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INVESTIGATION OF THE THERMOELASTIC STATE OF ANISOTROPIC BODIES OF ROTATION

Abstract: The paper presents a method for determining the stress-strain state of transversely isotropic bodies of revolution located in a stationary axisymmetric field of steady temperatures. The border of the body is free from effort and pinching. The problem is solved by the reverse method. The concept of the space of internal states is formulated and the scalar product is introduced in this space. Its orthonormalization is carried out and the desired state is a Fourier series in the elements of the orthonormal basis. The problem is reduced to determining the coefficients of these series. A rigorous solution of the test problem of thermoelasticity for a circular cylinder and an approximate solution of the problem for a body in the form of a stepped cylinder are presented. The resulting fields of characteristics of the stress-strain state are shown graphically. Analysis of the results is done.
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Introduction

Modern materials used in mechanical engineering, aircraft construction, such as polycrystalline metals, ceramics, and composite materials with significant anisotropy in terms of elastic properties, are often exposed to strong thermal effects. Determination of the stress-strain state of heated bodies due to the complex physical nature of materials is an urgent scientific problem.

The work [1] is devoted to the study of thermomechanical processes of the final deformation of anisotropic media. For a transversely isotropic cylinder, the boundary value problems of the theory of elasticity were solved with the participation of mass forces [2-5]. The peculiarity of the solution lies in the fact that the wake of the elastic field simultaneously satisfies the specified conditions on the boundary and inside the region. In the case of small deformations of

an elastic anisotropic body, stresses, deformations, and temperature are most often related using the Duhamel-Neumann equations, the derivation of which from the point of view of thermomechanics is given in Novatsky's monograph [6]. The problems of thermoelasticity for anisotropic bodies were considered in the books: B.E. Pobedri, A.S. Kravchuk, V. Novatsky.

In [7] an axisymmetric problem of static thermoelasticity for a transversely isotropic circular cylinder of finite length is considered. Using a special voltage function, the basic equation of the problem is derived. It is proved that the operator is symmetric and positive definite, and thus the solution of the original equation is reduced to the problem of the minimal functional.

The problems of determining the temperature field from the values of temperatures and heat fluxes

set at the boundary for isotropic homogeneous and inhomogeneous bodies were investigated by the method of boundary states in [8].

The work [9] is devoted to the study of stress fields in the problems of gradient thermoelasticity arising in the framework of gradient models of thermal conductivity that describe the thermal barrier properties of boundaries due to a more complete consideration of the conjugation conditions at the interfaces of composite layers. In [10], the stationary problem of the gradient theory of thermoelasticity for layered composite structures was considered. A solution is given to the problem of inhomogeneous temperature heating of a single-layer and two-layer structure.

In [11] using the generalized Fourier method, an axisymmetric thermoelastic boundary value problem for a transversely isotropic half-space with a spheroidal cavity was solved.

In [12] an inverse method was developed for determining the stress-strain state of an elastic isotropic body from continuous volumetric forces.

In [13] unrelated boundary value problems of thermoelasticity were solved for shallow shells of double curvature and constant torsion under conditions of convective heat transfer through the main surfaces with the external environment. The solutions are obtained by methods of single and double trigonometric series with variable coefficients.

Recently, three-dimensional asymmetric problems of thermoelasticity for isotropic bodies have been widely studied [14], [15]. In [16], an exact solution of an asymmetric boundary value problem of the theory of elasticity is constructed for a cylindrical reservoir with a liquid in a temperature field (an unconnected thermoelastic problem).

The aim of the work is to develop a mathematical model for solving thermoelasticity problems for transversely isotropic bodies of revolution located in a stationary axisymmetric field of steady temperatures. The model is based on the fundamentals of the integral overlay method, the method of boundary states and the inverse method.

1. Formulation of the problem

In the stationary problem of thermoelasticity, thermal boundary conditions reflect the effect of the environment on the surface S of the body and are written in the form of one of the following conditions:

1) the temperature T is set on the surface as a function of the coordinates

$$T = k(P) , \ P \in S ;$$

2) the normal component of the temperature gradient is set on the surface as a function of the coordinates

$$\partial T / \partial n = k(P) , \ P \in S ,$$

corresponding to the heat flux passing through the surface S;

3) a function describing free heat transfer is given on the surface:

$$(\partial / \partial n + \alpha)T = f(P) , \ P \in S ,$$

where α is some constant

4) mixed boundary thermal conditions are set on the surface, i.e. different boundary conditions are set on different sections S.

