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ON HARDY TYPE SPACES IN SOME DOMAINS IN $\mathbb{C}^n$ AND RELATED PROBLEMS

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We discuss some new problems in several new mixed norm Hardy type spaces in products of bounded pseudoconvex domains with smooth boundary in $\mathbb{C}^n$ and then prove some new sharp decomposition theorems for multifunctional Hardy type spaces in the unit ball and then we show also similar results in pseudoconvex and convex domains of finite type extending previously known assertions obtained by first author earlier in Bergman spaces under certain Poisson integral type condition which vanishes in one functional case. Some new (in particular sharp in the unit ball) embeddings for some new mixed norm Hardy spaces in bounded pseudoconvex domains will be also indicated. Some new extensions of Poisson integral in the unit ball and some new assertions concerning them will be indicated and discussed in product domains. Some related multifunctional results are also given.Some new embedding theorems are also provided in some new mixed norm Hardy spaces in unbounded tubular domains over symmetric cones.

Keywords: pseudoconvex, convex and tubular domains,embedding theorems, Hardy type spaces,Poisson type integral, product domains

1. On mixed norm Hardy type spaces in product domains in $\mathbb{C}^n$

In this section we discuss some problems in open new research area of Hardy type mixed norm spaces in product domains in $\mathbb{C}^n$. Some new spaces will be also defined and some interesting new problems will be posed.

Let $B^m = B\times\ldots\times B$ be the unit polyball and $S = \partial B$ be the unit sphere. We will denote by dV, dV_m the Lebegues measure on B, B^m and by $d\delta$ the normalized Lebegues measure on S, $S^m = S\times\ldots\times S$; $H(B\times\ldots\times B)$ is a space of all analytic functions on B^m. Let $\alpha > 1$.

Let further $\Gamma_\alpha(\xi) = \{z\in B\colon |1-\overline{\xi}z| < \alpha(1-|z|)\}$ be usual Luzin cone (see [6]).

Following [3] for $0 < p < \infty$ we define the analytic tent space $A_\infty H^p$(which is well known for $m = 1$) which consists of analytic functions f on $B\times\ldots\times B$, so that $\|f\|^p_{A_\infty H^p} =$

$$\int\limits_{S^m}\sup_{z_1\in\Gamma_{\alpha_1}(\xi_1)}\ldots\sup_{z_m\in\Gamma_{\alpha_m}(\xi_m)}|f(z_1,\ldots,z_m)|^p d\delta(\vec{\xi}) < \infty; \delta(\vec{\xi}) = \prod_{j=1}^m \delta(\xi_j).$$

Let also

$$M_\infty H^p=\Bigg\{f\in H(B^m):$$
$$:\left(\int\limits_S \sup_{z_1\in\Gamma_\alpha(\xi_1)}\dots\left(\int\limits_S\left(\sup_{z_m\in\Gamma_\alpha(\xi_m)}|f(z_1,\dots,z_m)|^P d\delta(\xi_m)\right)\dots d\delta(\xi_1)\right)\right)^{\frac{1}{p}}<\infty\Bigg\},$$

where $0<p<\infty$, $\alpha_j>1, j=1,\dots,m$. Note some more general $(A_\infty)H^{p_1\cdots p_m}$ type spaces for $p_j\varepsilon(0,\infty), j=1,\dots,m$ can be defined similarly like, for example, spaces with finite quazinorms

$$\left(\int\limits_S \sup_{z_m}\dots\left(\int\limits_S \sup_{z_1}|f|^{p_1}d\delta(\xi_1)\right)^{\frac{p_2}{p_1}}d\delta(\xi_m)\right)^{\frac{1}{p_m}},\ where \sup_{z_1}=\sup_{|z_1|<1}$$

.

Let futher also

$$M_\infty H^{\vec{p}}=\Bigg\{f\in H(B^m):$$
$$:\left(\int\limits_S \sup_{z_1\in\Gamma_\alpha(\xi_1)}\dots\left(\int\limits_S\left(\sup_{z_m\in\Gamma_\alpha(\xi_m)}|f(z_1,\dots,z_m)|^{P_1} d\delta(\xi_m)\right)^{\frac{p_2}{p_1}}\dots d\delta(\xi_1)\right)\right)^{\frac{1}{p_m}}<\infty\Bigg\},$$

where $0<p_j<\infty$, $\alpha_j>1, j=1,\dots,m$.

Similarly $A_\infty H^{\vec{P}}$ spaces can be defined.

These new Hardy type spaces in simpler case of unit disk were introduced recently in [3]. Some sharp embedding theorems were proved for these classes of functions there. We note as it was indicated in [3] these results are valid also in higher dimension namely in the unit ball. We formulate unit ball versions of some results from [3] below. Then we introduce similar spaces in bounded strictly pseudoconvex domains with smooth boundary and discuss possible extensions of these and other sharp results to such type product domains case also. We note this will be the partial goal of this paper to study such type objects.

