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APPLICATION OF APPROXIMATE METHODS FOR SOLVING HIGHER ORDER VOLTERA INTEGRO-DIFFERENTIAL EQUATIONS

Abstract: *The main aim of the present paper is to implement the homotopy perturbation method, Adomian decomposition method and variational iteration method for to an approximation and exact solution the higher order integro-differential equation Voltera. Implementation of these methods demonstrates the usefulness in finding exact solution for linear and nonlinear problems. Comparison is made between the exact solutions and the results of approximate methods in order to verify the accuracy of the results, revealing the fact that these methods are very effective and simple.*

Key words: *Voltera integro-differential equation, homotopy perturbation method, Adomian decomposition method, variational iteration method, approximate and exact solution.*

Language: *English*

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Introduction

Integro-differential equations have been studied in many works of researchers and scientists. Such equations can be found in applications to physics, mechanics, biology, and technology. A new perturbation method called Homotopy perturbation method (HPM) was proposed in [8-11] by He in 1997, and a systematical description was given in 2000 which is in fact, a coupling of the traditional perturbation method and Homotopy in topology. This new method was further developed and improved by He and applied to various linear and nonlinear problems. Below we consider only classical integro-differential equations. Voltera integro-differential equations arise in the mathematical modeling of various scientific phenomena. Nonlinear phenomena, which appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modeled by partial differential equations and by integral equations as well. This paper shows a comparative study between three traditional methods for analytic treatments of Voltera integro-differential equations. Homotopy perturbation method, Adomian decomposition method and variational iteration method, well-addressed in [1-17] has a constructive attraction that provides the exact and approximate solutions by computing only a few iterations. Homotopy perturbation method, Variational iteration method, Adomian decomposition method have been applied to analyze the behavior of the solution of Voltera integro-differential equations. Finally, a comparative study has been made among these methods.

Formulation of the problem.

The aim of the present paper is to implement the homotopy perturbation method, Adomian decomposition method and variational iteration methods for to an approximation and exact solution the Volterra integro-differential equation.

The mathematical formulations of many physical phenomena result into integro-differential equations. The standard *i*-th order Volterra integro-differential equation is of the form

$$y^{(i)}(x) = f(x) + \int_0^x K(x,s), F(y^{(i-1)}(s))ds,$$

$$0 < x < b \qquad (1)$$

where $y^{(i)}(x) = \frac{d^i y}{dx^i}$: $y^{(i)}(x)$ indicates the *i*-th order derivative of $y(x)$; $y(0), y'(0),..., y^{(i-1)}(0)$ are the initial conditions; F – is a nonlinear function, $K(x, s)$ is the kernel and $f(x)$ is a function of x; $y(x)$ and $f(x)$ are real and can be differentiated any number of times for $x \in [0, b]$ [8-11, 14-16].

Problem solving techniques.

Basic idea of homotopy perturbation method.

Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally.

Consider the nonlinear differential equation,

$$L(y)+N(y)=f(r),\ r\in\Omega. \qquad (2)$$

With boundary conditions, $B\left(y, \frac{\partial y}{\partial n}\right), r \in \Gamma$,

were L – a linear operator, N - a nonlinear operator, $f(r)$ – a known analytic function, B – a boundary operator, Γ – the boundary of the domain Ω. By Homotopy perturbation technique [He,1999] define a Homotopy $u(r,p)$: Ω x [0,1] $\to R$ this satisfies

$$H(u,p) = (1-p)[L(u) - L(y_0)] + p[L(u) - N(u) - f(r)] = 0 \quad \text{or} \qquad (3)$$

$$H(u,p) = L(u) - L(y_0) + pL(y_0) + p[N(u) - f(r)] = 0. \qquad (4)$$

Where, $r \in \Omega$, $p \in [0,1]$ is an embedding parameter and y_0 is an initial approximation, which satisfies the boundary conditions. Clearly

$$H(u,0) = L(u) - L(y_0) = 0, \quad H(u,1) = L(u) + N(u) - f(r) = 0.$$

As p changes from 0 to 1. Then $u(r,p)$ changes from $y_0(r)$ to $y(r)$. This is called a deformation and $L(u) - L(y_0)$, $L(u) + N(u) - f(r)$ are said to be Homotopy in topology. According to the HPM, the embedding parameter p can be used as a small parameter and assume that the solution of equation (3) and (4) can be expressed as a power series p, that is $u = u_0 + pu_1 + p^2u_2 + \dots$. For $p = 1$, the approximate solution of equation (2) therefore, can be expressed as $u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \dots$. The series is convergent in most cases and the convergence rate of the series depends on the nonlinear operator.

