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SECTION 1. Theoretical research in mathematics

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ON F-PRIME RINGS AND THEIR F-RINGS OF QUOTIENTS

Abstract: All rings are associative in what follows. We will also assume that all rings are unital and all ring homomorphisms preserve the identity. Throughout the paper, B stands for a unitary associative ring and f is an automorphism of B . The main purpose of the work is to describe the left f -ring of quotients $Q_f(B)$ of B . Our methods is not connected with right flat epimorphic hull, but we use a fruitful construction based on a direct limit to build the f -ring of quotients. The principal results to be given are Theorems 1 and 2 below. Theorem 1 establishes that $Q_f(B)$ is embeddable in the complete left ring of quotients $Q_{\max}(B)$ and the ring B is a subring of $Q_f(B)$. Theorem 2 asserts that the centres of the left and right f -ring of quotients rings coincides. The authors have intention to continue this study in their subsequent articles. Therefore, at the end of the paper, we formulate a hypothesis, for the proof of which we need the results of this work.

Key words: associative rings; f -rings of quotients; f -prime rings.

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Introduction

All rings in this paper are associative with multiplicative identity and all ring homomorphisms are assumed preserve the identity.

Definition 1. Let R be a ring and F be an injective ring endomorphism of R .

Let N be a subset of R . We say that N is an F -subset if $F^{-1}(N) = N$. Similarly, we shall say that an ideal N of R is an F -ideal if N is an F -subset (see [3], [4], [6], [14]). A ring R is said to be F -prime if the product of any two nonzero F -ideals of R is nonzero. In other words, ring R is F -prime if the product of any two F -ideals P and N is equal to the zero ideal if and only if either $P = 0$ or $N = 0$.

Lemma 1. Let A be a ring and f be an automorphism of A . Then the following conditions (1)–(5) are equivalent:

- (1) A is f -prime;
- (2) for any two left f -ideals I and J of A , the equality $IJ = 0$ implies that either $I = 0$ or $J = 0$;
- (3) $a \sum_{i \in \mathbb{Z}} Af^i(b) \neq 0$ for every $a, b \in A$;
- (4) $r_A(I) = 0$ for every nonzero f -ideals I of A ;
- (5) $\ell_A(I) = 0$ for every nonzero f -ideals I of A .

Materials and Methods

Throughout the sequel, B will denote a ring, f will stand for an automorphism of B and we will always assume that the ring B is f -prime. We denote by $\Phi(B)$ the set of all nonzero f -ideals of B .

Despite the fact that rings of rings for along time are classical objects of study, the rings of quotients have been actively studied recently (see,



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for example, [1], [2]). Now we define the left f -ring of quotients of the ring B .

We denote by M the set of all the pair of the form (I, α) where $I \in \Phi(B)$ and $\alpha: I \rightarrow B$ is a homomorphism of left B -modules. Since B is f -prime, we have $0 \neq IJ \subseteq I \cap J$ for all $I, J \in \Phi(B)$. Определим отношение эквивалентности θ на множестве M так: положим $(I, \alpha) \sim (J, \beta)$, если существует ненулевой f -идеал $K \subseteq I \cap J$ такой, что $\alpha(x) = \beta(x)$ для всех $x \in K$. Let us define the equivalence relation θ on the set M as follows: we put $(I, \alpha) \sim (J, \beta)$ if there exists a nonzero f -ideal $K \subseteq I \cap J$ such that $\alpha(x) = \beta(x)$ for all $x \in K$. We denote by $[I, \alpha]$ the equivalence class containing the pair (I, α) . Let $Q_f(B) = M/\theta$ (the set of all equivalence classes)

The set $Q_f(B)$ turn into a ring if we define on it the following operations:

$$\begin{aligned} [I, \alpha] + [J, \beta] &= [I \cap J, \alpha + \beta], \\ -[I, \alpha] &= [I, -\alpha], \\ [I, \alpha] \cdot [J, \beta] &= [I \cap J, \alpha \circ \beta]. \end{aligned}$$

Definition 2. The ring $Q_f(B)$ described above is called the left (Martindale) f -ring of quotients of B .

