigma_i$ (0±3) as shown in Fig. 3.

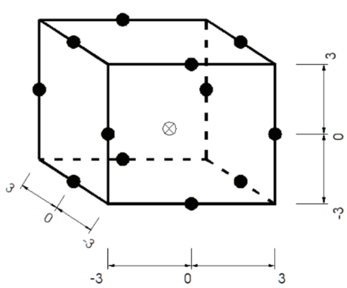


Fig. 3. The Box-Behnken design with three standard normal random variables.

Surrogate models: In statistical theory, surrogate models are often used that include the polynomial regression model, Kringing model, and radial basis functions [34]. The first and the second models are parametric models based on the assumed functional form of the response in terms of the design variables. The last model is non-parametric and uses different types of local models in different regions of the data to build up an overall model. Among these models, the polynomial regression model is often used to build a response surface function due to its calculation simplicity. In this study, the complete quadratic polynomial regression model is used for the responses of structure, in which all random variables X_{ci} are standard normal and assumed to be uncorrelated.

The complete quadratic polynomial regression model (The CQP model):

$$y(\mathbf{X}) = a_o + \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{ij} X_i X_j + \sum_{i=1}^{n} a_{ii} X_i^2 \quad (14)$$

According to the extension principle [1], the belief values of the fuzzy response of structures are received from the combination of belief values of input data. Hence, the surrogate model needs to respond to this constrained condition, such as in:

$$a_o = \hat{y}(\mathbf{x} = \mathbf{a}) \quad (15)$$

where $\hat{y}$(x=a) are displacements at the membership level μ=1 of the input data and are determined by classical FEM.

The remaining regression coefficients in Eq. (14) are determined by the least-squares method.

Error estimation and selecting reasonable design: Due to their ease of calculation, split sample and cross-validation methods [34] are always used to select a rational design. To reduce the variance of the error estimation, which usually appears in the split sample method, the cross-validation method is proposed. Differing from the split sample method, this method makes no distinction between the training and test data sets. In this study, leave-one-out cross-validation is applied where each response point is tested once and trained *k-1* times because the centre point was computed from Eq. (15). The error estimation of the j^{th} design (using $X^{(j)}$ as the test set) is determined by the formula:

$$GSE_j = (y_j - \hat{y}_j^{(-j)})^2 \quad (16)$$

where GSE_j is the square error estimation of the j^{th} design; y_j is output value at $X^{(j)}$ determined by classical FEM; and $\hat{y}_j^{(-j)}$ is the estimated value at $X^{(j)}$ of the j^{th} design.

The design uses the least-squares error estimation method to calculate the responses of the structures.

Determination for the responses of structures: The parameters of the responses of structures, including the mean and the deviation values, are directly calculated from the surrogate model. After using the properties for the n^{th} moments of the standard normal variable, we obtain:

- The mean value of the responses of structures:

$$m_r = E(y) = a_o + \sum_{i=1}^{n} a_{ii} \quad (17)$$

- The standard deviation value of the responses of structures:

$$\sigma_r = \sigma(y) = \sqrt{E(y^2) - (E(y))^2} \quad (18)$$

where $E(y^2)$ is determined as follows:

$$E(y^2) = (a_o)^2 + \sum_{i=1}^{n}(a_i)^2 + 3\sum_{i=1}^{n}(a_{ii})^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_{ij})^2 + 2\sum_{i=1}^{n-1}\sum_{k=i+1}^{n} a_{ii}a_{kk} + 2a_o\sum_{i=1}^{n} a_{ii} \quad (19)$$

Results and discussion

Numerical example

Consider a two-bar truss with a spring support in Fig. 4, where the elastic modulus of the material is $\tilde{E}$, the loads are $\tilde{P}_1$

and $\tilde{P}_2$, and the spring stiffness $\tilde{k}$ are symmetric triangular fuzzy numbers: $\tilde{E}=(210,20)_{LR}$ GPa; $\tilde{P}_1=(30,3)_{LR}$ kN; $\tilde{P}_2=(25,3)_{LR}$ kN; $\tilde{k}=(2000,400)_{LR}$ kN/m.

The cross-sectional area of both bars is $A_1=A_2=5.10^{-4}$ m^2.

Required: Determine the horizontal displacement u_1 and the vertical displacement v_1.