Basic idea of Adomian decomposition method.

We usually represent the solution $y(x)$ a general nonlinear equation in the following form $Ly(x) + Ry(x) + Ny(x) = f(x)$. Invers operator L with $L^{-1} = \int_0^x (\cdot)dx$.

Equation can be written as $y(x) = L^{-1}[f(x)] - L^{-1}[Ry(x)] - L^{-1}[Ny(x)]$. The decomposition method represents the solution of equation as the following infinite series

$y(x)=\sum_{n=0}^{\infty} y_n(x)$. The nonlinear operator $Ny = g(y)$ is decomposed as $Ny=\sum_{n=0}^{\infty} A_n(x)$. Where A_n are Adomian polynomial which are defined as,

$$A_n=\frac{1}{n!}\frac{d^n}{d\xi^n}N\left[\sum_{m=0}^{\infty}\xi^m y_m(x)\right]_{\xi=0},\ n=0,1,2,\dots .$$

Therefore, we have

$$y=\sum_{n=0}^{\infty} y_n(x)=L^{-1}(f)-L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right)-L^{-1}\left(\sum_{n=0}^{\infty} A_n(x)\right).$$

Consequently, it can be written as,

$$y_0=L^{-1}(f),\ y_1=-L^{-1}(R(y_0))-L^{-1}(A_0),\ y_2=-L^{-1}(R(y_1))-L^{-1}(A_1),\dots .$$

Consequently the solution of (1) in a series form follows immediately by using $y(x)=\sum_{n=0}^{\infty} y_n(x)$. As indicated earlier, the series obtained may yield the exact solution in a closed form, or a truncated $\sum_{n=1}^{m} y_n(x)$ series may be used if a numerical approximation is desired.

Basic idea of variational iteration method.

We illustrate the basic concept of variational iteration method, we consider the following general nonlinear differential equation given in the form

$$Ly(x)+Ny(x)=f(x),$$

where L is a linear operator, N is a nonlinear operator, and $f(x)$ is a known analytical function. We can construct a correction functional according to the variational method as:

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda\left[Ly_n(s)+N\tilde{y}_n(s)-f(s)\right]ds, \tag{5}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, and $\tilde{y}_n$ un is considered as a restricted variation, namely $\delta\tilde{y}_n=0$. The exact solution is thus given by $y(x)=\lim_{n\to\infty} y_n(x)$ [14, 15].

In the following examples, we will illustrate the usefulness and effectiveness of the proposed techniques.

Illustrative Examples.

The following are examples that demonstrate the effectiveness of the methods.

Example 1. Consider third-order Voltera integro-differential equation [14, 15]

$$y'''(x)=2y(x)+1+x-\frac{x^2}{2}-\int_0^x y(s)ds,$$

with initial conditions $y(0)=0,\ y'(0)=0,\ y''(0)=1,$; the exact solution is $y(x)=-1-x+e^x$.

Application of homotopy perturbation method.

A Homotopy can be readily constructed as follows

$$H(u,p)=u_n'''(x)-2u_n(x)-1-x+\frac{x^2}{2}+\int_0^x u_n(s)ds=0. \tag{6}$$

Substituting $u = u_0 + pu_1 + p^2u_2 + \dots$ into (6) and rearranging the resulting equation based on power of p-terms, one has

$$p^0:\ u_0'''(x)-1-x+\frac{x^2}{2}=0;$$

$$p^1:\ u_1'''(x)-2u_0(x)+\int_0^x u_0(s)ds=0;$$

$$p^2:\ u_2'''(x)-2u_1(x)+\int_0^x u_1(s)ds=0\ \dots$$

With the following conditions $u_n(0)=0,\ u_n'(0)=0,\ u_n''(0)=1,\ n=0,1,2,\dots$.