Let in an undeformed and unstressed state a transversely isotropic body bounded by one or several coaxial surfaces of revolution has a temperature of T_1 as a function of coordinates r, z. As a result of the influence of any factors (external loads, internal heat sources, surface heating), the body temperature changed and became T_2, then the temperature increment will be $T = \Delta T = T_2 - T_1$. The surface of the body is free from pinching and external stress. We will assume that a change in temperature does not lead to a change in the elastic and thermal constants of the material. The limiting values of the temperature function at the points of the boundary are used as the boundary temperature conditions.

A change in temperature is accompanied by the occurrence of displacements, deformations and stresses, which must be determined.

Due to axial symmetry, the displacement vector component $v = 0$, the stress tensor components $\sigma_{r\theta} = \sigma_{z\theta} = 0$ and the strain tensor $\gamma_{r\theta} = \gamma_{z\theta} = 0$.

In cylindrical coordinates, the sought axisymmetric (ie, depending only on the coordinates z and r) temperature displacements, deformations, and stresses must satisfy the following resolving equations [17].

1. Differential equilibrium equations:

$$\sigma_{r\theta} = \sigma_{z\theta} = 0 . \qquad (1)$$

2. Generalized Hooke's Law:

$$\varepsilon_z = \frac{1}{E_z}\left[\sigma_z - v_z(\sigma_r + \sigma_\theta)\right] + \alpha_z T ;$$

$$\varepsilon_r = \frac{1}{E_r}(\sigma_r - v_r\sigma_\theta) - \frac{v_z}{E_z}\sigma_z + \alpha_r T ; \qquad (2)$$

$$\varepsilon_\theta = \frac{1}{E_r}(\sigma_\theta - v_r\sigma_r) - \frac{v_z}{E_z}\sigma_z + \alpha_r T ;$$

$$\gamma_{zr} = \frac{1}{G_z}\tau_{zr} .$$

3. The Cauchy relations:

$$\varepsilon_z = \frac{\partial w}{\partial z}; \varepsilon_r = \frac{\partial u}{\partial r};$$

$$\varepsilon_\theta = \frac{u}{r}; \gamma_{zr} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}. \qquad (3)$$

Here the designations are introduced: σ_{ij} – stress tensor components; ε_{ij} – strain tensor components; u, w – components of the displacement vector in the direction of the r and z axes; α_z, α_r – coefficients of thermal expansion in the direction of

the z and r axes; T is the temperature; E_r, E_z – modules of elasticity in the direction of the plane of isotropy and normal to it; v_r – Poisson's ratio, characterizing compression in the plane of isotropy, when stretched in this plane, v_z – the same, but when stretched in the direction normal to the plane of isotropy; G_r and G_z - shear modules for the plane of isotropy and any perpendicular to it.

The main difficulty is the construction of the basis of internal states, which is based on a general, fundamental or particular solution for the environment. The procedure for constructing the basis of internal states in the case of thermal deformations is described below.

Orthonormalization of the basis is carried out according to the developed recursive-matrix Gram-Schmidt algorithm [19], where as cross scalar products are taken (for example, for the first and second states):

$$(\xi_1, \xi_2) = \int_V T_0^{(1)} T_0^{(2)}\, dV .$$

The sought thermoelastic state is the Fourier series:

$$\xi = \sum_{k=1}^{\infty} c_k \xi_k ;$$

or in expanded form:

$$\mathbf{u} = \sum_{k=1}^{\infty} c_k \mathbf{u}^{(k)} ; \quad \varepsilon_{ij} = \sum_{k=1}^{\infty} c_k\, \varepsilon_{ij}^{(k)} ;$$

$$\sigma_{ij} = \sum_{k=1}^{\infty} c_k\, \sigma_{ij}^{(k)} ; \quad \grave{O} = \sum_{k=1}^{\infty} c_k \grave{O}_0^{(k)} . \qquad (5)$$

where ξ_k are the elements of the orthonormal basis of the internal state Ξ, c_k are the Fourier coefficients, which are calculated as follows:

$$c_k = \int_V T\, T_0^{(k)}\, dV , \qquad (6)$$

where $T_0^{(k)}$ is the temperature in the basic element ξ_k, T is a given temperature field.

3. Building a basis of internal states

The temperature field giving the temperature value at any point of the body will be considered known.