Remark. One can formalize the construction of $Q_f(B)$ as the direct limit

$$Q_f(B) = \varinjlim (\text{Hom}_B({}_B I, {}_B B) : I \in \Phi(B)).$$

If every ideal of B is an f -ideal, then $Q_f(B)$ coincides with the left ring of quotients in the sense of Martindale.

We define analogously the right f -ring of quotients of B (in the sense of Martindale) as follows:

$$Q_f^r(B) = \varinjlim (\text{Hom}_B(I_B, B_B) : I \in \Phi(B)).$$

Lemma 2. Let $I \in \Phi(B)$, J be a left ideal of B and $\alpha, \beta: I \rightarrow B$ be a homomorphism of left B -modules. Suppose that $\alpha(x) = \beta(x)$ for all $x \in I \cap J$. Then $\alpha \equiv \beta$.

Proof. Take any $b \in J$. If $a \in I$ then $ab \in I \cap J$ and $\alpha(ab) = \beta(ab)$. Hence, $a(\alpha(b) - \beta(b)) = 0$ for all $a \in I$. But $r_A(I) = 0$ by Lemma 1. It follows that $\alpha(b) = \beta(b)$ for all $b \in J$. QED.

The above f -ring of quotients $Q_f(B)$ possesses some properties of the usual Martindale ring of quotients. In particular, we have the following theorem.

Theorem 1. The f -ring of quotients $Q_f(B)$ is embeddable in the maximal left ring of quotients $Q_{max}(B)$ and the ring B is a subring of the ring $Q_f(B)$.

Proof. If $I \in \Phi(B)$ then $r_B(I) = 0$ by Lemma 1 and, hence, I is a dense right ideal of B . Therefore, to each element $q = [I, \alpha] \in Q_f(B)$ can be match up

with the element $\varphi(q) \in Q_{max}(B)$ such that $x\varphi(q) = \alpha(x)$ for all elements $x \in I$.

One can easily verify that the mapping $q \mapsto \varphi(q)$ is a correctly defined injective ring homomorphism. Furthermore, according to the construction $\varphi(Q_f(B)) = \{q \in Q_f(B) : \text{there exists an ideal } I \in \Phi(B) \text{ such that } Iq \subseteq B\}$. Each element $a \in B$ defines the homomorphism of left B -modules $\alpha_a: x \mapsto xa$ ($x \in B$).

Since $B \in \Phi(B)$, the equivalence class $[B, \alpha_a]$ belongs to $Q_f(B)$. This gives a natural embedding $B \subseteq Q_f(B)$. QED.

In what follows we identify the rings $Q_f(B)$ and $\varphi(Q_f(B))$.

Let $q \in Q_f(B)$. We denote by $J(q)$ the sum of all f -ideals I of B such that $Iq \subseteq B$. Observe that $J(q)$ itself is also an f -ideal and $J(q) \cdot q \subseteq B$. Таким образом, $J(q)$ – наибольший идеал среди f -идеалов I кольца B , обладающих свойством $Iq \subseteq B$.

Thus, $J(q)$ is the largest ideal among f -ideals I of B having the property $Iq \subseteq B$.

If $q \in Q_{max}(B)$ then we set $D(q) = \{a \in Q_{max}(B) : aq \in B\}$. As is well known in the theory of rings of quotients, the automorphism f can be uniquely extended to an automorphism of $Q_{max}(B)$. We will denote this extension by the same symbol f . Since $f(D(q)) \cdot f(q) = f(D(q)q) \in f(B) = B$ and $f^{-1}(D(f(q))) \cdot q = f^{-1}(D(f(q)) \cdot f(q)) \in f^{-1}(B) = B$, we get $f(D(q)) = D(f(q))$.

Let us consider any element $q \in Q_f(B)$ and note the following: if $x \in J(q)$ then both $f(x)$ and $f^{-1}(x)$ lie in $J(q)$, and therefore we have that $xf(q) = f(f^{-1}(x)q) \in B$ and $xf^{-1}(q) = f^{-1}(f(x)q) \in B$. It follows that $J(q)f(q) \subseteq B$ and $J(q)f^{-1}(q) \subseteq B$. Consequently, $f(Q_f(B)) \subseteq Q_f(B)$ and $f^{-1}(Q_f(B)) \subseteq Q_f(B)$. Thus, f can be regarded as an automorphism of $Q_f(B)$.

