

COMPLEX VARIANCE IN MODERN ECONOMETRICS¹

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Svetunkov S. G.

Complex Variance in Modern Econometrics

One of the modern trends in economics is the use of elements of the theory of functions of complex variables. When constructing complex-valued econometric models, researchers come across the fact that in mathematical statistics the section associated with processing a complex random variable is based on the hypothesis on independence of the real and imaginary parts of complex variables. This hypothesis leads to the necessity of calculating the actual characteristics of complex random variables, including variance. As shown in the article, this assumption significantly limits the possibilities of modern econometrics. Therefore, the article substantiates the need to use complex variance in econometrics. The analysis of the properties of complex variance and the meaning of its real and imaginary parts is carried out. It is shown how, using a complex variance, to estimate the confidence limits for a complex random variable. Since the use of complex variance in econometrics and mathematical statistics is proposed for the first time ever, the article discusses the formation of complex-valued correlation and regression analysis, sections of which will be used in econometrics of complex variables.

Keywords: econometrics, complex-valued econometric models, complex variance, correlation moment, complex pair correlation coefficient

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Svetunkov Sergey G. – Doctor of Sciences (Economics), Professor, Professor of the Graduate School of Management and Business of Peter the Great St. Petersburg Polytechnic University (29 Politekhnichna Str., St. Petersburg, 195251, Russia)

E-mail: sergey@svetunkov.ru

ORCID: 0000-0001-6251-7644

Researcher ID: N-1787-2013

Scopus Author ID: 56652307100

УДК 338.27.015 (075.8)
JEL Classification: C40; C46Светушков С. Г. Комплексна дисперсія
в сучасній економіці

Однією з сучасних тенденцій в економічній науці є використання елементів теорії функцій комплексних змінних. При побудові комплекснозначних економічних моделей дослідники стикаються з тим, що в математичній статистиці розділ, пов'язаний з обробкою комплексної випадкової змінної, базується на гіпотезі про незалежність дійсної і уявної частин комплексних змінних. Ця гіпотеза призводить до необхідності обчислення дійсних характеристик комплексних випадкових величин, в тому числі і дисперсії. Як показано в статті, таке припущення істотно обмежує можливості сучасної економіки. Тому в статті обґрунтовується необхідність використання в економіці комплексної дисперсії. Здійснюється аналіз властивостей комплексної дисперсії і сенсу її дійсної і уявної частин. Показується, як, використовуючи комплексну дисперсію, оцінити довірчі границі для комплексної випадкової величини. Оскільки застосування комплексної дисперсії в економіці та математичній статистиці пропонується вперше, то в цій статті обговорюються питання формування комплекснозначного кореляційного аналізу та регресійного аналізу, розділи яких будуть використовуватися в економіці комплексних змінних.

Ключові слова: економіка, комплекснозначні економічні моделі, комплексна дисперсія, кореляційний момент, комплексний коефіцієнт парної кореляції.

УДК 338.27.015 (075.8)
JEL Classification: C40; C46Светушков С. Г. Комплексная дисперсия в современной
эконометрике

Одной из современных тенденций в экономической науке является использование элементов теории функций комплексных переменных. При построении комплекснозначных эконометрических моделей исследователи сталкиваются с тем, что в математической статистике раздел, связанный с обработкой комплексной случайной переменной, базируется на гипотезе о независимости действительной и мнимой частях комплексных переменных. Эта гипотеза приводит к необходимости вычисления действительных характеристик комплексных случайных величин, в том числе и дисперсии. Как показано в статье, такое предположение существенно ограничивает возможности современной эконометрики. Поэтому в статье обосновывается необходимость использования в эконометрике комплексной дисперсии. Осуществляется анализ свойств комплексной дисперсии и смысла её действительной и мнимой частей. Показывается, как, используя комплексную дисперсию, оценить доверительные границы для комплексной случайной величины. Поскольку применение комплексной дисперсии в эконометрике и математической статистике предлагается впервые, то в данной статье обсуждаются вопросы формирования комплекснозначного корреляционного анализа и регрессионного анализа, разделы которых будут использоваться в эконометрике комплексных переменных.

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Светуцьков Сергій Геннадійович – доктор економічних наук, професор, професор Вищої школи управління та бізнесу Санкт-Петербурзького політехнічного університету Петра Великого (вул. Політехнічна, 29, Санкт-Петербург, 195251, Росія)

E-mail: sergey@svetunkov.ru

ORCID: 0000-0001-6251-7644

Researcher ID: N-1787-2013

Scopus Author ID: 56652307100

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E-mail: sergey@svetunkov.ru

ORCID: 0000-0001-6251-7644

Researcher ID: N-1787-2013

Scopus Author ID: 56652307100

Introduction. Econometric methods and models gain popularity and are frequently used in economic research. However, mathematical methods applied in economics do not fully satisfy the needs of the researchers. A large number of economic problems still remain unsolved. This encourages scientists to continuously search for new methods of developing economic and mathematical models which will help to carry out economic research more efficiently.

