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APPLICATION OF APPROXIMATE METHODS FOR SOLVING HIGHER ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Abstract*: The main aim of the present paper is to implement the Adomian decomposition method and variational iteration methods for to an approximation and exact solution the higher order integro-differential equation Fredholm. Implementation of these methods demonstrates the useful-ness in finding exact solution for linear and nonlinear problems. Comparison is made between the exact solutions and the results of approximate methods in order to verify the accuracy of the results, revealing the fact that these methods are very effective and simple.*

Key words*: Fredholm integro-differential equation, Adomian decomposition method, variational iteration method, approximate and exact solution.*

Language*: English*

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Introduction

The recent years have seen significant development in the use of various methods for the numerical, approximate and analytic solution of the linear and nonlinear integro-differential equation. Over the last decades several analytical/approximate methods have been developed to solve linear and nonlinear integro-differential equations. Some of these techniques include variational iteration method [2-5, 9, 11, 16-20], homotopy perturbation method (HPM) [1, 2, 6-8, 10, 12, 14, 16, 18, 19], Adomian decomposition method (ADM) [13, 14, 16-19, homotopy analysis method (HAM) [16-19] etc. [16-19]. Linear and Nonlinear phenomena play an important role in varios fields of science and engineering, such as chemical kinetics, fluid dynamics, engineering problems and biological models. Most models of real life problems are still very difficult to solve. There-fore, approximate-analytical solutions such as homotopy perturbation method, Adomian decomposition method, variational iteration method etc were introduced [11, 14, 15, 18, 19]. These methods is the most effective and convenient ones for both linear and nonlinear equations. The aim of the present paper is to implement the Adomian decomposition method and variational iteration methods for to an approximation and exact solution the Fredholm integro-differential equation.

Formulation of the problem.

The aim of the present paper is to implement the Adomian decomposition method and variational iteration methods for to an approximation and exact solution the Fredholm integro-differential equation.

The mathematical formulations of many physical phenomena result into integro-differential equations. The standard *i*th order Fredholm integro-differential equation is of the form

$$y^{(i)}(x) = f(x) + \int_a^b K(x,s)F(y^{(i)}(s))ds\,, \qquad (1)$$

where $y^{(i)}(x) = \frac{d^i y}{dx^i}$: $y^{(i)}(x)$ indicates the *i*-th order derivative of $y(x)$; $y(0), y'(0), ..., y^{(i-1)}(0)$ are the initial conditions; F – is a nonlinear function, $K(x, s)$ is the kernel and $f(x)$ is a function of x; $y(x)$ and $f(x)$ are real and can be differentiated any number of times for $x \in [a, b]$ [14, 15].

Problem solving techniques.

Basic idea of Adomian decomposition method.

We usually represent the solution $y(x)$ a general nonlinear equation in the following form $Ly(x) + Ry(x) + Ny(x) = f(x)$, were L, R – a linear operator, N - a nonlinear operator, $f(r)$ – a known analytic function. Invers operator L with $L^{-1} = \int_0^x (\cdot)dx$. Equation can be written as $y(x) = L^{-1}[f(x)] - L^{-1}[Ry(x)] - L^{-1}[Ny(x)]$. The decomposition method represents the solution of equation as the following infinite series $y(x) = \sum_{n=0}^{\infty} y_n(x)$. The nonlinear operator $Ny = g(y)$ is decomposed as $Ny = \sum_{n=0}^{\infty} A_n(x)$. Where A_n are Adomian polynomial which are defined as,

$$A_n = \frac{1}{n!}\frac{d^n}{d\xi^n} g\left[\sum_{m=0}^{\infty} \xi^m y_m(x)\right]_{\xi=0}, \quad n = 0,1,2,... .$$

Therefore, we have

$$y = \sum_{n=0}^{\infty} y_n(x) = L^{-1}(f) - L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n(x)\right)$$

. Consequently, it can be written as, $y_0 = L^{-1}(f)$,

$y_1 = L^{-1}(R(y_0)) - L^{-1}(A_0)$,

$y_2 = -L^{-1}(R(y_1)) - L^{-1}(A_1)$,.....

Consequently the solution of (1) in a series form follows immediately by using $y(x) = \sum_{n=0}^{\infty} y_n(x)$. As indicated earlier, the series obtained may yield the exact solution in a closed form, or a truncated $\sum_{n=1}^{m} y_n(x)$ series may be used if a numerical approximation is desired.