Proposition 1. The f -ring of quotients $Q_f(B)$ introduced above has the following properties.

(1). For any elements $q_1, q_2, \dots, q_n \in Q_f(B)$, there exists an ideal $I \in \Phi(B)$ such that $Iq_i \subseteq B$ for all $i = 1, 2, \dots, n$.

(2). If $Iq = 0$ for some ideal $I \in \Phi(B)$ and some element $q \in Q_f(B)$, then $q = 0$.

(3). If $qI = 0$ for some ideal $I \in \Phi(B)$ and some element $q \in Q_f(B)$, then $q = 0$.

(4). If $I \in \Phi(B)$ and $\gamma: I \rightarrow B$ is a homomorphism of left B -modules, then there is an element $q \in Q_f(B)$ such that $\gamma(x) = xq$ for all elements $x \in I$.

(5). $Q_f(B)$ is an f -prime ring.

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Proof. (1). As the required ideal, we can take the ideal

$$I = J(q_1) \cap J(q_2) \cap \dots \cap J(q_n).$$

It works, because B is an f -prime ring and $0 \neq J(q_1)J(q_2) \cdots J(q_n) \subseteq J(q_1) \cap J(q_2) \cap \dots \cap J(q_n)$.

(2). This follows from Lemma 2 and the definition of the f -ring of quotients.

(3). Observe that $P = \sum_{i \in \mathbb{Z}} J(q) f^i(q)$ is the left f -ideal of B and $PI = \sum_{i \in \mathbb{Z}} J(q) f^i(qI) = 0$. Since is an f -prime ring, then $P = 0$. Therefore, $J(q)q = 0$ and $q = 0$.

(4). This assertion follows immediately from Definition 2.

(5). Let a and b be two non-zero elements of $Q_f(B)$. Then it follows from (2) that $\sum_{i \in \mathbb{Z}} J(a) f^i(a)$ and $\sum_{i \in \mathbb{Z}} J(b) f^i(b)$ are non-zero left f -ideals. Since B is an f -prime ring, we get that $\sum_{i \in \mathbb{Z}} J(a) f^i(a) \cdot \sum_{i \in \mathbb{Z}} J(b) f^i(b) \neq 0$. Hence, it follows that $a \cdot \sum_{i \in \mathbb{Z}} J(b) f^i(b) \neq 0$ and, by Lemma 1, $Q_f(B)$ is an f -prime ring. QED.

We denote by $C(B)$ and $C^r(B)$ the center of the left maximal and right maximal rings of quotients of B , correspondingly. One more piece of notation: $C_f(B) = \{c \in C(B) : f(b) = b\}$ and $C_f^r(B) = \{c \in C^r(B) : f(b) = b\}$.

Theorem 2.

(1). If $q \in C_f(B)$, then the set $D(q) = \{b \in B \mid bq \in B\}$ is an f -ideal.

(2). $C_f(B)$ is the center of $Q_f(B)$.

(3). If $q = [I, \alpha] \in Q_f(B)$, then $q \in C_f(B)$ if and only if α is a homomorphism of bimodules over the ring B .

(4). The rings $C_f(B)$ and $C_f^r(B)$ are isomorphic.

Proof. (1). Let $a \in B$. Then $qa \in B$ if and only if $f(qa) = f(a)q \in B$. Therefore, $D(q) \in \Phi(B)$.

Assertion (2) follows from assertion (1) since $C_f(B)$ is the centralizer of the set B in $Q_{max}(B)$.

(3). If α is a homomorphism of bimodules over the ring B , then, as is easily seen, $q \in C(B)$ and therefore $q \in C_f(B)$. The converse follows from assertion (1).

(4). Using assertion (3), one can prove that both rings $C_f(B)$ and $C_f^r(B)$ are isomorphic to the ring $\{c \in \lim_{\rightarrow} (\text{Hom}_B({}_B I_B, {}_B B_B) : I \in \Phi(B)) \mid f(c) = c\}$. QED.