We denote various positive constants in this paper by C,C_1,C_2,C,C_α ect.

Theorem A Let $\mu_1,\dots,\mu_m$ be a set of positive Borel measures on B. Let also $0<p_i;q_i<\infty, i=1,\dots,m$ be such that $\sum\limits_{i=1}^{m}\left(\dfrac{q_i}{p_i}\right)=1$.

Then the following conditions are equivalent.

1) $\int\limits_B\dots\int\limits_B\prod\limits_{i=1}^{n}|f_i(z_1,..,z_m)|^{q_i}d\mu_1(z_1)\dots d\mu_m(z_m)\le c\prod\limits_{i=1}^{n}||f_i||^{q_i}_{A_\infty H^{P_i}}$;

2) $\int\limits_B\dots\int\limits_B\prod\limits_{i=1}^{n}|f_i(z_1,\dots,z_m)|^{q_i}d\mu_1(z_1)\dots d\mu_m(z_m)\le\tilde{c}\prod\limits_{i=1}^{n}||f_i||^{q_i}_{M_\infty H^{P_i}}$;

3) $\left(\sup\limits_{a\varepsilon B^m}\right)\int\limits_{B^m}\prod\limits_{k=1}^{m}\dfrac{(1-|a_k|)d\mu_1(z_1)\dots d\mu_m(z_m)}{|(1-\langle a_k,z_k\rangle)|^{n+1}}<\infty.$

Proof of theorem A. For $A_\infty H^P$ spaces we provide simple unit disk proof for $m=2, n=2$ case, the unit ball case can be obtained by repetition of arguments based on same estimates, but in the ball (see [5] for these estimates). The $m>2, n>2, n\in\mathbb{N}, M_\infty H^P$ cases need small changes.

We have for positive Borel μ measure in (see in [5]) $D_1=\{|z|<1\}$ and f function $\int\limits_{D_1}|f(z)|d\mu(z)\le c\int\limits_{T}[C_1(\mu)(\xi)]\times(A_\infty(f))(\xi)dm(\xi)$ (in the ball this is valid also see [5])).

We then apply this by z_1, then by z_2 and then we use Fubini's theorem.

We get as a result the following obvious estimates in product domain $D_1^2=D_1\times D_1$.

$$\int\limits_{D_1^2}|\phi(z_1,z_2)|\times d\mu_1(z_1)d\mu_2(z_2)\le(|\phi|=|\phi_1|^{q_1}|\phi_2|^{q_2})\le$$

$$\le c\left(\sup_{I\subset T}\frac{1}{I}\int\limits_{S(I)}d\mu_1(z)\right)\times\left(\sup_{I\subset T}\frac{1}{I}\int\limits_{S(I)}d\mu_2(z)\right)\times$$

$$\times\left(\int\limits_{T^2}(A_\infty)_{z_1}(A_\infty)_{z_2}(\phi)dm(\xi_1)dm(\xi_2)\right),$$

where $S(I)$ is a usual Carleson box, (see [3]); and $T=\{z:|z|=1\}$, I is an arc and $\Gamma_\alpha(\xi)=\left\{z\in D_1;|1-\vec{\xi}z|<\alpha(1-|z|)\right\}$ and $A_\infty(f)(\xi)=\sup\limits_{z\in\Gamma_\alpha(\xi)}|f(z)|$; $C_1(\mu)(\xi)=$
$=\left(\sup\limits_{\xi\in I}\right)\left(\frac{1}{|I|}\right)\times\left(\int\limits_{S(I)}d\mu(z)\right)$.

The rest easily follows from Holder's inequality. To prove the reverse we put $(f_i)(z)=\left(\frac{(1-|a|)}{(1-\bar{a}z)^2}\right)^{\frac{1}{p_i}}, z\in D, a\in D_1^m; i=1,2,...,m; (1-\bar{a}z)^2=$
$=\prod\limits_{i=1}^{m}(1-a_kz_k)^2; a_k,z_k\in D_1$, and apply standard arguments (see [3]).

Theorem B Let $\mu_1,..,\mu_m$ be a set of positive Borel measures on B. Let also $s>0, p>0, q>0, s+\dfrac{q}{p}=1$. Then the following conditions are equivalent.