One of the promising trends in elaborating the economic research tools is turning to the methods of the theory of functions of complex variables. Only first hesitant steps have been taken in this research area of economics so far. There are few instances of using complex variables in the economic studies but they do exist.

For example, when modeling economic dynamics, scientists feel the need to use complex variables. These variables sometimes result from characteristic roots of equations [5; 6; 11]. Another application of complex variables is complex mathematical models: for example, when spectral analysis in economics is used, and the given economic data are regarded as random signals [7]. Yet more often economists face the situations when complex variables are used in mathematical modeling of discount cash flows [1; 4]. However, these cases are not further developed and discussed in science, since their practical application requires appropriate recommendations on statistical treatment of random complex variables. Indeed science has not provided such recommendations yet.

The theory of functions of complex variables contains a section called z -transformation of Laurent. It is widely used by mathematicians for solving complex differential equations. This is why one could expect a more frequent use of z -transformation of Laurent in economic modeling. However, there are very few attempts of this kind in economics [10; 18].

One of the works written by Ben Tamari [21], who used elementarily function of complex variables to study the balances of cash flow at the macro level, stands out from other random economic studies on complex variables. However, this work remained unnoticed by economists. Moreover, it presented studying the features of the function of a complex variable, endowed with some characteristics of real economy, to a greater extent rather than studying of economic qualities based on the corresponding model. Furthermore, the possibility of practical use of Ben Tamari's conclusions and recommendations does not seem obvious.

The situation began to change when the book themed to complex-valued modeling in economics and finance was published [20]. The models of complex variables are considered as major kinds of mathematical equations here. Some recommendations on econometrics of complex variables are presented in this book, whereas a key point on variance of complex variables is not discussed. Thus, the probative value of all the econometric models considered in this book does not prove to be very strong.

The problem with the use of models of complex variables is caused by the fact that the branch of mathematical statistics studying random complex variables seems undeveloped. Scientists paid attention to statistical treatment of complex variables dynamics in the 50-60s of the 20th century. R. Wooding (1956) was the first to formulate this task and offer the approach to presenting the complex random value based on the normal distribution [24]. R. Arens [2] (1957) and I. S. Reed [16] (1962) introduced major concepts and characteristics of the random normally distributed complex variable, such as mathematical mean value, moment coefficients (including correlation coefficient), covariance, variance, etc. It was a priori supposed that the case of the independent normally distributed real and imaginary parts was under discussion, and only N. R. Goodman (1963) formulated the hypothesis more clearly [9]. From that time on scientists began to consider the distribution of complex random value as an aggregate of two independent normally distributed random values – real and imaginary parts. It was W. Feller (1966) who systematized these statements [8]. Today this assumption about the autonomy of the real and imaginary parts of a complex random variable serves as a key prerequisite for mathematical statistics of a complex random variable.

As the scope of functions of modeling using complex variables in various scientific spheres widened, scientists faced the necessity to develop the body of mathematical statistics, which could allow to cope with it. Since this function is of interest not only for economists but for specialists dealing with complex variables in other sciences, based on the real part of the complex random value variance, an adapted least-squares method was proposed [22]. However, at this point the development of the statistical tools of the complex random value stopped – neither tools for calculating complex-related correlations nor tools for determining the confidence limits or other instruments of statistical treatment of random complex variables have been offered so far.

Since statistical data treatment in the form of the complex random value is of current interest for economics and for such an important branch of economics as econometrics, scientists feel an objective need in elaborating the existing and appropriate mathematical tools.

Materials and Methods

Basic complex-valued model in economics

Firstly, let us present our understanding of how complex-valued models can be applied in solving economic problems.

The complex variable itself can be regarded as a model characterizing the qualities of an object in a more complex way. This is due to the fact that it consists of two real variables rather than of one variable as it is characteristic of real variable models.

When considering such economic indicator as, for example, gross profit G we understand that it allows estimating only one aspect of the complex economic phenomenon – the output. This is why it is a typical situation when making decisions nobody considers themselves satisfied with the criterion of the maximum gross profit. To have more clear understanding of the situation and make the right decision, additional indicators of the output are taken into account. In the real economy such crucial economic indicator as the cost of goods manufactured C is considered. Then, after considering the gross profit and the cost of goods manufactured, profit margin can be calculated. As the profit margin is the economic indicator which helps to estimate both the cost of the goods manufactured and the output, i.e., serves as the indicator of production efficiency, it is used as another additional criterion when making an economic decision.

