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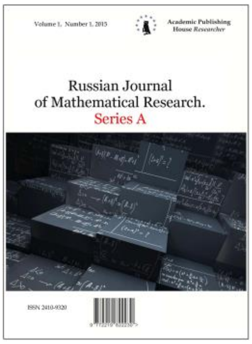

Proofs of Andrica and Legendre Conjectures

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Abstract

In this research proof of Legendre conjecture is presented. The proof is based on a property possessed exclusively by all prime numbers. That is, the positive square-root of any prime number is an irrational number that always lies between two consecutive positive integers. This property excludes the number one from the set of prime numbers. Not all composite numbers possess this sure property possessed by all prime numbers. It is this special property of prime numbers special property of prime numbers that makes Legendre conjecture a sure law for all prime numbers.

In the process of seeking to prove Legendre's conjecture the prime gap problem is resolved and Riemann hypothesis reviewed in the light of these findings.

Keywords: proof of Legendre's conjecture, an exclusive property of prime numbers, number theory, Riemann hypothesis, on differences between consecutive primes.

1. Introduction

The properties of prime numbers have been studied for many centuries. Euclid gave the first proof of infinity of primes. Euler gave a proof which connected primes to the zeta function. Then there was the Gauss and Legendre's formulation of the prime number theorem and its proof by Hadamard and de la Vallee Poussin. Riemann further came with some hypothesis about the roots of the Riemann-zeta function.

Many others have contributed towards prime number theory.

Legendre's conjecture, proposed by Adrien-Marie Legendre states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers. Up to 2017 the conjecture had neither been proved nor disproved.

In this research a method will be presented of proving Legendre's conjecture. The proof is based on an exclusive properties of prime numbers not generally shared with composite numbers. A square root property of prime numbers will be discussed that also implies the truthfulness of Legendre's conjecture.

Relevance

The research aims at furthering our understanding of the prime gap problem as proposed in Legendre, Oppermann, Andrica conjectures and even Riemann hypothesis.

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2. Discussion

Theorem

The square-root of every prime number is an irrational number in the gap between two consecutive integers.

Proof

First by definition a prime number is a natural number greater than 1 that cannot be formed by multiplying two smaller natural numbers each of which is greater than one. The primality property of prime numbers does not permit them to be factorized to two identical natural factors ($n \times n$), thus the square-root of a prime number is always an irrational number. All irrational numbers, including square-roots of prime numbers lie between two consecutive integers.

Thus:

$$1 < \sqrt{2} < 2$$
$$1 < \sqrt{3} < 2$$
$$2 < \sqrt{5} < 3$$
$$2 < \sqrt{7} < 3$$
$$3 < \sqrt{11} < 4$$
$$3 < \sqrt{13} < 4$$
$$4 < \sqrt{17} < 5$$
$$4 < \sqrt{19} < 5$$
$$4 < \sqrt{23} < 5......$$
$$n < \sqrt{p_i} < n+1$$

Deriving Legendre's conjecture

If the square-root of a general prime number p_i lies between two consecutive integers, n and n+1, then it follows mathematically that the prime number p_i lies between two consecutive square numbers. That is to say:

$$n < \sqrt{p_i} < n+1 \equiv n^2 < p_i < (n+1)^2$$

Proof of Legendre's conjecture

In order to prove Legendre's conjecture there is need to show that between every pair of consecutive positive integers there exists a radical number that is equal to the positive square-root of a prime number.

The above results from the theorem also suggest that the gap between square-roots of consecutive primes is less than 1. That is:

$$\sqrt{p_{j+1}} - \sqrt{p_j} < 1 \rightarrow p_{j+1} < 1 + p_j + 2\sqrt{p_j}$$
$$\rightarrow p_{j+1} - p_j < 1 + 2\sqrt{p_j}$$

(1)

Proof of Legendre conjecture thus involves resolving the prime gap problem. The inequality 1 is a statement of Andrica conjecture. The conjecture suggests a smaller gap between successive primes than Legendre conjecture. Thus proof of Andrica conjecture would also imply validity of Legendre conjecture.

The above inequality 1 suggests that the maximum possible gap tends to infinity as primes tend to infinity.