Basic idea of variational iteration method.

The correction functional for the integro-differential equation (1) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left[y^{(i)}(\xi) - f(\xi) + \int_a^b K(\xi,s)y(s)ds\right]d\xi\,. \qquad (2)$$

As presented before, the variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier $\lambda(\xi)$ that can be identified optimally via integration by parts and by using a restricted variation. Having $\lambda(\xi)$ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $y_{n+1}(x)$, $n \geq 0$ of the solution $y(x)$. The zeroth approximation y_0 can be any selective function. However, using the given initial values $y(0), y'(0),...$ are preferably used for the selective zeroth approximation y_0 as will be seen later. Consequently, the solution is given by $y(x) = \lim_{n\to\infty} y_n(x)$.

In the following examples, we will illustrate the usefulness and effectiveness of the proposed techniques.

Illustrative Examples.

The following are examples that demonstrate the effectiveness of the methods.

Example 1. Consider third- order Fredholm integro-differential equation [14, 15]

$$y'''(x) = 1 - e + e^x + \int_0^1 y(s)ds\,,$$

with initial conditions $y(0) = y'(0) = y''(0) = 1$,

the exact solution is $y(x) = e^x$.

Application of Adomian decomposition method.

Using $y(x)=\sum_{n=0}^{\infty} y_n(x)$ and the recurrence relation we obtained: we start by setting the zeroth component $y_0'''(x)=1-e+e^x$, so that the first component is obtained by $y_1'''(x)=\int_0^1 y_0(s)ds$; $y_2'''(x)=\int_0^1 y_1(s)ds$; … . Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot)=\int\int\int(\cdot)dxdxdx$, and using the given initial condition we obtain

$$y(x)=\frac{1-e}{6}x^3+e^x+\frac{1}{6}x^3\int_0^1 y(s)ds.$$

Hence, taking into account the boundary conditions, we have

$$y_0(x)=\frac{1-e}{(3!)^1 4^0}x^3+e^x;$$

$$y_1(x)=-\frac{23(1-e)}{(3!)^2 4^1}x^3;$$

$$y_2(x)=-\frac{23(1-e)}{(3!)^3 4^2}x^3; \ldots.$$

This gives the solution in the series form

$$y(x)=\sum_{n=0}^{\infty} y_n(x)=e^x+(1-e)x^3\sum_{n=0}^{\infty}\frac{1}{(3!)^{n+1}4^n}=e^x.$$

Application of variational iteration method.

Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads $\lambda=-(s-x)^2/2$. So we can construct a variational iteration form for (2) in the form:

$$y_{n+1}(x)=y_n(x)-\int_0^x\frac{(s-x)^2}{2}\left[y_n'''(s)+\cos s-s-s\int_0^{\pi/2} y''(p)dp\right]ds.$$

We start by setting the zeroth component

$$y_0(x)=y(0)+xy'(0)+\frac{x^2}{2}y''(0)=1+x+\frac{x^2}{2}.$$

That will lead to the following successive approximations:

$$y_1(x)=e^x+\left(\frac{4}{9}-\frac{e}{6}\right)x^3;$$

$$y_2(x)=e^x+\left(\frac{1}{54}-\frac{e}{144}\right)x^3;$$

$$y_3(x)=e^x+\left(\frac{1}{1296}-\frac{e}{3456}\right)x^3; \;\ldots.$$

If $y_1(x)=e^x+\left(\frac{4}{9}-\frac{e}{6}\right)x^3\approx e^x$, then

$y_2(x)=e^x$; $y_3(x)=e^x$; … .

So we obtain the following approximate solution $y(x)=\lim_{n\to\infty} y_n(x)=e^x$, which is the exact solution of the problem: $y(x)=e^x$.

Example 2. Consider third- order Fredholm integro-differential equation [14, 15]

$$y'''(x)=-\cos x+x+\int_0^{\pi/2} xy''(s)ds,$$

with initial conditions $y(0)=0,\ y'(0)=1,\ y''(0)=0$, the exact solution is $y(x)=\sin x$.

Application of Adomian decomposition method.