To make a decision in a real economic situation when describing some production process based on real variable models, scientists have to model both the gross profit and the cost of goods manufactured. As building two models is quite inconvenient and cost-inefficient, one model is constructed by adding the profit and the operating expenses to find out the gross output. It is the gross output which is considered as the major output in economic and mathematical modeling.

The need to model two economic variables — the gross profit and the operating expenses — simultaneously is easily met if the output is regarded as a complex number. In this case this complex number itself acts as a model that reflects the output. For the case under discussion it can be presented in the following way:

$$Z = C + iG. \tag{1}$$

In this case i is an imaginary unit which possesses quality $i^2 = -1$.

Thus, when we consider and model a new number Z , we take into account both the gross profit G and the operating expenses C , since they are indispensable characteristics of the complex number. This means that while carrying out some actions with one complex variable, the researcher deals with two real variables. Therefore, the use of the complex variable like (1) as a model integrating two economic variables allows to obtain a much more compact formula, on the one hand, and to include more detailed information about the modeled object in the economic and mathematical model, on the other hand, as well as to consider them in relation to each other.

However, if these were the only innovations introduced in economic and mathematical modeling by the use of complex variables, it would possibly not pay to do that. Economic indicators and processes modeled by using complex variables are considerably broader than it may seem. In fact, if we just add real and imaginary part of the variable (1), we will calculate the known indicator – gross profit:

$$Q = C + G. \tag{2}$$

Any complex number can be presented in trigonometric or exponential form. For model (1) we have:

$$Z = C + iG = R(\cos\theta + i\sin\theta) = Re^{i\theta}. \tag{3}$$

As it is easy to see, the tangent of vectorial angle θ of this complex number represents the profit margin as:

$$\frac{G}{C} = r = \operatorname{tg}\theta. \tag{4}$$

And the modulus of this complex number adds a new characteristic, which can be called the 'scale' of production:

$$R = \sqrt{C^2 + G^2}. \tag{5}$$

This means that by modeling the behavior of only one complex variable the researcher can simultaneously take into account and use a number of additional indicators derived from them. In the case under review we can use as many as four crucial economic indicators: gross profit G , operating expenses C , gross output Q and profit margin r .

Therefore, even simple representation of economic indicators and factors in the form of the complex number (1) opens a lot of new prospects for the researcher and economic and mathematical modeling. However, mathematical operations with the complex number provide a nontrivial result for the operations with real numbers. Using these new mathematical tools for economics leads to the growth of an economic modeling toolbox because, unlike real variable models, complex-valued models describe the interrelation between variables differently. It is often easier to describe very complex interrelations between real variables by means of models and methods of the theory of functions of complex variables rather than using real variable models.

A basic complex-valued model represents functional relationship between one complex variable $y_r + iy_i$ and another complex variable $x_r + ix_i$:

$$y_r + iy_i = f(x_r + ix_i). \tag{6}$$

Since any complex variable can be presented as a graphic point on the plane of Cartesian coordinate system, equality (6) means that one point on the complex variables plane x is assigned to another point (in some cases several points) on the complex variables plane y .

The basic model (6) in accordance with the qualities of complex numbers can be represented as the system of two equalities of real variables:

$$\begin{cases} y_r = f_r(x_r; x_i) \\ y_i = f_i(x_r; x_i) \end{cases}. \tag{7}$$

For example, an elementary linear complex-valued function

$$y_r + iy_i = (a_0 + ia_1)(x_r + ix_i). \quad (8)$$

can be presented as the system of two equalities:

$$\begin{cases} y_r = a_0x_r - a_1x_i \\ y_i = a_0x_i - a_1x_r \end{cases} \quad (9)$$

Hence it follows that the changing of one factor of complex variable x results in the changing of both real and imaginary parts of complex variable y . This means that functions of complex variables are considered multifactor by convention.

A special case of basic model (6) is the complex argument model when only the real part is found on the left of the equality:

$$y_r = f(x_r + ix_i). \quad (10)$$

In general terms this model can be represented as the complex-valued function with the imaginary part equal to zero:

$$y_r + i0 = f(x_r + ix_i). \quad (11)$$

To solve a great number of economic problems, the function inverse to function (10) is of interest:

$$x_r + ix_i = f(y). \quad (12)$$

This is the complex-valued function with the actual argument. A simple example of this function can be the function of the form:

$$x_r + ix_i = y^{(a_0+ia_1)}. \quad (13)$$

In this case the changing of one variable y determines the simultaneous changing of two variables x_r and x_i . When modeling such situation by using real variable models, it is necessary to use the system of two equations. Compact character of modeling complex processes is a distinct advantage of the complex-valued models, however, not the only one.