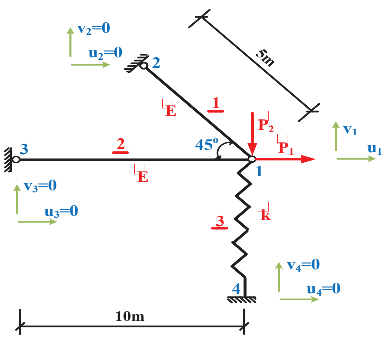


Fig. 4. Two-bar truss with spring support.

One realizes that the weighting factor of 0.5 represents the equivalent characteristic according to two principles: The principle of the insufficient reason when going from fuzzy numbers to random variables and the principle of maximum specificity when going from random variable to fuzzy number. Hence, in this study, we utilize the weighting factor to calculate the responses of the structure. The other weighting factor values are used to survey the variation of the responses of the structure.

The mean values m_r and the standard deviation values σ_r of the horizontal displacement u_l and the vertical displacement v_l for each value of the weighting factor γ are shown in Table 1.

Table 1. The mean values m_r and the standard deviation values σ_r of the horizontal displacement u_1 and the vertical displacement v_1.

The weighting factor γ	The horizontal displacement u_1 (m)		The vertical displacement v_1 (m)	
	The mean value m_r	*The standard deviation value σ_r*	*The mean value m_r*	*The standard deviation value σ_r*
0	0.0007425	0.000218	-0.0013835	0.000310
0.1	0.0007395	0.000036	-0.0013735	0.000248
0.2	0.0007397	0.000068	-0.0013793	0.000096
0.3	0.000740	0.000102	-0.0013799	0.000144
0.4	0.0007406	0.000136	-0.0013807	0.000192
0.5	0.0007411	0.000161	-0.0013815	0.000229
0.6	0.0007421	0.000204	-0.0013830	0.000290
0.7	0.0007431	0.000239	-0.0013840	0.000343
0.8	0.0007451	0.000273	-0.0013862	0.000391
0.9	0.0007464	0.000310	-0.0013878	0.000439
1	0.000740	0.000098	-0.0013798	0.000138

Tables 2 and 3 show the results of fuzzy horizontal displacement $\tilde{u}_1$ and fuzzy vertical displacement $\tilde{v}_1$ using the α-optimization method, respectively. Membership functions of fuzzy horizontal displacement $\tilde{u}_1$ and fuzzy vertical displacement $\tilde{v}_1$ are depicted in Figs. 5 and 6, respectively.

Table 2. The result of fuzzy horizontal displacement $\tilde{u}_1$ using the α-optimization method.

α-cuts	The α-optimization method	
	Lower u_1 (m)	*Upper u_1 (m)*
0.0	0.000188	0.001352
0.2	0.000294	0.001224
0.4	0.000402	0.001098
0.6	0.000512	0.000976
0.8	0.000624	0.000856
1.0	0.000739	0.000739

Table 3. The result of fuzzy vertical displacement $\tilde{v}_1$ using the α-optimization method.

α-cuts	The α-optimization method	
	Lower v_1 (m)	*Upper v_1 (m)*
0.0	-0.002283	-0.000675
0.2	-0.002083	-0.000803
0.4	-0.001893	-0.000937
0.6	-0.001713	-0.001078
0.8	-0.001542	-0.001225
1.0	-0.001379	-0.001379

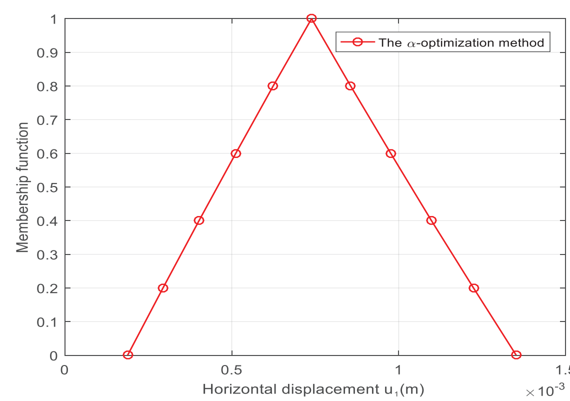


Fig. 5. Fuzzy horizontal displacement $\tilde{u}_1$.

