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OPTIMAL CONTROL PROBLEM ASSOCIATED WITH HEAT TRANSFER PROCESS

***Abstract**: In this work, we consider boundary control problem for heat exchange process. It is supposed that on the boundary of this domain the heat exchange according to Newton's law takes place. In the part of the bound of the given region it is given value of the derivative of the solution with the respect to the normal and it is required to find admissible control to get average value of solution. By the mathematical method it is proved that like this control exist.*

***Key words**: Heat conduction equation, admissible control, integral equation, smooth function, mathematical model, Laplace transform.*

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Introduction

Consider the following mathematical model of the heat exchange process along the interval $0 < x < l$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < l, t > 0, \tag{1.1}$$

with boundary conditions

$$u|_{x=0} = \mu(t), u|_{x=l} = 0, \tag{1.2}$$

and initial condition

$$u|_{t=0} = 0. \tag{1.3}$$

Let $M > 0$ be some given constant. We say that the function $\mu(t)$ is an admissible control if this function is differentiable on the half-line $t \geq 0$ and satisfies the following constraints

$$\mu(0) = 0, |\mu(t)| \leq M, t \geq 0. \tag{1.4}$$

In the present work we consider the following problem.

Problem. Let the function $\varphi(x)$ satisfies conditions

$$\int_0^l \varphi(x)dx = 1, \varphi(x) \geq 0, \varphi'(x) \leq 0.$$

Set

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k \sin\frac{k\pi x}{l}, 0 \leq x \leq l. \tag{1.5}$$

For a given function $\theta(t)$ problem consist in looking for the admissible control $\mu(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem (1.1)-(1.3) exists, is unique and for all $t > 0$ satisfies the following equation

$$\int_0^l \varphi(x)u(x,t)dx = \theta(t). \tag{1.6}$$

One of the urgent problems for the equations of mathematical physics is the problem of mathematical modeling of processes associated with various partial differential equations. In particular, mathematical modeling of the heat exchange process and the control of this process. Control in this situation is made by changing the heat flux entering to the region under consideration from a part of is boundary. It is natural to achieve temperature in the whole area. Therefore, it is important to control the boundary flow to reach the average temperature in any part of the area.

We recall that the time-optimal control problem for partial differential equations of parabolic type was first concerned in [7]. More recent results concerned with this problem were established in [1]-[6], [8], [9], [14] and [15]. Detailed information on the problems of optimal control for distributed parameter systems is given in the monographs [10] and [13].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [11]. The practical approaches to optimal control of the heat equation are described in publications like [12].

2. THE MAIN INTEGRAL EQUATION

We find out the solution to the problem (1.1)-(1.3) by Fourier method. Consider the following Green function:

$$G(x,y,t)=\frac{2}{l}\sum_{k=1}^{\infty}e^{-(k\pi/l)^2 t}\sin\frac{k\pi x}{l}\sin\frac{k\pi y}{l}.$$

Set

$$w(x,t)=\frac{l-x}{l}\mu(t), x\in(0,l), t\geq 0,$$

and assume that the solution $u(x,t)$ has the form:

$$u(x,t)=w(x,t)+v(x,t). \qquad (2.1)$$

It follows from (1.1)-(1.3) that the function $v(x,t)$ satisfies equation

$$\frac{\partial v(x,t)}{\partial t}=\frac{\partial^2 v(x,t)}{\partial x^2}-\frac{l-x}{l}\mu'(t), 0<x<l, t>0,$$

with boundary conditions

$$v|_{x=0}=0, v|_{x=l}=0,$$

and initial condition

$$v|_{t=0}=0.$$

Consequently,

$$v(x,t)=$$

$$=-\frac{1}{l}\int_0^t\mu'(s)ds\int_0^l G(x,y,t-s)(l-y)dy. \qquad (2.2)$$

Proposition 2.1. Let $\mu(t)$ be a smooth function on the half-line $t\geq 0$. Then the function

$$u(x,t)=\frac{l-x}{l}\mu(t)-$$

$$-\int_0^t\mu'(s)ds\int_0^l G(x,y,t-s)\frac{(l-y)}{l}dy$$

is the solution of the initial-boundary value problem (1.1)-(1.3).

Proof. The proof comes from (2.1) and (2.2) (see, e.g. [16], [17]).