The steady-state temperature field $T_0^{pl}(z, y)$ of a flat state with no heat sources inside satisfies the heat equation [17]:

2. Solution method

Method of boundary states [18] and the inverse method [12] are similar in structure; both use the concept of space of internal states of the environment

$$\Xi = \{\xi_1, \xi_1, ... \xi_k, ...\}, \qquad (4)$$

whose basis is orthogonalized. The desired state is expanded into a Fourier series in terms of the elements of the orthonormal basis, and the task is to find the coefficients of this linear combination. The difference lies in the choice of the orthogonalizer and in the expression for dot products.

Sets are accepted as internal state ξ_k :

$$\xi_k = \left\{ \left\{ u^{(k)}, v^{(k)}, w^{(k)} \right\}, \left\{ \varepsilon_z^{(k)}, \varepsilon_r^{(k)}, \varepsilon_\theta^{(k)}, \varepsilon_{zr}^{(k)}, \varepsilon_{z\theta}^{(k)}, \varepsilon_{r\theta}^{(k)} \right\}, \left\{ \sigma_z^{(k)}, \sigma_r^{(k)}, \sigma_\theta^{(k)}, \tau_{zr}^{(k)}, \tau_{z\theta}^{(k)}, \tau_{r\theta}^{(k)} \right\}, T_0^{(k)} \right\}.$$

$$\left(k_z \frac{\partial^2}{\partial z^2} + k_r \frac{\partial^2}{\partial y^2} \right) T_0^{\,pl}(z, y) = 0 ,$$

where k_z and k_r are the coefficients of thermal conductivity in the direction of the z and r axes;

$$T_0^{pl} = \frac{g_0}{E_z} \mathrm{Re}[\varphi_0'(\varsigma_0)], \quad \varsigma_0 = z / \gamma_0 + iy ,$$

$$\gamma_0 = \sqrt{k_z / z_r} . \qquad (7)$$

Displacements and stresses of plane states corresponding to the temperature field [17]:

$$u_z^{pl} = \mathrm{Re}[\, p_0 \varphi_0(\varsigma_0)] ; \quad u_y^{pl} = \mathrm{Re}[iq_0 \varphi_0(\varsigma_0)] ;$$

$$u_\eta^{pl} = 0 ;$$

$$\sigma_z^{pl} = -\mathrm{Re}[\gamma_0^2 \varphi_0'(\varsigma_0)] ; \quad \sigma_y^{pl} = \mathrm{Re}[\varphi_0'(\varsigma_0)] ; \quad (8)$$

$$\sigma_\eta^{pl} = \mathrm{Re}[(1 - \varepsilon_0) \varphi_0'(\varsigma_0)] ; \quad \sigma_{zy}^{pl} = -\mathrm{Re}[\gamma_0 \varphi_0'(\varsigma_0)]$$
,

where g_0, p_0, q_0, ε_0 are constants depending on elastic and thermomechanical constants; $\varphi_0(\varsigma_0)$ – some analytical function of the variable ς_0 .

In [17], based on the method of integral superposition, a relationship was established between the spatial stressed and deformed states of an elastic transversely isotropic body and certain auxiliary two-dimensional states, the components of which depend on two coordinates z and y (variables). Plane deformation arising in infinite cylinders with axis η, having at each point a plane of elastic symmetry parallel to the zy plane, is used as plane auxiliary states.

The transition to an axisymmetric spatial state in cylindrical coordinates is carried out according to the dependencies [20]:

$$\sigma_z = \frac{1}{\pi} \int_{-r}^{r} \frac{\sigma_z^{pl}}{\sqrt{r^2 - y^2}}\, dy ;$$

$$\sigma_{zr} = \frac{1}{\pi} \int_{-r}^{r} \frac{\sigma_{zy}^{pl}}{r\sqrt{r^2 - y^2}}\, dy ; \quad \sigma_{z\theta} = \sigma_{r\theta} = 0 ;$$

$$\sigma_r - \sigma_\theta = \frac{1}{\pi} \int_{-r}^{r} \frac{(\sigma_y^{pl} - \sigma_\eta^{pl})(2y^2 - r^2)}{r^2 \sqrt{r^2 - y^2}} dy ;$$

$$\sigma_r + \sigma_\theta = \frac{1}{\pi} \int_{-r}^{r} \frac{(\sigma_y^{pl} + \sigma_\eta^{pl})}{\sqrt{r^2 - y^2}} dy ; \quad (9)$$

$$u = \frac{1}{\pi} \int_{-r}^{r} \frac{u_y^{pl}}{r\sqrt{r^2 - y^2}} dy ; \quad w = \frac{1}{\pi} \int_{-r}^{r} \frac{u_z^{pl}}{r\sqrt{r^2 - y^2}} dy ;$$

$$v = 0 .$$

The basis of space (4) can be constructed by assigning the following values to the function φ_0 in (7), (8) sequentially: $\varphi_0 = \varsigma_0^n$, $n = 1, 2, 3...$ and, according to (9), to carry out the transition to a spatial axisymmetric temperature state, forming a finite-dimensional basis.