Simple complex-valued models have very complex alternatives in the field of real variables. These functions of real variables often prove so complex that there is no point in putting them in practice. Moreover, when some mathematical operations are carried out with two real variables, only these two variables are involved. However, when similar operations are done with two complex numbers, for example, multiplying one complex number by another complex number, as many as four real numbers are involved in this mathematical operation.

To make it clear, it should be admitted that everything mentioned above does not imply the following: mathematical operations to be carried out with complex variables are better than the operations done with real variables, and the complex-valued models are better than the real variable models. Nowise! Everything presented above should be interpreted in the following way: mathematical operations with complex economic variables provide *different* results, and mathematical models of complex economic variables model *different* economic processes. In some cases the complex-valued models will better describe economic processes than the real variable ones, whereas in other cases they will work worse.

However, the presenting of two economic indicators in the form of the complex number, as it was demonstrated with the output (1), does provide economists with the opportunity

to use the theory of functions of complex variables for economic modeling.

Variance of a complex variable

Nowadays mathematical statistics deals with variance of independent constituents – real and imaginary parts – rather than variance of a complex variable in general. In this case statistical characteristics of the complex random variable are regarded as real values. To calculate the real characteristics of the complex random value, this value is multiplied by the complex conjugate. This procedure, as is known, allows to find out the real characteristic of the complex number. The variance of a complex variable is presented as the mathematical expectation of the squared absolute value of the corresponding centered variable [3; 14; 19; 23]:

$$\begin{aligned} D(z) &= M[|z|] = M[|x_r + ix_i|] = \\ &= M[(x_r + x_i)(x_r - x_i)] = \quad , \quad (14) \\ &= M[x_r^2] + M[x_i^2], \end{aligned}$$

where

$$M[x_r^2] = M[(x_r - \bar{x}_r)^2] = D(x_r), \quad (15)$$

$$M[x_i^2] = M[(x_i - \bar{x}_i)^2] = D(x_i). \quad (16)$$

Viz:

$$D(z) = D(x_r) + D(x_i). \quad (17)$$

However, such interpretation of the complex variable imposes the limits to statistical data treatment. Let us illustrate it by determining the correlations between two complex random variables. Actually, we can calculate the pair correlation coefficient of the variables by using the correlation moment and the variance:

$$r_{XY} = \frac{\mu_{XY}}{\sigma_x \sigma_y}. \quad (18)$$

Let us take this approach to calculate the correlation between the complex random variables. The correlation moment is represented as a real value using one of the variable in the conjugate form, whereas the variance is calculated as real characteristics in accordance with (14) [13; 14]. It should be noted that the correlation moment that is calculated as

$$\mu_{XY} = M[(x_r + ix_i)(y_r - iy_i)] \quad (19)$$

is not the real number. It is the complex number, since when multiplying and grouping the summands, we obtain:

$$\begin{aligned} \mu_{XY} &= M[x_r y_r] + M[x_i y_i] + i(M[x_i y_r] - M[x_r y_i]) = \\ &\mu_{x_r y_r} + \mu_{x_i y_i} + i(\mu_{x_i y_r} - \mu_{x_r y_i}). \end{aligned} \quad (20)$$

And only in the case when $z_x = z_y$ the latter summand (20) with the imaginary part equals zero, and the correlation moment becomes a real number. In all other cases the correlation moment will be of complex type, thus the pair correlation coefficient between two complex random variables will be the complex value. Keeping this in mind, scientists claim that they calculate the absolute value of the correlation moment, i.e. instead of (19) they use $\text{Re}(\mu_{XY})$ [13].

However, let us find the pair correlation coefficient by using (19).

For the sake of simplicity, we assume that the discrete sequence of the complex random value z centered in relation to its arithmetic mean is under review, hence:

$$z - \bar{z} = x_r - \bar{x}_r + i(x_i - \bar{x}_i) \Leftrightarrow z = x_r + ix_i. \quad (21)$$

Sample value of the correlation coefficient (18) when using the real-valued variance (14) and the correlation moment (19) is of the form:

$$r = \frac{\mu_{XY}}{\sigma_X \sigma_Y} = \frac{\sum(y_i y_r + x_i x_r) + i(\sum(x_r y_i - y_r x_i))}{\sqrt{\sum(y_r^2 + y_i^2) \sum(x_r^2 + x_i^2)}}. \quad (22)$$

On the other hand, we should keep in mind that the pair correlation coefficient as related to real numbers was suggested in the 90s of the 19th century by K. Pearson to estimate linear interrelation of complex variables. It was defined as the geometric mean of regressions y by x and x by y [15]:

$$r = \pm \sqrt{a_1 b_1}, \quad (23)$$

where the proportionality factors of simple regressions a_1 and b_1 are found by using the least-squares method.