That is:

$$g_j = p_{j+1} - p_j < 1 + 2\sqrt{p_j} = 1 + p_j^{1/2+\log 2/\log p_i}$$

(2)

$$\lim_{j\to\infty} g_j \to \infty$$

$$\to \frac{g_j}{p_j} < \frac{1+2\sqrt{p_j}}{p_j}$$

(3)

$$\to \lim_{j\to\infty} \frac{g_j}{p_j} = 0$$

(4)

$$\lim_{j\to\infty} g_j < p_j^{1/2}$$

(5)

Thus the average gap in between two primes is given by the inequality below:

$$\frac{\sum g_j}{n} < \frac{1}{n}\sum_{j=1}^{n}(1+2\sqrt{p_i})$$

(6)

Thus the number of primes less than a given natural number x is given by the inequality:

$$\pi(x) > \frac{x}{\frac{1}{n}\sum_{i=1}^{n}(1+2\sqrt{p_i})} \wedge p_i \le x$$

(7)

Hoheisel (1930) was the first to show that there is a constant θ<1 such that:

$$\pi(x+x^\theta) - \pi(x) \sim \frac{x^\theta}{\log(x)} as : x \to \infty$$

Hence showing that

$$g_n < p_n^{\theta}$$

(8)

Hoheisel obtained the possible value 32999/33000 for θ. These results were later improved by Heilbronn to θ = ¾ +ε, for any ε>0 and by Chudakov. Other major improvements were introduced by Ingham, Huxley (1972), Baker, Harman and Pintz (2001).

Ingham showed that for some positive constant c,

$$\zeta(1/2+it) = O(t^c)$$

then:

$$\pi(x+x^\theta) - \pi(x) \sim \frac{x\theta}{\log(x)}$$

For any $\theta > {(1+4c)}/{(2+4c)} = 1/2 + \frac{2c}{2+4c}$

The Lindelöf hypothesis implies that Ingham's formula holds for c any positive number.

If we take if $c = \log 2/\log p_i$ then

$$(\theta > {(1+4\log 2/\log p_i)}/{(2+4\log 2/\log p_i)})$$

This would mean that

$$g_i < 1 + p_j^{(1+4\log 2/\log p_j)/(2+4\log 2/\log p_j)} = 1 + p_j^{1/2 + \frac{2\log 2/\log p_j}{2+4\log 2/\log p_j}} < 1 + p_j^{1/2+\log 2/\log p_j} = 1 + 2\sqrt{p_j}$$

(9)

The inequality 8 is in agreement with inequality 2. The inequality 1 was assumed from observations made from the proposed theorem in section 2. No proof was presented for deriving the inequality; instead the Lindelöf hypothesis was used in conjunction with Ingham's formula to arrive at inequality 9. As such the above proposed proof has a gap because it is based on an unproved hypothesis. There is need for a more rigorous proof of the equality 1 above.

Analytical solution of the gap between consecutive primes and proof of Andrica and Legendre conjectures

Consider two consecutive primes, p_{j+1}, p_j

If the primes are expressible in terms of a variable y such that:

$p_{i+1} = 2y + 2n + 1 \wedge p_i = 2y + 1$

Then the gap between the two primes is 2n and:

$$y = \frac{p_j - 1}{2}$$

$$g_j = p_{j+1} - p_j = 2n \rightarrow n = \frac{p_{j+1} - p_j}{2}$$

$$\sqrt{p_{j+1}} - \sqrt{p_j} = \sqrt{(2y+2n+1)} - \sqrt{(2y+1)}$$

$$= (2y+2n+1)^{1/2} - (2y+1)^{1/2}$$

$$= \{(2y)^{1/2} + \tfrac{1}{2}(2y)^{-1/2}(2n+1) - \tfrac{1}{8}(2y)^{-3/2}(2n+1)^2 + \ldots\} - \{(2y)^{1/2} + \tfrac{1}{2}(2y)^{-1/2} - \tfrac{1}{8}(2y)^{-3/2} + \ldots\}$$

$$< (2y)^{-1/2} n = \frac{g_j}{2\sqrt{p_j - 1}} = \frac{p_{j+1} - p_j}{2\sqrt{p_j - 1}} = \frac{(\sqrt{p_{j+1}} + \sqrt{p_j})(\sqrt{p_{j+1}} - \sqrt{p_j})}{2\sqrt{p_i - 1}}$$