4. Test problem

We will test the method by studying the thermoelastic state of a transtropic cylinder made of coarse dark gray siltstone [21]. After the dimensionless procedure, the analogy of which is given in [22], the elastic characteristics of the material: $E_z = 6.21$; $E_r = 5.68$; $G_z = 2.55$; $v_z = 0.22$; $v_r = 0.24$. The body occupies an area: $D_2 = \{(z,r) \mid 0 \le r \le 1, \ -2 \le z \le 2\}$. Let us set the following dimensionless thermomechanical characteristics of a hypothetical transversely isotropic material: $k_z = 1.6$, $k_r = 6.5$, $\alpha_z = 6.7$, $\alpha_r = 8.6$. Preset temperature field $Ò = z + 2$.

The orthonormalized basis of the temperature functions T^0 in (5) is presented in Table 1

Table 1. Orthonormal basis set of temperature functions

	T_0
ξ_1	-0.70711
ξ_2	$-0.61237 z$
ξ_3	$0.75375 + 0.07294 r^2 - 0.59266 z^2$
ξ_4	$1.28853 z + 0.21475 r^2 z - 0.58163 z^3$
ξ_5	$-0.6645 - 0.21216 r^2 - 0.01278 r^4 + 1.7238 z^2 + 0.41552 r^2 z^2 - 0.56268 z^4$

Fourier coefficients (6) $c_k \in \{-2.82843, -1.633, 0, 0, 0, ...\}$.

The decision is strict. Reconstructed components of the temperature elastic field (5):

$$u = -4.01471 r - 2.00736 rz ;$$

$$w = -0.95057 r^2 + 30.8935 z + 7.72338 z^2 ;$$

$$\sigma_r = -143.267 - 71.6334 z ;$$

$$\sigma_\theta = -143.267 - 71.6334 z ;$$

$$\sigma_z = 39.8667 + 19.9333 z ; \quad \tau_{rz} = -9.96667 r ;$$

$$\tau_{z\theta} = \tau_{r\theta} = 0 ; \quad T = z + 2 .$$

5. Design problem

Let us investigate the thermoelastic equilibrium of a transversely isotropic stepped cylinder with the same elastic and thermomechanical characteristics. Set temperature field $T = z^2$. The body occupies an area:

$$D = D_1 + D_2 ; D_1 = \{(z,r) \mid 0 \le r \le 2, \ 0 \le z \le 2\};$$

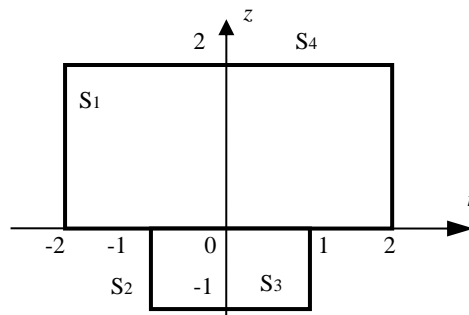$$D_2 = \{(z,r) \mid 0 \le r \le 1, \ -1 \le z \le 0\} .$$

Fig. 1. Meridian section of a body of revolution

When determining the elastic field from temperature, an orthonormal basis of 20 elements was used. The solution representing series (6) satisfies all equations (1) – (4).

In Figure 2 shows a graph illustrating the "saturation" of the Bessel sum (the left side of the Bessel inequality). This is an indirect sign of the convergence of the solution.
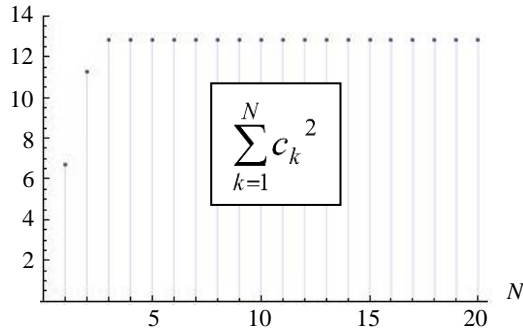

$$\sum_{k=1}^{N} c_k^2$$

Fig. 2. Bessel sum

Figure 3 illustrates the comparison of the reconstructed temperature field with a given field.