To find the formula for calculating the sample value of the pair correlation coefficient for the case of the two complex variables using K. Pearson's approach (23), we will handle that variant of the least-squares method which results from the assumption about the real character of the variance of a complex variable [22]. The complex regression coefficient of linear relationship between complex variable Y and other complex variable X by using this approach to the least-squares method will be calculated in the following way:

$$a = \frac{\sum(y_r + iy_i)(x_r - ix_i)}{\sum(x_r + ix_i)(x_r - ix_i)} = \frac{\sum(y_r + iy_i)(x_r - ix_i)}{\sum(x_r^2 + x_i^2)}. \quad (24)$$

The inverse relationship between complex variable X and the other complex random variable Y , represented in the linear form, has the following formula for calculating the complex regression coefficient found by using the least-squares method:

$$b = \frac{\sum(x_r + ix_i)(y_r - iy_i)}{\sum(y_r + iy_i)(y_r - iy_i)} = \frac{\sum(x_r + ix_i)(y_r - iy_i)}{\sum(y_r^2 + y_i^2)}. \quad (25)$$

By using these formulae of estimating the sample values of proportionality coefficients of regression lines Y by X and X by Y in (23), we obtain the formula for calculating the sample value of the complex pair correlation coefficient:

$$r = \sqrt{a_1 b_1} = \frac{\sum(x_r y_r + x_i y_i) + i \sum(x_i y_r - x_r y_i)}{\sqrt{\sum(y_r^2 + y_i^2) \sum(x_r^2 + x_i^2)}}. \quad (26)$$

Now let us compare formula (22) with formula (26). This should be one and the same coefficient, which is calculated based on the same basic prerequisites. However, if the denominators of formulae (21) and (26) coincide, their numerators differ fundamentally from each other. These are different formulae that are used to calculate different coefficients, and these

coefficients will give different values for one and the same sequence. This is why it seems unclear: whether we should use formula (22) or formula (26), or none of these formulae can be used? The obtained result is contradictory, and it does not allow scientists to form the body of complex correlations.

Even if we agree with the suggestions made by the followers of the conception of the real character of the complex variance and use their real parts [13] instead of complex characteristics, the conflict will not be solved.

In reality, for (22) we have:

$$\begin{aligned} \operatorname{Re}\left(\frac{\mu_{XY}}{\sigma_X \sigma_Y}\right) &= \operatorname{Re}\left(\frac{\sum(y_i y_r + x_i x_r) + i(\sum(x_r y_i - y_r x_i))}{\sqrt{\sum(y_r^2 + y_i^2) \sum(x_r^2 + x_i^2)}}\right) = \\ &= \frac{\sum(y_i y_r + x_i x_r)}{\sqrt{\sum(y_r^2 + y_i^2) \sum(x_r^2 + x_i^2)}}, \end{aligned} \quad (27)$$

and for (26) we obtain:

$$\begin{aligned} \operatorname{Re}(\sqrt{a_1 b_1}) &= \operatorname{Re}\left(\frac{\sum(x_r y_r + x_i y_i) + i \sum(x_i y_r - x_r y_i)}{\sqrt{\sum(y_r^2 + y_i^2) \sum(x_r^2 + x_i^2)}}\right) = \\ &= \frac{\sum(x_r y_r + x_i y_i)}{\sqrt{\sum(y_r^2 + y_i^2) \sum(x_r^2 + x_i^2)}}. \end{aligned} \quad (28)$$

As can be seen from two results compared, different formulae and a contradictory result are obtained again. This is why P. Schreier and L. Scharf note that so far the research carried out in the area of the correlation of complex random variables has produced deplorable results [17].

Exactly the same problems arise in the field of statistical hypotheses and other branches of mathematical statistics of complex variables, which to a certain extent rely upon an important variability measure – variance. Since economists come across the problem of interrelation (direct or indirect) between the factor and the indicator when considering some object or phenomenon, the assumption that the variances of the real and imaginary parts are not interrelated is rarely met. Thus, the hypothesis saying that the variance of the complex random value should always be real cannot be taken as a **basic one in econometrics**. On the contrary, the variance of an economic complex random variable should be presented as a complex characteristic of the variability of a random complex sequence.

Then the complex variance of the complex random value can be represented in the following way:

$$D_c(z) = M[z^2] = M[|z|e^{i2\theta}] = M[|z|\cos 2\theta] + iM[|z|\sin 2\theta], \quad (29)$$

where $\theta = \operatorname{arctg} \frac{x_i}{x_r}, k = 0, 1, 2, \dots$.