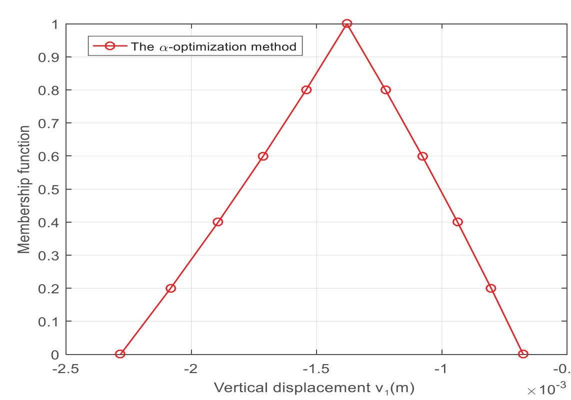


Fig. 6. Fuzzy vertical displacement $\tilde{v}_1$.

To verify the accuracy of the proposed methods, the mean value m_r and the confidence interval $[m_r\text{-}h.\sigma_r, m_r\text{+}h.\sigma_r]$ are compared with the belief value and the support of fuzzy response of the structure using the α-optimization method, respectively. Because the non-linearity terms are comprised of the responses of structure, the load effects (the output data) such as displacements and internal forces exhibit non-Gaussian probabilistic characteristics. Hence, the interval $[m_r\text{-}3.2\sigma_r, m_r\text{+}3.2\sigma_r]$, which corresponds to the largest 99% confidence interval for the Laplace, the uniform, and the triangular probability distributions [29] is chosen. Tables 4 and 5 show the percentage errors between the mean values at each value of the weighting factor γ and the belief values for the horizontal displacement u_1 and the vertical displacement v_1, respectively. The differences between the results of the confidence intervals and that of the support of fuzzy displacements $\tilde{u}_1$ and $\tilde{v}_1$ are calculated in Tables 6 and 7, respectively, including percentage errors in the lower bound, the upper bound, and the width of solutions. The formulae for calculating these percentage errors are presented in [4].

Table 4. The percentage errors between the mean and belief values for the horizontal displacement u_1.

The weighting factor γ	The mean value m_r (m)	The belief value (m)	Error (%)
0	0.0007425	0.000739	-0.48864
0.1	0.0007395	0.000739	-0.07360
0.2	0.0007397	0.000739	-0.10331
0.3	0.000740	0.000739	-0.15292
0.4	0.0007406	0.000739	-0.22886
0.5	0.0007411	0.000739	-0.29201
0.6	0.0007421	0.000739	-0.43162
0.7	0.0007431	0.000739	-0.57072
0.8	0.0007451	0.000739	-0.84072
0.9	0.0007464	0.000739	-1.00942
1	0.000740	0.000739	-0.14791

Table 5. The percentage errors between the mean and belief values for the vertical displacement v_1.

The weighting factor γ	The mean value m_r (m)	The belief value (m)	Error (%)
0	-0.0013835	-0.001379	-0.30406
0.1	-0.0013735	-0.001379	0.41886
0.2	-0.0013793	-0.001379	-0.00017
0.3	-0.0013799	0.000739	-0.04119
0.4	-0.0013807	-0.001379	-0.09851
0.5	-0.0013815	-0.001379	-0.26862
0.6	-0.0013830	-0.001379	-0.34153
0.7	-0.0013840	-0.001379	-0.40643
0.8	-0.0013862	-0.001379	-0.50283
0.9	-0.0013878	-0.001379	-0.61848
1	-0.0013798	-0.001379	-0.03628

Table 6. The percentage errors between the confidence intervals and support of fuzzy horizontal displacement $\tilde{u}_1$.

The weighting factor γ	The confidence interval $[m_r\text{-}3.2\sigma_r, m_r\text{+}3.2\sigma_r]$		The support of fuzzy displacement $\tilde{u}_1$		Error LB (%)	Error UB (%)	Error width (%)
	Lower (m)	*Upper (m)*	*Lower (m)*	*Upper (m)*			
0	0.000044	0.001441	0.000188	0.001352	76.57	6.57	19.96
0.1	0.000625	0.000853	0.000188	0.001352	233.53	36.88	80.42
0.2	0.000522	0.000958	0.000214	0.001320	143.91	27.43	60.56
0.3	0.000414	0.001066	0.000214	0.001320	93.70	19.24	41.08
0.4	0.000307	0.001175	0.000240	0.001288	27.60	8.78	17.12
0.5	*0.000224*	*0.001258*	*0.000240*	*0.001288*	*6.66*	*2.31*	*1.31*
0.6	0.000088	0.001396	0.000267	0.001256	66.93	11.18	32.26
0.7	-0.000023	0.001509	0.000267	0.001256	108.62	20.20	54.97
0.8	-0.000128	0.001618	0.000294	0.001224	143.53	32.21	87.67
0.9	-0.000247	0.001740	0.000320	0.001192	177.15	45.95	127.93
1	0.000427	0.001053	0.000320	0.001192	33.37	11.69	28.25

Table 7. The percentage errors between the confidence intervals and the support of fuzzy vertical displacement $\tilde{v}_1$.