$$\frac{\sqrt{p_{j+1}} + \sqrt{p_j}}{2\sqrt{p_j - 1}} < 2$$

$$\frac{\sqrt{p_{j+1}} - \sqrt{p_j}}{2\sqrt{p_j - 1}} < \frac{1}{2}$$

$$\lim_{p_j \to \infty} \frac{\sqrt{p_{j+1}} + \sqrt{p_j}}{2\sqrt{p_j - 1}} = 1$$

$$\lim_{p_j \to \infty} \frac{\sqrt{p_{j+1}} - \sqrt{p_j}}{2\sqrt{p_j - 1}} = 0$$

$$\sqrt{p_{j+1}} - \sqrt{p_j} < \frac{g_j}{2\sqrt{p_j - 1}} < 1$$

(10)

To clarify the derivation of inequality 10, it should be noted that:

$$\{(2y)^{1/2}+\tfrac{1}{2}(2y)^{-1/2}(2n+1)-\tfrac{1}{8}(2y)^{-3/2}(2n+1)^2+\ldots\}-\{(2y)^{1/2}+\tfrac{1}{2}(2y)^{-1/2}-\tfrac{1}{8}(2y)^{-3/2}+\ldots\}<$$

$$\tfrac{1}{2}(2y)^{-1/2}(2n+1)\}-\tfrac{1}{2}(2y)^{-1/2}=\tfrac{1}{2}(2y)^{-1/2}$$

It should be noted again that:

$$(a+b)^n-(a+1)^n=\{a^n+na^{n-1}b+\frac{n(n-1)a^{n-2}b^2}{2!}+.....\}-\{a^n+na^{n-1}+\frac{n(n-1)a^{n-2}}{2!}+.....\}$$

There is need to improve on the above inequality 10

Theorem (gap between successive prime numbers)

The size of the gap between successive prime numbers is governed by the inequality

$$p_{j+1}>\frac{g_j^{\;2}}{4}+g_i+1$$

(11)

Proof

The above theorem proposes that there exists a Diophantine equation governing gaps between successive primes given by:

$$g_j^{\;2}+4g_j+4(p_{j+1}+1-s)=0$$

$$s\in N$$

(12)

The solution of the above Diophantine equation is given by:

$$g_j=-2+\sqrt{4-(p_{j+1}+1-s)}$$

$$s>p_{j+1}$$

(13)

$$p_2=3;s=9\wedge g_1=1$$

For $p_3=5;s=16\wedge g_2=2$

$$p_4=7;s=20\wedge g_3=2....$$

The above theorem is valid since it enables generation of a Diophantine quadratic equation that accounts for gaps between successive primes.

The inequality 10 can be rewritten for an easy proof of Andrica and Goldbach conjecture using the above theorem.

$$\sqrt{p_{j+1}}-\sqrt{p_j}<\frac{g_j}{2\sqrt{p_j-1}}=\frac{g_j}{2\sqrt{p_{j+1}-g_j-1}}=\frac{1}{2\sqrt{\dfrac{p_{j+1}-g_j-1}{g_j}}}$$

$$p_{j+1}>\frac{g_j^{\;2}}{4}+g_j+1\rightarrow\sqrt{p_{j+1}}-\sqrt{p_j}<1$$

Thus Andrica conjecture is proved. From the foregoing analysis the Andrica conjecture follows from the Legendre conjecture since it narrows the gap between the primes proposed in the Legendre conjecture.

By the theorem 11 above if

1) $g_i=1\rightarrow p_2>2.25$

2) $g_i=2\Rightarrow p_{i+1}>3$

3) $g_i=4\Rightarrow p_{i+1}>9$

$g_i = 6 \Rightarrow p_{i+1} > 16$

Further analysis

From inequality 10:

$$g_j < 2\sqrt{p_j - 1}$$

(14)

Thus the gap between primes can be further reduced than that stipulated by Andrica Conjecture

Thus the average gap between primes is given by the inequality:

$$g < \frac{1}{n}\sum_{j=1}^{j=n}\sqrt{p_j - 1}$$

(14)

Thus:

$$\frac{1}{10}\sum_{j=1}^{j=10}\sqrt{p_j - 1} = \frac{1}{10}(\sqrt{1} + \sqrt{2} + \sqrt{4} + \sqrt{6} + ... + \sqrt{28}) = 3.180830383$$

Thus the inequality 14 predicts that the average gap of prime numbers between 1 and 30 is less than 3.181. The prime number theorem stipulates that the average gap between 1 and 30 is about equal to $\ln 30 = 3.401197382$. Thus the results above generally agree with the prime number theorem.