This comparison is key to assess the accuracy of the solution across the entire area.

a   b
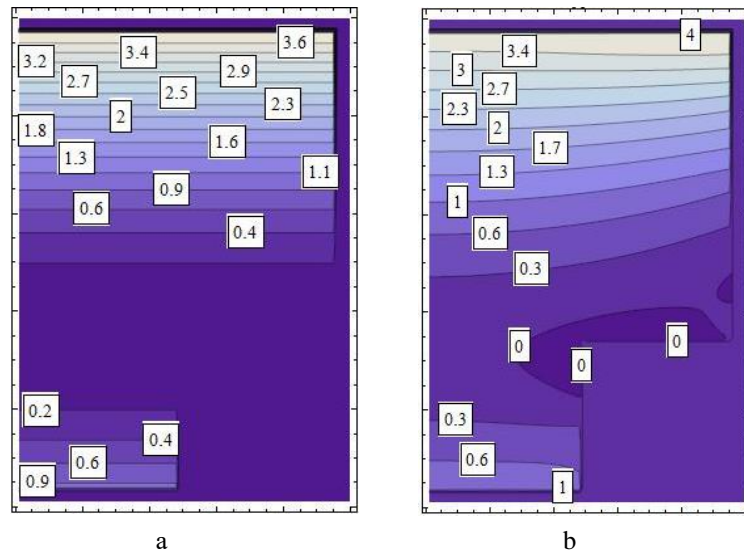
Fig. 3. Temperature field: a - preset; b - restored

Let us compare the temperature at the S_1 boundary (Fig. 4). The dashed line on the graph is the specified values; solid - recovered values.
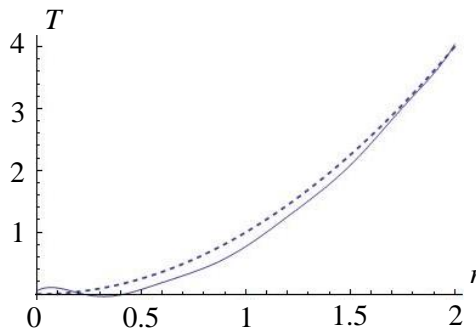

Fig. 4. Verification of temperature at the border S_1

As seen from Fig. 3 and 4, the reconstructed temperature field coincides with the preset ones within the range of a certain accuracy ($\pm 10\%$ of the preset value at any point in the region).