Not that

$$\frac{l-x}{l}=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi x}{l}, 0\leq x\leq l, \qquad (2.3)$$

and according to Parceval equation,

$$\int_0^l\varphi(x)\frac{l-x}{l}dx=\frac{l}{\pi}\sum_{k=1}^{\infty}\frac{\varphi_k}{k}. \qquad (2.4)$$

Taking into consideration (2.3), we get

$$\int_0^l G(x,y,t-s)\frac{(l-y)}{l}dy=$$

$$=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-(k\pi/l)^2(t-s)}\sin\frac{k\pi x}{l}.$$

According to Pareceval equation and (1.6) we can write

$$\int_0^l\varphi(x)u(x,t)dx=\mu(t)\int_0^l\varphi(x)\frac{l-x}{l}dx-$$

$$-\frac{l}{\pi}\sum_{k=1}^{\infty}\frac{\varphi_k}{k}\left(\int_0^t e^{-(k\pi/l)^2(t-s)}\mu'(s)ds\right)=$$

$$=\frac{\pi}{l}\sum_{k=1}^{\infty}\varphi_k\cdot k\int_0^t e^{-\left(\frac{k\pi}{l)^2(t-s)}\right)}\mu(s)\,ds=$$

$$=\int_0^t B(t-s)\mu(s)ds,$$

where

$$B(t)=\frac{\pi}{l}\sum_{k=1}^{\infty}\varphi_k\cdot k\cdot e^{-\lambda_k t}, \lambda_k=\left(\frac{k\pi}{l}\right)^2, \qquad (2.5)$$

and

$$\varphi_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi x}{l}dx. \qquad (2.6)$$

Then we get the main integral equation

$$\int_0^t B(t-s)\mu(s)ds=\theta(t), t>0. \qquad (2.7)$$

Lemma. Let $g(x)$ assume that this function is decreasing and non-negative on $[0,\infty)$. Then the following inequality holds

$$\int_0^{k\pi}g(y)\sin y\,dy\geq 0. \qquad (2.8)$$

Proof. The proof easy of lemma (see, e.g. [6]).

Proposition 2.2. For $\{\varphi_k\}_{k=1}^{\infty}$ defined by (2.6) the following estimate holds

$$0\leq\varphi_k\leq\frac{C}{k}, k=1,2,3,\ldots.$$

Proof. If we substitute $x\cdot\frac{k\pi}{l}$ into $y$in the (2.8) inequality, we have the following inequality

$$\int_0^l\varphi(x)\sin\frac{k\pi x}{l}dx\geq 0$$

from this we get

$$\varphi_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi x}{l}dx\geq 0, k=1,2,\ldots. \qquad (2.9)$$

From (2.6), we can write

$$\varphi_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi x}{l}dx=$$

$$=-\frac{2}{l}\varphi(x)\frac{l}{k\pi}\cos\frac{k\pi x}{l}\Big|_{x=0}^{x=l}+$$

$$+\frac{2}{k\pi}\int_0^{l\int\frac{k\pi x}{l}\frac{2\varphi(0)}{k\pi}(1-(-1)^k)\frac{o(1)}{k}}\varphi'(x)\cos$$

Then we obtain

$$0\leq\varphi_k\leq\frac{C}{k}.$$

Proposition 2.2 proved.

Proposition 2.3. For $B(t)$ defined by (2.5) the following estimate

$$0<B(t)\leq\frac{C_0}{\sqrt{t}},$$

is valid.

Proof. From (2.5) and (2.9), we get

$$B(t)>0,$$

and according to Proposition 2.2, we may write (see, e.g. [6])

$$B(t)\leq C_0\sum_{k=1}^{\infty}e^{-(k\pi/l)^2 t}\leq\frac{C_0}{\sqrt{t}}.$$

Proposition 2.3 proved.

3. SOLUTION OF THE PROBLEM

We consider Voltaire integral equation

$$\int_0^t B(t-s)\mu(s)ds = \theta(t), t > 0. \qquad (3.1)$$

Theorem 3.1. Let there be a constant M_0, then for any function $\theta(t) \in W_2^2(-\infty,+\infty), \theta(t) = 0$, $t \le 0$ that satisfies the

$$\|\theta(t)\|^2_{W_2^2(R_+)} \le M_0, \qquad (3.2)$$

condition, there exists an admissible control $\mu(t)$ that satisfies condition (1.6).

The solve equation (3.1), we use the Laplace transform method. We introduce the notation

$$\tilde{\mu}(p) = \int_0^\infty e^{-pt}\mu(t)dt.$$

Using the convolution property from equation (3.1) we obtain

$$\tilde{B}(p)\tilde{\mu}(p) = \tilde{\theta}(p),$$

consequently

$$\tilde{\mu}(p) = \frac{\tilde{\theta}(p)}{\tilde{B}(p)}, \text{ where } p = a + i\xi, a > 0,$$

and

$$\mu(t) = \frac{1}{2\pi i}\int_{a-i\xi}^{a+i\xi}\frac{\tilde{\theta}(p)}{\tilde{B}(p)}e^{pt}dp =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\theta}(a+i\xi)}{\tilde{B}(a+i\xi)}e^{(a+i\xi)t}d\xi. \qquad (3.3)$$