The weighting factor γ	The confidence interval $[m_r\text{-}3.2\sigma_r, m_r\text{+}3.2\sigma_r]$		The support of fuzzy displacement $\tilde{v}_1$		Error LB (%)	Error UB (%)	Error width (%)
	Lower (m)	*Upper (m)*	*Lower (m)*	*Upper (m)*			
0	-0.002376	-0.000391	-0.002283	-0.000675	4.04	42.03	23.37
0.1	-0.002167	-0.000580	-0.002283	-0.000675	5.09	14.09	1.31
0.2	-0.001685	-0.001073	-0.002283	-0.000675	26.20	59.07	61.97
0.3	-0.001841	-0.000919	-0.002283	-0.000675	19.39	36.20	42.71
0.4	-0.001996	-0.000765	-0.002283	-0.000675	12.58	13.38	23.47
0.5	*-0.002115*	*-0.000648*	*-0.002283*	*-0.000675*	*7.37*	*4.02*	*8.77*
0.6	-0.002312	-0.000454	-0.002283	-0.000675	1.24	32.67	15.46
0.7	-0.002481	-0.000287	-0.002283	-0.000675	8.64	57.41	36.34
0.8	-0.002636	-0.000137	-0.002283	-0.000675	15.44	79.77	55.38
0.9	-0.002793	0.000017	-0.002283	-0.000675	22.31	102.57	74.70
1	-0.001822	-0.000938	-0.002283	-0.000675	20.23	39.00	45.08

Discussion

Based on the results of the above example, the following discussions are presented:

- The mean values of the displacements obtained by the proposed method are close to the belief values of fuzzy displacements by the α-optimization method. Due to the difference between the mathematics of fuzzy sets and that of random parameters, this demonstrates that the complete quadratic polynomial in which all equivalent random variables are standard normal random variables using Eqs. (14) and (15) is a reasonable regression model to determine the displacements of the structure.

- The confidence intervals of the displacements with the weighting factor γ=0.5 approximate the supports of fuzzy displacements (the percentage errors are less than 10%). One realizes that a deviation of 0.476 for the equivalent normal variable, which corresponds to the weighting factor γ=0.5, is a plausible selection for calculating the responses of structures. This also points out that the proposed method is reliable. Indeed, when the weighting factor γ=0.728 corresponding to the method [25], the average percentage of errors is about 50%, which is much more than the largest percentage errors with the weighting factor γ=0.5. Once the spread of the fuzzy spring stiffness k=200 kN/m, which is the same as the example presented in [35], the percentage errors between the confidence intervals of the displacements corresponding to the weighting factor γ=0.5 and the support of fuzzy displacements are also less than 10%.

- The number of computations obtained by the proposed method is less than that of the vertex method [12]. Indeed, in the illustrative example, the proposed method requires 25 deterministic finite element problems while 81 deterministic finite element problems are needed for analysis by the vertex method. This especially reveals that besides achieving reasonable accuracy, the proposed method is also a solution to reduce the number of computations.

Conclusions

This paper presents a new method for analysing FFE by using the transformation between fuzzy and random variables. The novel formulae for determining equivalent normal random variables are established and explored in detail. By using the standard normal random variables in the quadratic polynomial regression models and selecting the suitable response surface model amongst the regression models, reasonable accuracy can be achieved for the responses of structures. Simultaneously, by applying suitable experimental designs, a reduction in the number of computations is demonstrated. A weighting factor of 0.5 is a reasonable selection to calculate the responses of structures. Thorough surveys of more complex examples will be presented in future research. Additionally, the present study will be extended to asymmetric triangular fuzzy numbers.

COMPETING INTERESTS

The authors declare that there is no conflict of interest regarding the publication of this article.

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