Using $y(x)=\sum_{n=0}^{\infty} y_n(x)$ and the recurrence relation we obtained: we start by setting the zeroth component $y_0'''(x)=-\cos x+x$, so that the first component is obtained by $y_1'''(x)=x\int_0^{\pi/2} y_0''(s)ds$; $y_2'''(x)=x\int_0^{\pi/2} y_1''(s)ds$; … . Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot)=\int\int\int(\cdot)dxdxdx$, and using the given initial condition we obtain

$$y(x)=\sin x+\frac{x^4}{4!}+\frac{x^4}{4!}\int_0^{\pi/2} y''(s)ds.$$

Hence, taking into account the boundary conditions, we have

$$y_0(x)=\sin x+\frac{x^4}{4!};\quad y_1(x)=-\frac{1}{24}x^4+\frac{1}{1152}x^4\pi^3;\quad y_2(x)=-\frac{1}{1152}x^4\pi^3+\frac{1}{55296}x^4\pi^6;\ \ldots .$$

This gives the solution in the series form

$$y(x)=\sum_{n=0}^{\infty}y_n(x)=\sin x+\frac{1}{24}x^4-\frac{1}{24}x^4+\frac{1}{1152}x^4\pi^3-\frac{1}{1152}x^4\pi^3+\frac{1}{55296}x^4\pi^6-\ldots=\sin x .$$

Application of variational iteration method.

Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads $\lambda=-(s-x)^2/2$. So we can construct a variational iteration form for (2) in the form:

$$y_{n+1}(x)=y_n(x)-\int_0^x\frac{(s-x)^2}{2}\left[y_n'''(s)+\cos s-s-s\int_0^{\pi/2}y''(p)dp\right]ds.$$

We start by setting the zeroth component

$$y_0(x)=y(0)+xy'(0)+\frac{x^2}{2}y''(0)=x .$$

That will lead to the following successive approximations:

$$y_1(x)=\sin x+\frac{1}{24}x^4\approx\sin x;\quad y_2(x)=\sin x;$$

$$y_3(x)=\sin x;\ \ldots .$$

So we obtain the following approximate solution $y(x)=\lim\limits_{n\to\infty}y_n(x)=\sin x$, which is the exact solution of the problem: $y(x)=\sin x$.

Example 3. Consider the third-order linear integro-differential equation [14, 15]

$$y'''(x)=\sin(x)-x-\int_0^{\pi/2}xsy'(s)ds ,$$

with initial conditions $y(0)=1,\ y'(0)=0,\ y''(0)=-1$; the exact solution is $y(x)=\cos(x)$.

Application of Adomian decomposition method.

Using $y(x)=\sum_{n=0}^{\infty}y_n(x)$ and the recurrence relation we obtained: we start by setting the zeroth component $y_0'''(x)=\sin x-x$, so that the first component is obtained by $y_1'''(x)=-x\int_0^{\pi/2}sy_0'(s)ds$; $y_2'''(x)=-x\int_0^{\pi/2}sy_1'(s)ds$; … . Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot)=\int\int\int(\cdot)dxdxdx$, and using the given initial condition we obtain

$$y(x)=\cos x-\frac{x^4}{4!}-\frac{x^4}{4!}\int_0^{\pi/2}sy'(s)ds .$$

Hence, taking into account the boundary conditions, we have

$$y_0(x)=\cos x-\frac{x^4}{4!};$$

$$y_1(x)=\frac{1}{24}x^4+\frac{1}{23040}x^4\pi^5;$$

$$y_2(x)=-\frac{1}{23040}x^4\pi^5-\frac{1}{22118400}x^4\pi^{10};$$

$$y_3(x)=\frac{1}{22118400}x^4\pi^{10}+\frac{1}{21233664000}x^4\pi^{15};$$

$$\ldots .$$

This gives the solution in the series form

$$y(x)=\sum_{n=0}^{\infty}y_n(x)=\cos x-\frac{1}{24}x^4+\frac{1}{24}x^4+\frac{1}{23040}x^4\pi^5-$$

$$-\frac{1}{23040}x^4\pi^5-\frac{1}{22118400}x^4\pi^{10}+\ldots=\cos x.$$

Application of variational iteration method.

Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads $\lambda = -(s-x)^2/2$. So we can construct a variational iteration form for (2) in the form:

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^2}{2}\left[y_n'''(s) - \sin s + s + s\int_0^{\pi/2} py'(p)dp\right]ds.$$

We start by setting the zeroth component $y_0(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) = 1 - \frac{x^2}{2}$.