Not that

$$\int_0^\infty e^{-\lambda_k t}e^{-pt}\,dt = \frac{1}{p+\lambda_k}.$$

We can write

$$\tilde{B}(p) = \int_0^\infty B(t)e^{-pt}dt = \frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k}{p+\lambda_k},$$

and

$$\tilde{B}(a+i\xi) = \frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k}{a+i\xi+\lambda_k} =$$

$$= \widetilde{Re\,B}\,(a+i\xi) + \widetilde{Im\,B}\,(a+i\xi),$$

where,

$$\widetilde{Re\,B}\,(a+i\xi) = \frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2},$$

$$\widetilde{Im\,B}\,(a+i\xi) = -\frac{\pi\cdot\xi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k}{(a+\lambda_k)^2+\xi^2}.$$

We know that

$$\frac{1}{(a+\lambda_k)^2+\xi^2} \ge \frac{1}{1+\xi^2}\cdot\frac{1}{(a+\lambda_k)^2+1}.$$

Consequently,

$$\left|\widetilde{Re\,B}\,(a+i\xi)\right| = \frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2} \ge$$

$$\ge \frac{1}{1+\xi^2}\frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k(a+\lambda_k)}{(a+\lambda_k)^2+1} = \frac{C_{1a}}{1+\xi^2}, \qquad (3.4)$$

and

$$\left|\widetilde{Im\,B}\,(a+i\xi)\right| = \frac{\pi\cdot|\xi|}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k}{(a+\lambda_k)^2+\xi^2} \ge$$

$$\ge \frac{|\xi|}{1+\xi^2}\frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k}{(a+\lambda_k)^2+1} = \frac{|\xi|C_{2a}}{1+\xi^2}, \qquad (3.5)$$

where

$$C_{1a} = \frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k(a+\lambda_k)}{(a+\lambda_k)^2+1},$$

$$C_{2a} = \frac{\pi}{l}\sum_{k=1}^{\infty}\frac{k\cdot\varphi_k}{(a+\lambda_k)^2+1}.$$

From (3.4) and (3.5), we have

$$\left|\tilde{B}(a+i\xi)\right| \ge \frac{C_{3a}}{\sqrt{1+\xi^2}}, \qquad (3.6)$$

where C_{3a} is expressed through C_{1a} and C_{2a}.

Then, when $a \to 0$from (3.3) we obtain the equality

$$\mu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\theta}(i\xi)}{\tilde{B}(i\xi)}e^{i\xi t}d\xi. \qquad (3.7)$$

Theorem 3.2. Let $\theta(t) \in W_2^2(-\infty,+\infty), \theta(t) = 0$ at $t \le 0$. Then for the image of the function $\theta(t)$ the inequality

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\xi)\right|\sqrt{1+\xi^2}d\xi < \infty$$

is valid.

Proof. Using the integral representation of the image of a given function $\theta(t)$ and integration by parts we get

$$\tilde{\theta}(a+i\xi) = \int_0^\infty e^{-(a+i\xi)t}\theta(t)dt =$$

$$= \theta(t)\frac{e^{-(a+i\xi)t}}{-a-i\xi}\bigg|_{t=0}^{t=\infty} + \frac{1}{a+i\xi}\int_0^\infty e^{-(a+i\xi)t}\theta'(t)dt,$$

then

$$(a+i\xi)\tilde{\theta}(a+i\xi) = \int_0^\infty e^{-(a+i\xi)t}\theta'(t)dt.$$

Therefore, when $a \to 0$we have

$$i\xi\tilde{\theta}(i\xi) = \int_0^\infty e^{-i\xi t}\theta'(t)dt,$$

and

$$(i\xi)^2\tilde{\theta}(i\xi) = \int_0^\infty e^{-i\xi t}\theta''(t)dt.$$

Consequently,

$$\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\xi)\right|\sqrt{1+\xi^2}d\xi = \int_{-\infty}^{+\infty}\frac{\left|\tilde{\theta}(i\xi)\right|(1+\xi^2)}{\sqrt{1+\xi^2}}d\xi \le$$

$$\le \left(\int_{-\infty}^{+\infty}\left|\tilde{\theta}(i\xi)\right|^2(1+\xi^2)^2d\xi\right)^{\frac{1}{2}}\cdot$$

$$\cdot\left(\int_{-\infty}^{+\infty}\frac{d\xi}{1+\xi^2}\right)^{1/2} \le C\|\theta\|^2_{W_2^2(R_+)}.$$

Theorem 3.2 proved.

Proof of Theorem 3.1. According to (3.6) and theorem 3.2 we can write

$$|\mu(t)| \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\tilde{\theta}(i\xi)|}{|\tilde{B}(i\xi)|}d\xi \leq$$

$$\leq \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty}|\tilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi < \infty.$$

Then, we get

$$|\mu(t)| \leq \frac{1}{2\pi C_0}C\|\theta\|^2_{W_2^2(R_+)} \leq \frac{CM_0}{2\pi C_0} = M.$$

Theorem 3.1 proved.

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