As will readily be observed, variance (14) is a special case of variance (29), namely – when vectorial angle θ between the real part and imaginary part of the complex variable is equal to $\theta = \pi k, k = 0, 1, 2, \dots$, i.e. the real part and imaginary part are not interdependent.

How can the assumed hypothesis suggesting that the relationship between the real and imaginary parts, and that the variance of the complex variable should be considered as the complex value, help in solving applied econometric problems? To answer this question, let us turn back to the calculation of correlations between complex random values using two methods, which resulted in an impasse, if we assume that the variance of the complex variable is a real number.

We shall consider all the characteristics of the complex random value as complex numbers. This is why we shall not resort to their artificial transformation into real numbers of these characteristics by multiplying the complex number by its conjugate. Let us represent the correlation moment of two random complex variables as a complex number:

$$\begin{aligned} \mu_{XY} &= M[x_r y_r] - M[x_i y_i] + i(M[x_i y_r] + M[x_r y_i]) = \\ &= \mu_{x_r y_r} - \mu_{x_i y_i} + i(\mu_{x_i y_r} + \mu_{x_r y_i}). \end{aligned} \quad (30)$$

If we apply the values of complex variance (27) and complex correlation moment (30) to the formula for calculating pair correlation coefficient (18), we obtain:

$$\begin{aligned} r_{XY} &= \frac{\mu_{XY}}{\sigma_X \sigma_Y} = \frac{\sum(x_r + ix_i)(y_r + iy_i)}{\sqrt{\sum(x_r + ix_i)^2 \sum(y_r + iy_i)^2}} = \\ &= \frac{\sum(x_r y_r - x_i y_i) + i \sum(x_r y_i + x_i y_r)}{\sqrt{\sum(x_r + ix_i)^2 \sum(y_r + iy_i)^2}}. \end{aligned} \quad (31)$$

The obtained formula for the calculation of the sample value of the pair correlation coefficient of two random variables (20) does not coincide with any of the previously derived formulae (22) and (26), when all the characteristics were considered to be real ones.

We shall calculate the complex pair correlation coefficient using the second method — as the geometric mean of sample values of the regression coefficients. In order to find this coefficient, let us formulate the criterion of the least-squares method. It is to be recalled that assuming that the variance of the random complex variable is a real value, the least-squares method means in fact searching for such regression coefficients whereby:

$$M[(\epsilon_r + i\epsilon_i)(\epsilon_r - i\epsilon_i)] = M[\epsilon_r^2 + \epsilon_i^2] \rightarrow \min. \quad (32)$$

where $\epsilon_r = y_r - \hat{y}_r$, $\epsilon_i = y_i - \hat{y}_i$ are approximation errors.

In the case of the complex variance of the complex random value, the least-squares method is reduced to searching for other coefficients whereby [20, p. 83]:

$$M[(\epsilon_r + i\epsilon_i)^2] = M[\epsilon_r^2 + i2\epsilon_r \epsilon_i - \epsilon_i^2] \rightarrow \min. \quad (33)$$

Here one can pay attention to the interrelation between criteria (32) and (33). In view of this let us present the complex approximation error in the exponential form:

$$\epsilon_r + i\epsilon_i = Re^{\theta} = \sqrt{\epsilon_r^2 + \epsilon_i^2} e^{i \arctg \frac{\epsilon_i}{\epsilon_r}}. \quad (34)$$

With regard to this, criterion (32) takes the form:

$$M[(\epsilon_r + i\epsilon_i)(\epsilon_r - i\epsilon_i)] = M[R^2] \rightarrow \min, \quad (35)$$

and criterion (33) is as follows:

$$M[(\epsilon_r + i\epsilon_i)^2] = M[R^2 e^{i2\theta}] \rightarrow \min. \quad (36)$$

Therefore, criterion of the least-squares method (32) suggested by G. N. Tavares and L. M. Tavares, is a special case of criterion (33) — when the vectorial angle of the complex approximation error is equal to zero.

Now, using criterion (33) in relation to the complex regression coefficient of complex number X by complex number Y denoted as a , using the least-squares method with criterion (33) we obtain the following formula [20, p. 103 – 112]:

$$a = \frac{\sum(x_r + ix_i)(y_r + iy_i)}{\sum(x_r + ix_i)(x_r + ix_i)}. \quad (37)$$

The complex coefficient of proportionality b of the inverse regression can also be calculated by using criterion (33) as:

$$b = \frac{\sum(x_r + ix_i)(y_r + iy_i)}{\sum(y_r + iy_i)(y_r + iy_i)}. \quad (38)$$

Now when plugging these coefficients in the formula for calculating the pair correlation coefficient (23), we obtain:

$$\begin{aligned} r_{XY} &= \sqrt{a_1 b_1} = \frac{\sum(x_r + ix_i)(y_r + iy_i)}{\sqrt{\sum(x_r + ix_i)^2 \sum(y_r + iy_i)^2}} = \\ &= \frac{\sum(x_r y_r - x_i y_i) + i \sum(x_r y_i + x_i y_r)}{\sqrt{\sum(x_r + ix_i)^2 \sum(y_r + iy_i)^2}}. \end{aligned} \quad (39)$$

As can be seen, the same formula of the complex pair correlation coefficient (31) as in the case of its calculation through the complex correlation moment (29) is obtained.