That will lead to the following successive approximations:

$$y_1(x) = \cos x - \frac{1}{24}x^4 + \frac{1}{576}\pi^3 x^4 \approx \cos x;$$

$$y_2(x) = \cos x; \quad y_3(x) = \cos x; \quad \ldots .$$

So we obtain the following approximate solution $y(x) = \lim_{n\to\infty} y_n(x) = \cos x$, which is the exact solution of the problem: $y(x) = \cos x$.

Example 4. Consider fourth- order Fredholm integro-differential equation [14, 15]

$$y^{(4)}(x) = \frac{1}{4} + (1 - 2\ln 2)x - \frac{6}{(1+x)^4} + \int_0^1 (x-s)y(s)ds,$$

with initial conditions $y(0)=0,\ y'(0)=1,\ y''(0)=-1,\ y'''(0)=2$; the exact solution is $y(x) = \ln(1+x)$.

Application of Adomian decomposition method.

Using $y(x) = \sum_{n=0}^{\infty} y_n(x)$ and the recurrence relation we obtained: we start by setting the zeroth component

$y_0^{(4)}(x) = \frac{1}{4} + (1-2\ln 2)x - \frac{6}{(1+x)^4}$, so that the first component is obtained by

$$y_1^{(4)}(x) = \int_0^1 (x-s)y_0(s)ds;$$

$y_2^{(4)}(x) = \int_0^1 (x-s)y_1(s)ds$; … . Applying the three-fold integral operator L^{-1} defined by, $L^{-1}(\cdot) = \int\int\int\int(\cdot)dxdxdxdx$, and using the given initial condition we obtain

$$y(x) = \frac{1}{4\cdot 4!}x^4 + \frac{1}{5!}(1-2\ln 2)x^5 + \ln(1+x) + \frac{x^5}{5!}\int_0^1 y(s)ds - \frac{x^4}{4!}\int_0^1 sy(s)ds.$$

Hence, taking into account the boundary conditions, we have

$$y_0(x) = \frac{1}{96}x^4 + \frac{1}{120}(1-2\ln 2)x^5 + \ln(1+x);$$

$$y_1(x) = \left(\frac{1}{10080}\ln 2 - \frac{5099}{483840}\right)x^4 + \left(\frac{719}{43200}\ln 2 - \frac{287}{34560}\right)x^5;$$

$$y_2(x) = -\left(\frac{181}{181440}\ln 2 - \frac{8543}{69672960}\right)x^4 + \left(\frac{5096}{217728000}\ln 2 - \frac{12671}{435456000}\right)x^5; \ \ldots .$$

This gives the solution in the series form

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = \ln(1+x) + \alpha_n x^4 + \beta_n x^5 = \ln(1+x),\ \lim_{n\to\infty}\alpha_n = 0,\ \lim_{n\to\infty}\beta_n = 0.$$

After the fourth iteration, the maximum absolute error is less than 10^{-12}, but the maximum absolute error decreases with increasing iteration.

Application of variational iteration method.

Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads $\lambda = (s-x)^3/6$. So we can construct a variational iteration form for (2) in the form:

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^3}{6}\left[y_4^{(4)}(s) - \frac{1}{4} - (1-2\ln 2)s + \frac{6}{(1+s)^4} + \int_0^1 (s-p)y(p)dp \right] ds.$$

We start by setting the zeroth component

$$y_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

That will lead to the following successive approximations:

$$y_1(x) = \ln(1+x) - 0.001041666667x^4 + 0.00025310255x^5;$$

$$y_2(x) = \ln(1+x) + 0.000005727233x^4 - 0.00000138458x^5;$$

$$y_3(x) = \ln(1+x) - 3.15295\cdot 10^{-8}x^4 + 7.62234\cdot 10^{-9}x^5; \quad \ldots .$$

So we obtain the following approximate solution $y(x) = \lim_{n\to\infty} y_n(x) = \ln(1+x)$, which is the exact solution of the problem: $y(x) = \ln(1+x)$.

Conclusion.

This results shows a comparative study between variational iteration method and Adomian decomposition method of solving Fredholm integro-differential equations. The main advantage of these methods are the fact that they provide its user with an analytical approximation, in many cases an exact solution in rapidly convergent sequence with elegantly computed terms. Also these methods handle linear and non-linear equations in a straightforward manner. These methods provide an effective and efficient way of solving a wide range of linear and nonlinear integro-differential equations. Illustrative examples are given to demonstrate the validity, accuracy and correctness of the proposed methods. The error between the approximate solution and exact solution decreases when the degree of approximation increases.

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