With the effective initial approximation solution can be written as follows

$$u_0(x)=\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5;$$

$$u_1(x)=\frac{1}{60}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7-\frac{1}{13440}x^8+\frac{1}{362880}x^9;$$

$$u_2(x)=\frac{1}{10080}x^8+\frac{1}{362880}x^{10}-\frac{1}{5702400}x^{11}+\frac{1}{95800320}x^{12}-\frac{1}{6227020800}x^{13};\ \dots.$$

After the fourth iteration, the absolute error is less than 10^{-10}. In the same manner, the rest of components were obtained using the Maple package

$$y(x)=\lim_{p\to 1} u = u_0+u_1+u_2+\dots=$$

$$-1-x+\sum_{n=0}^{\infty}\frac{1}{n!}x^n=-1-x+e^x.$$

Application of Adomian decomposition method.

Using $y(x)=\sum_{n=0}^{\infty}y_n(x)$ and the recurrence relation we obtained: we start by setting the zeroth component $y_0'''(x)=1+x-\frac{x^2}{2}$, so that the first component is obtained by

$$y_1'''(x)=2y_0(x)-\int_0^x y_0(s)ds;$$

$$y_2'''(x)=2y_1(x)-\int_0^x y_1(s)ds;\ \dots.$$

Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot)=\int\int\int(\cdot)dxdxdx$.

Hence, taking into account the boundary conditions, we have

$$y_0(x)=\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4;\quad y_1(x)=\frac{1}{12}x^4+\frac{1}{120}x^5-\frac{1}{240}x^6+\frac{1}{5040}x^7;$$

$$y_2(x)=\frac{1}{180}x^6-\frac{1}{5760}x^8+\frac{1}{72576}x^9-\frac{1}{3628800}x^{10};\ \dots.$$

This gives the solution in the series form

$$y(x)=\sum_{n=0}^{\infty}y_n(x)=-1-x+\sum_{n=0}^{\infty}\frac{1}{n!}x^n=-1-x+e^x.$$

Application of variational iteration method.

Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads $\lambda=-(s-x)^2/2$. So we can construct a variational iteration form for (5) in the form:

$$y_{n+1}(x)=y_n(x)-\int_0^x\frac{(s-x)^2}{2}\left[y_n'''(s)-2y_n(s)-1-s+\frac{s^2}{2}+\int_0^s y(p)dp\right]ds.$$

We start by setting the zeroth component

$$y_0(x)=y(0)+xy'(0)+\frac{x^2}{2}y''(0)=\frac{x^2}{2}.$$

That will lead to the following successive approximations:

$$y_1(x)=\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5-\frac{1}{720}x^6;$$

$$y_2(x)=\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7+\frac{1}{40320}x^8+\dots;\ \dots.$$

After the fourth iteration, the maximum absolute error is less than 10^{-11}, but the maximum absolute error decreases with increasing iteration.

So we obtain the following approximate solution $y_n(x) = -1 - x + \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, which is the exact solution of the problem: $y(x) = -1 - x + e^x$.

Example 2. In the following example, we consider linear boundary value problem for the integro-differential equation [14, 15]

$$y''(x) = -\cos x + \frac{1}{4}\sin 2x + \frac{1}{2}x + \int_0^x y^2(s)ds,$$

with initial conditions $y(0) = 1,\ y'(0) = 0$; the exact solution is $y(x) = \cos x$.

Application of homotopy perturbation method.