This means that the obtained result is not contradictory. Both pair correlation coefficient (31) between two random complex variables calculated through variance and correlation moment and the pair correlation coefficient calculated through the geometric mean of linear regression have one and the same form.

This in turn means that our hypothesis about the need to use complex variance and other complex characteristics of complex variables in statistics of complex random variables, is confirmed.

Now let us turn to the analysis of the properties of complex variance of complex random value (27), the use of which in econometrics of complex random variables has just been justified. To illustrate it, let us write complex variance in the arithmetic form:

$$D_c(z) = M[z^2] = M[x_r^2] - M[x_i^2] + i2M[x_r x_i]. \quad (40)$$

Complex variance, depending on the form of its real and imaginary parts, can be a complex, real, or imaginary value — the variety of its values correspond to the variety of the properties of the complex random variable. In addition, complex variance can be both positive and negative. Let us consider these options and properties of the complex random value sequence, for which these options of complex variance are valid.

Firstly, let us pay attention to the imaginary part of complex variance (40):

$$\text{Im}[D_c(z)] = 2M[x_r x_i]. \quad (41)$$

It has a simple meaning — it is a double covariance between the real and imaginary parts of the random complex variable. If there is no correlation between variables, the variable covariance is equal to zero. This means that the imaginary part of the complex variance serves the basis for the assumption about the presence or absence of correlation between the real and imaginary parts of the random complex variable.

The real part of the complex variance of the random complex value is also meaningful for the researcher:

$$\text{Re}[D_c(z)] = M[x_r^2] - M[x_i^2]. \quad (42)$$

As can be seen, it characterizes the degree of distinction between the variance of the real part of a random complex variable and the variance of the imaginary part of the given variable. This is why in case when both types of variance are equal to each other, the real part of the complex variance is equal to zero. If the variance of the real part of the complex variable is larger than that of the imaginary part of the complex variable, real part (40) of the complex variance will be positive. Otherwise, it will be negative.

It is noteworthy that the justification of the complex character of the complex random value does not refuse the possibility to use variance in real form — it can be applied as an additional characteristic of the process under research, since the real variance characterizes the variability measure of the absolute value of the complex variance.

Results. We shall illustrate how the major characteristics of the complex random value can be determined by using the suggested method and procedure for estimating the complex variance of the complex random value. To compare, we shall use simultaneously the method and the procedure which result from the assumption about the autonomy of the real and imaginary parts of the complex variable and about real character of the variance.

To exemplify, we shall use the data provided by the Central Bank of Russia on the rate of two currencies — the euro and the US dollar — to the rouble for the period from 2 January 2017 to 13 April 2017. These data are in open access on the website of the Central Bank of Russia: http://www.cbr.ru/currency_base/daily.aspx.

We shall consider the complex random variable of the currencies. We shall relate the euro value in roubles (y_{rt}) to the real part of this variable, and the US dollar value in roubles (y_{it}) to its imaginary part:

$$Z_t = y_{rt} + iy_{it}.$$

The arithmetical mean of this complex variable for the period under review is equal to:

$$(\bar{L} + i\bar{K}) = 62,25 + i58,35. \quad (43)$$

The sample value of the real variance of this complex sequence for the given observation period as the real variability measure is calculated by using formula (14). We obtain:

$$\sigma_z^2 = 3.29; \quad \sigma = 1.81. \quad (44)$$

It is hardly possible to understand what variance (44) and its standard deviation mean as one cannot compare the real

number with the complex one. We can obtain the information about the sequence variability based on the variance of each of its constituents:

$$\sigma_r^2 = 1.84; \quad \sigma = 1.36; \quad \sigma_i^2 = 1.49; \quad \sigma = 1.22. \quad (45)$$

Now let us calculate the sample value of the complex variance of the considered complex value by using formula (40):

$$\sigma_{complex}^2 = 0.35 + i3.09. \quad (46)$$

Based on the obtained results, we can make certain conclusions about the character of the sequence under review. Firstly, since the real part of the complex variable is positive, we can conclude that the variance of the imaginary part (the rouble in relation to the dollar) for the observed period is smaller than that of its real part (the rouble in relation to the euro). This can be confirmed by comparing the variance of the real and imaginary parts of the considered variable, which were calculated above (45).