From inequality 13 the gap between primes can be represented by the equation:

$$(n(g_j + \lambda))^{ix} = (p_j - 1)^{\frac{1}{2}+ix}$$

(15)

Thus from equation 15:

$$p_j = (n(g_j + \lambda))^{\frac{ix}{\frac{1}{2}+ix}} + 1$$

(16)

$$\frac{1}{1 - p_j^{-(\frac{1}{2}+ix)}} = \frac{1}{1 - ((n(g_j + \lambda))^{\frac{ix}{\frac{1}{2}+ix}} + 1)^{-(\frac{1}{2}+ix)}}$$

(17)

$$\zeta(s) = \prod_{j=1}^{j=x}\frac{1}{1 - ((n(g_j + \lambda))^{\frac{ix}{\frac{1}{2}+ix}} + 1)^{-(\frac{1}{2}+ix)}}$$

(18)

Thus the gap stipulated in inequality 13 implies Riemann hypothesis.

From the results of equation 13:

$$\log_e x = \frac{\vartheta}{n}\sum_{j=1}^{n}\sqrt{p_j - 1} \to$$

$$x^{\frac{n}{\theta}} = e^{\sum_{j=1}^{n}\sqrt{p_j - 1}} \to$$

$$x = \sqrt[\frac{n}{\theta}]{e^{\sum_{j=1}^{n}\sqrt{p_i - 1}}} = (e^{\sum_{j=1}^{n}\sqrt{p_j - 1}})^{\frac{\vartheta}{n}} = e^{\frac{\vartheta}{n}\sum_{j=1}^{n}\sqrt{p_j - 1}}$$

(19)

Using equation 19:

$$\varsigma(s)=\sum_{x=1}^{\infty}\frac{1}{\left(\sqrt[\frac{\vartheta}{n}]{e^{\sum_{j=1}^{n}\sqrt{p_j-1}}}\right)^s}=\sum_{x=1}^{\infty}\left(\frac{1}{\left(e^{\sum_{j=1}^{n}\sqrt{p_j-1}}\right)^{\frac{\vartheta}{n}}}\right)^s=\sum_{x=1}^{\infty}e^{-s\frac{\theta}{n}\sum_{j=1}^{n}\sqrt{p_j-1}}=\sum_{k=1}^{\infty}e^{-s\ln x}$$

(20)

If we take $s=\frac{1}{2}+it$ then the equation 20 takes the form:

$$\varsigma(s)=\sum_{x=1}^{\infty}e^{-(\frac{1}{2}+it)\frac{\theta}{n}\sum_{j=1}^{n}\sqrt{p_j-1}}=\sum_{x=1}^{\infty}e^{-(\frac{1}{2}+it)g_x}=\sum_{x=1}^{\infty}e^{-g_k/2}(\cos g_x t+i\sin g_x t)$$

$$=\sum_{x=1}^{\infty}e^{-\ln x/2}(\cos t\ln x+i\sin t\ln x)=\sum_{x=1}^{\infty}\frac{(\cos(t\ln x)+i\sin(t\ln x))}{x^{1/2}}$$

(21)

In the functional trigonometric series of the form 21 above the non-trivial zeroes of Riemann hypothesis can be derived and Riemann hypothesis verified.

3. Conclusion

Prime numbers have an exclusive property of which Legendre conjecture is a derivative. The square-root of every prime number is an irrational number between two consecutive integers. This property can be used to prove both Legendre's and Andrica conjecture through provable derivative inequality. It is possible to reduce the gap between further to less than that stipulated in Andrica conjecture. A function has been achieved that accounts the non-trivial zeroes of the Riemann hypothesis.

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