A Homotopy can be readily constructed as follows

$$H(u,p) = u''(x) = -\cos x + p\left(\frac{1}{4}\sin 2x + \frac{1}{2}x\right) + p\int_0^x u^2(s)ds = 0. \tag{7}$$

Substituting $u = u_0 + pu_1 + p^2u_2 + \ldots$ into (7) and rearranging the resulting equation based on power of p-terms, one has

$$p^0:\ y_0''(x) = -\cos x;$$

$$p^1:\ y_1''(x) = \frac{1}{4}\sin 2x + \frac{1}{2}x - \int_0^x y_0^2(s)ds;$$

$$p^2:\ y_2''(x) = -\int_0^x 2y_0(s)y_1(s)ds;$$

$$p^3:\ y_3''(x) = -\int_0^x (2y_0(s)y_2(s) + y_1^2(s))ds;$$

$$p^4:\ y_4''(x) = -\int_0^x (2y_0(s)y_3(s) + 2y_1(s)y_2(s))ds;\ \ldots\ .$$

Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot) = \int_0^x\int_0^x (\cdot)dxdx$.

Hence, taking into account the boundary conditions, we have $y_0(x) = \cos x$; $y_1(x) = 0$; $y_2(x) = 0$; … .

This gives the solution in the series form $y(x) = \sum_{n=0}^{\infty} y_n(x) = \cos x$.

Application of Adomian decomposition method.

Using $y(x) = \sum_{n=0}^{\infty} y_n(x)$ and the recurrence relation we obtained: we start by setting the zeroth component $y_0''(x) = -\cos x + \frac{1}{4}\sin 2x + \frac{1}{2}x$, so that the first component is obtained by $y_n''(x) = \int_0^x \left(\sum_{n=0}^{\infty} A_n(s)\right)ds,\ n \geq 1$. Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot) = \int_0^x\int_0^x (\cdot)dxdx$. Hence, taking into account the boundary conditions, we have

$$y_0(x) = -1 + \frac{1}{8}x + \frac{1}{12}x^3 + \cos x - \frac{1}{16}\sin 2x;$$

$$y_2(x) = -\frac{15899}{1728} - \frac{15329}{8192}x + \frac{35}{96}x^2 + \frac{769}{3072}x^3 - \frac{1}{96}x^4 + \frac{1}{3840}x^5 - \frac{1}{720}x^6 + \frac{1}{10080}x^7 +$$
$$+\frac{1}{72576}x^9 + 2\sin x + \frac{147}{16}\cos x - \frac{71}{1024}\sin 2x + \frac{1}{64}\cos 2x - \frac{1}{432}\cos 3x + \frac{1}{32768}\sin 4x \quad ; \ \dots .$$
$$+\frac{23}{4}x\sin x + \frac{5}{512}x\cos 2x + \dots .$$

This gives the solution in the series form $y(x) = \sum_{n=0}^{\infty} y_n(x) = \cos x$.

Application of variational iteration method.

Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads $\lambda = (s-x)^3/6$. So we can construct a variational iteration form for (5) in the form:

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x)\left[y_n''(s) + \cos s - \frac{1}{4}\sin 2s - \frac{1}{2}s + \int_0^s y^2(p)dp \right] ds.$$

We start by setting the zeroth component $y_0(x) = y(0) + xy'(0) = 1$.

That will lead to the following successive approximations:

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{60}x^5 - \frac{1}{720}x^6 + \frac{1}{630}x^7 + \frac{1}{40320}x^8 + \dots;$$
$$y_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{8064}x^8 - \dots .;$$
$$y_3(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$$

After the fourth iteration, the maximum absolute error is less than 10^{-10}, but the maximum absolute error decreases with increasing iteration.

So we obtain the following approximate solution $y_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, which is the exact solution of the problem:

$y(x) = \lim_{n\to\infty} y_n(x) = \cos x$.

Conclusion.

This results shows a comparative study between homotopy perturbation method, variational iteration method and Adomian decomposition method of solving Voltera integro-differential equations. The main advantage of these methods are the fact that they provide its user with an analytical approximation, in many cases an exact solution in rapidly convergent sequence with elegantly computed terms. Also these methods handle linear and non-linear equations in a straightforward manner. These methods provide an effective and efficient way of solving a wide range of linear and nonlinear integro-differential equations. Illustrative examples are given to demonstrate the validity, accuracy and correctness of the proposed methods. The error between the approximate solution and exact solution decreases when the degree of approximation increases.

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