The contours of the obtained components of the stress-strain state are shown in Fig. 5 a – d. Due to axial symmetry, region $\{(z,r)|\ 0 \le r \le 2,\ -1 \le z \le 2\}$ is depicted.
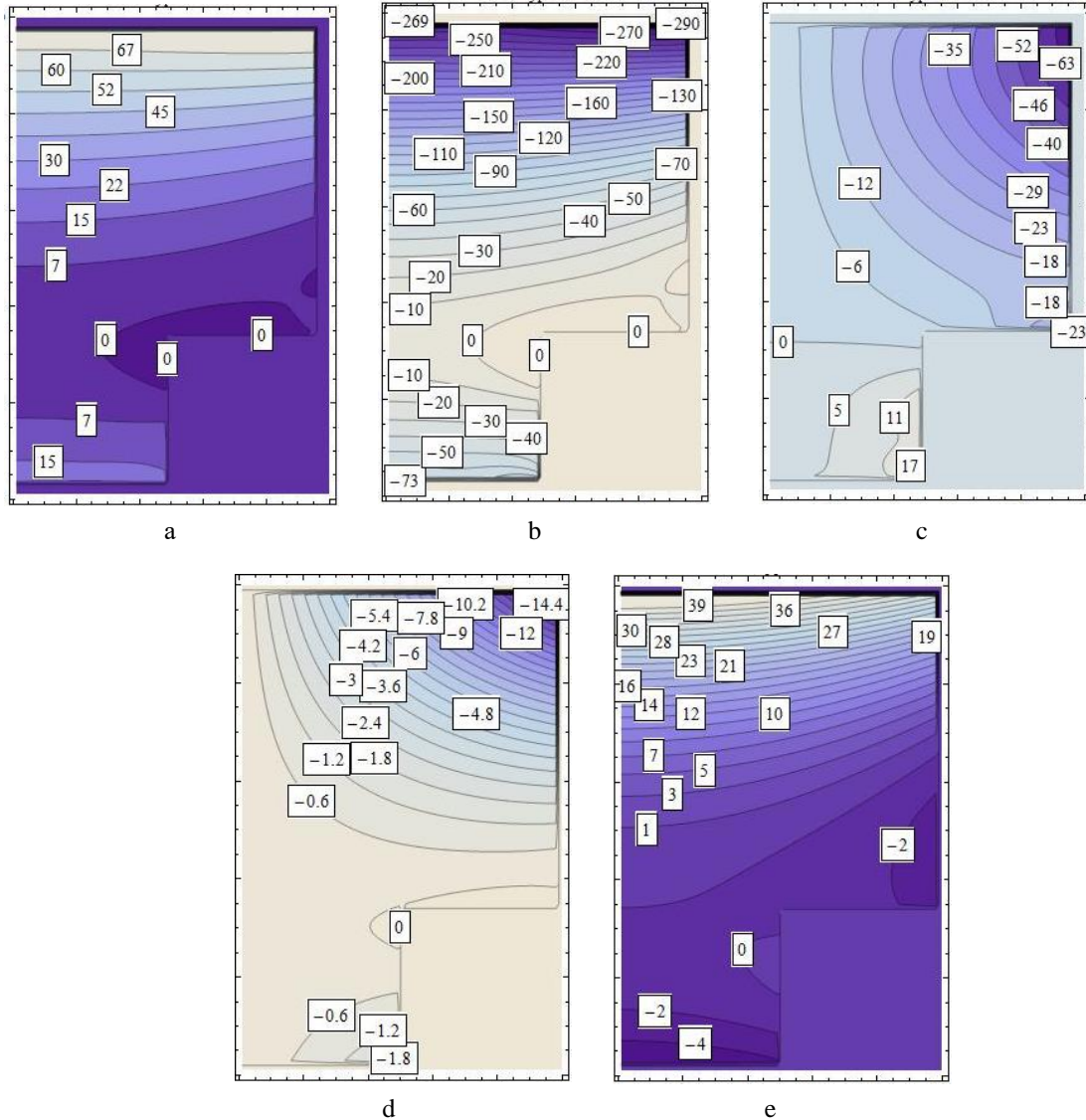

a

b

c

d

e

Fig. 5. Isolines: a – stress σ_{zz} ; b – stress σ_{rr} ;; c – stress σ_{zr} ; d – displacement u; e – displacements w

The question of the convergence of the solution when the basis increases was investigated. With an increase in the number of used elements of the basis of internal states, an oscillation was observed in the vicinity of the singular boundary, which continued to grow and "creep" into the depth of the region with an increase in the number of used basis elements, while the Fourier coefficients constantly decrease. For example, the found temperature values on the surface S_1 with 43 retained basis elements are shown in Fig. 6. Naturally, the solution becomes unusable in this case. Overcoming these difficulties requires further research, but here, however, as an approximate result, the obtained state was chosen, the reconstructed temperature of which corresponded to the given.
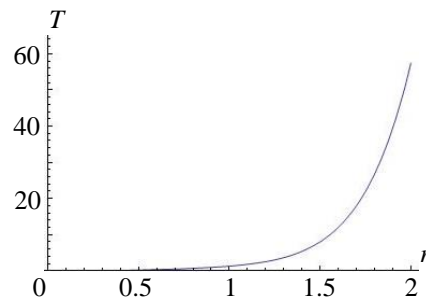
Fig. 6. Temperature at the S$_1$ boundary at 43 Fourier coefficients

Finally, we can say that the convergence of the solution mainly depends on the boundary of the body and the temperature distribution function.

The proposed approach, which is, in fact, a development of the inverse method, has shown its effectiveness in terms of solving thermoelastic axisymmetric problems for transversely isotropic bodies of revolution. The advantage of the presented approach is that the most laborious calculations, namely the construction of an orthonormal basis, are performed once for a body of a certain configuration.

Then this basis can be used to solve various thermoelastic problems for this body. The main advantage over numerical methods is that in its structure the method operates with quadratures, which are taken by means of computer algebra with absolute precision. This eliminates another reason for the formation of the resulting calculation error associated with the intermediate nature of the numerical calculation. Also, the proposed approach allows you to get an analytical solution to problems.

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