Since the imaginary part of the complex variance is not equal to zero and is quite a significant value, we can make a conclusion that there is a certain interrelation between the real and the imaginary parts of the complex variable. Thus, the assumption about their independence from each other, as it is a priori done today in modern mathematical statistics, seems to be wrong.

In order to estimate the variability of the analyzed complex variable, let us calculate the value of the standard deviation of the given sequence as the square root of the complex variance (46):

$$\sigma_{complex} = 1.32 + i1.18. \quad (47)$$

By simply comparing complex standard deviation (47) with standard deviations of the real and imaginary parts (45), we can notice that they differ from each other.

Now we can form confidence limits for the complex variable under discussion. First, let us assume that the real and imaginary parts of the complex variable are independent from each other. Then we shall use standard deviations of the real and imaginary parts (45). Since we have 65 observations at our disposal, with the confidence level for t -statistics (Student statistics) at the level of 5 %, we shall obtain $t=1.996$. For the real part we obtain:

$$61.91 < y_r < 62.58. \quad (48)$$

Similarly, we can find the confidence limits for the imaginary part:

$$58.00 < y_i < 58.70. \quad (49)$$

Or in the complex-valued form:

$$61.91 + i58.00 < (y_r + iy_i) < 62.58 + i58.70. \quad (50)$$

Now let us calculate the confidence limits for the complex random variable by using the complex variance or to be more exact — the standard deviation (47). In general, confidence limits can be estimated in the following way:

$$\begin{aligned} (\bar{y}_r + i\bar{y}_i) - t_{\alpha, N} \frac{\sigma_{complex}}{\sqrt{n}} < (y_r + iy_i) < \\ < (\bar{y}_r + i\bar{y}_i) + t_{\alpha, N} \frac{\sigma_{complex}}{\sqrt{n}}. \end{aligned} \quad (51)$$

For the sequence under review we obtain:

$$62.08 + i58.21 < (y_r + iy_i) < 62.41 + i58.50. \quad (52)$$

It can be seen by comparing (48) and (49) with (51) that the found confidence limits are close to each other but still differ from each other. The first variation of the confidence limits is formed by assuming that such interrelation does exist. However, since from (46) it follows that the real and imaginary parts of the considered complex variable correlate to each other, it seems more reasonable to use complex variance and complex standard deviation for estimating confidence limits rather than their real analogue.

Discussion. The prevailing in mathematical statistics assumption on the autonomous character of the real and imaginary parts of the complex random value limited the toolbox of modern econometrics of complex variables. If further development of statistical methods for the treatment of sequences of complex variables seemed impossible without the prerequisite on the independence of the real and imaginary parts of these variables, turning to another idea — about possible correlation between them — allows to elaborate this toolbox. Introducing the complex variance into scientific discourse allows to improve the tools of mathematical statistics of complex random variables, first of all, those ones that are widely used in solving various econometric problems — correlation and regression analysis.

First and foremost, it is necessary to point out the use of the least-squares method for estimating the parameters of the economic model of the complex random variable. Formal grounds for this method (33) have been obtained by using the complex variance (29), but the qualities of the least-squares method like efficiency, unbiasedness and consistency require further research. Considering the application of the least-squares method under conditions when distribution of at least one element (real or imaginary) of the complex random value differs from normal distribution, is of special interest.

Obtaining consistent results for developing the formula of the complex pair correlation coefficient (31) allows to focus on the qualities of this coefficient. It seems clear that in case when two random complex variables are in linear function relationship, the complex pair correlation coefficient turns into a real number equal to one in absolute value. All the rest cases of correlation between random complex variables require thorough research. In particular, the meaning of the imaginary part of the complex pair correlation coefficient is to be considered.

Developing the approach on determining the confidence limits of statistical estimates of complex variables (50) is of great importance. The article deals with the simplest case of estimating the confidence limits of the sample value of the arithmetical mean of a complex random value, both parts of which are supposed to be normally distributed. Studying the cases when distribution of the real and imaginary parts is not identical is of current interest. For example, the real part is distributed according to the normal distribution law, and the imaginary one is of lognormal distribution. Extending this approach for sample values of coefficient estimates of econometric models have yet to be done.

Considering the situation when there can be random interrelation between the real and imaginary parts of the complex random value significantly increased the chances of elaborating

the tools of complex-valued econometrics. The paper illustrated these chances by using the basic characteristic — complex variance of the random complex variable (29). As it can be seen from the obtained results, such approach opens up additional opportunities for economists to study such a complex object as economics.

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