Theorem 3. Let B be an f -prime ring and $0 \neq q \in Q_f(B)$. If $q f(a) = aq$ for all elements $a \in B$, then q is an invertible element of the ring $Q_f(B)$ and f is an inner automorphism of the ring $Q_f(B)$ defined by the element q :

$$f(p) = q^{-1} p q \quad (\forall p \in Q_f(B)).$$

Proof. For brevity, we denote by Q the ring of quotients $Q_f(B)$. Let p be an arbitrary element of the ring Q and let k be an integer. Then for each elements $a \in B$, we have a chain of equalities:

$$f^k(q) f(a) = f^k(q f^{1-k}(a)) = f^k(f^{-k}(a) q) = a f^k(q). \quad (1)$$

We recall that $J(p)$ denotes the largest among the f -ideals I of B with the property $I p \subseteq B$. It follows from (1) that for all elements $a \in J(p)$ the following equalities hold:

$$a f^k(q) f(p) = f^k(q) f(a) f(p) = f^k(q) f(ap) = a p f^k(q).$$

$$\text{Hence } J(p)(f^k(q) f(p) - p f^k(q)) = 0.$$

Therefore, $f^k(q) f(p) - p f^k(q) = 0$ by Proposition 1. Consequently,

$$f^k(q) f(p) = p f^k(q) \quad (\forall p \in Q \quad \forall k \in \mathbb{Z}) \quad (2)$$

and in particular,

$$f^k(q) Q = Q f^k(q) \quad (3)$$

The ring Q is f -prime by Proposition 1. Therefore, it follows from Lemma 1 that $a \sum_{i \in \mathbb{Z}} Q f^i(b) \neq 0$ for any two non-zero elements $a, b \in Q$. Hence, $q \sum_{i \in \mathbb{Z}} f^i(q) Q = q \sum_{i \in \mathbb{Z}} Q f^i(q) \neq 0$ by (3), and therefore $q f^k(q) \neq 0$ for some integer k . We set $u = q f^k(q)$ and note that, by virtue of (2), we obtain that

$$p u = p q f^k(q) = q f(p) f^k(q) = q f^k(q) f^2(p) = u f^2(p) \quad (4)$$

for all $p \in Q$. Moreover, equality (1) leads to the invariance of the element u :

$$u = q f^k(q) = f^k(q) f(q) = f(q) f(f^k(q)) = f(u).$$

So, we have proved that

- 1) $uQ = Qu$ (this follows from equality (4));
- 2) $f(u) = u$.

To complete the proof, we need the following lemma.

Lemma 3 ([9, Lemma 1.2]). If an element $u \in Q$ satisfies conditions 1) and 2), then u is an invertible element in the ring Q .

To finish the proof, it remains to note that q also turns out to be an invertible element of the ring Q . Indeed, $q f^k(q) u^{-1} = u u^{-1} = 1$ and $u^{-1} f^{k-1}(q) q = u^{-1} q f^k(q) = u^{-1} u = 1$. It follows from (2) that $q f(p) = p q$ for all elements $p \in Q$. Since q is invertible, $f(p) = q^{-1} p q \quad (\forall p \in Q)$. QED.

Now we formulate an open problem:

Let R be a ring an f -prime ring, and A be the Cohn-Jordan extension of R by means of f (see [10], [11], [12], [14]). The one can prove that A is also f -prime. It would be interesting to clarify the relationship between the f -rings of quotients of rings R and A .



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Conclusion

At the end of the present work, let us make the following hypothesis, which will be verified in the subsequent author's works:

Let R be a ring and F be an injective endomorphism of R . Suppose that the skew polynomial ring $R[x, F]$ is semiprime. Let (A, f) be the Cohn-Jordan extension of the pair (R, f) . This is extension described in [3], [4], and [5]. We denote by

Q the maximal left ring of quotients of A and denote by \mathfrak{D}_* the orthogonal completion of the center of the ring of oblique Laurent polynomials $Q\langle x, f \rangle$ in the maximal left ring of quotients of $Q\langle x, f \rangle$. Then the extended centroid of the ring $R[x, F]$ is isomorphic to the complete left classical ring of quotients $\mathfrak{D}_*^{-1}\mathfrak{D}_*$.

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