1)

$$\left(\sup_{a\varepsilon B^m}\right)\int\limits_B...\int\limits_B|f(z_1,...,z_m)|^q\times\left(\prod_{k=1}^{m}\frac{(1-|a_k|)}{|(1-\langle\bar{a}_k,z_k\rangle)|^{n+1}}\right)^S d\mu_1(z_1)...d\mu_m(z_m)\le$$

$$\le c||f||^q_{A_\infty H^P};$$

2)

$$\left(\sup_{a\varepsilon B^m}\right)\int\limits_B....\int\limits_B|f(z_1,...,z_m)|^q\times\left(\prod_{k=1}^{m}\frac{(1-|a_k|)}{|1-\langle a_k,z_k\rangle|^{n+1}}\right)^S d\mu_1(z_1)...d\mu_m(z_m)\le$$

$$\le c_1||f||^q_{M_\infty H^P};$$

3)

$$\left(\sup_{a\varepsilon B^m}\right)\int\limits_B \dots \int\limits_B \prod_{k=1}^{m} \frac{(1-|a_k|)d\mu_1(z_1)\dots d\mu_m(z_m)}{\prod\limits_{k=1}^{m}(|1-\langle a_k, z_k\rangle|)^{n+1}} < \infty.$$

We refer interested reader to [3] for proof of theorem B in the unit disk, for the case of unit ball proofs are the same based on some results from [5]. Moreover it is interestiing to note two implications 2) => 3) and 1) => 3) in both theorems can be easily similarly extended to more general $M_\infty H^{\vec{P}}$,$A_\infty H^{\vec{P}}$ spaces.

Indeed consider larger spaces of more general form with general quazinorms on

$$X_1 \times \dots \times X_m; L^{\vec{p}}(X) = L^{\vec{p}}(X_1 \times \dots \times X_m); X_i \subset \mathbb{C}^n;$$

$$i = 1, \dots, m; \|f\|_{L^{\vec{p}}} = \|\dots\|f\|_{L^{p_1}(X_1)} \dots\|_{L^{p_m}(X_m)}; 0 < p_i < \infty, i = 1, \dots, m.$$

We note that for any function $f(z_1, \dots, z_m) = \prod\limits_{i=1}^{m}(f_i(z_i)); z_i \in (X_i)$; we have that $\|f\|_{L^{\vec{p}}(X)} = \prod\limits_{i=1}^{m}\|f_i\|_{L^{p_i}(X_i)}$; and then we must only repeat our arguments given in less general $(p_i = p), i = 1, \dots, m$ cases to get one implication (necessary condition on measure).

We further need some known definitions and theorems from the theory of analytic functions in bounded strictly pseudoconvex domains with smooth boundary.
Especially we are interested in Hardy H^p spaces in such domains to pass some classical facts to the case of product domains.

Let D be bounded strictly pseudoconvex domains with smooth boundary, ν_ξ be outer unit normal to the boundary of D, $\partial D = \Gamma, d\xi$ or $d\delta(\xi)$ is a Lebegues measure on ∂D.

Let f be analytic then, $f\varepsilon H^p$ (Hardy space in $\mathbb{D}$); $0 < p < \infty$ if

$$(\overline{\lim_{\varepsilon\to 0}})\int\limits_{\partial D}|f(\xi - \varepsilon\nu_\xi)|^p(d(\xi)) < \infty$$

(with natural extension for product domain mixed norm (quazinorm) case, we define $\widetilde{M}_\infty H^{\vec{P}}, \widetilde{A}_\infty H^{\vec{P}}, p_j > 1, j = 1, \dots m$ function spaces as we did in the ball.

The natural question is to extend these sharp results (theorems A, B) to bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$ and other maybe even unbounded domains, for example, to look at similar Hardy type spaces in tube domains. (see [8])

<u>Theorem C</u> (see [4]) Let D be bounded, $\partial D = \Gamma \in C^2$, then

$$\int\limits_\Gamma \sup_{z\in A_\alpha(\xi)}|f(z)|^p d(\xi) \le c_{p,\alpha}\overline{\lim_{\varepsilon\to 0}}\int\limits_\Gamma |f(\xi - \varepsilon\nu_\xi)|^p d\xi$$

if $\alpha > 0$, where $A_\alpha(\xi) = \left\{z \in D : |(z-\xi, \nu_\xi)| \leqslant (1+\alpha)\delta_\xi(z); |\xi - z|^2 < \alpha\delta_\xi(z)\right\}$, and where $\delta_\xi(z) = (min)\left\{(dist)(z, \partial D), (dist)(z, T_\xi)\right\}$, where $(T_\xi)(\partial D)$ is so called tangent plane region to $\partial D, D\varepsilon\mathbb{C}^n$.

We provide a famous Hormander result, sharp embedding for Hardy class in bounded pseudoconvex domains.

Let $B(z,r)$ be Bergman (or Kobayashi) ball in bounded pseudoconvex domain D. We say μ positive Borel measure is Carleson if $\mu(B(z,r)) \leq c(\delta(z)^n); n \geq 0, z \in D, r > 0$. In theorem D we give another (equivalent) description of such measures.

Theorem D (see [4]). Let μ be positive measure in bounded strictly pseudoconvex domain D. Let

$$\left(\int\limits_{A(\xi,\delta)} d\mu\right) \leqslant C(\delta^n); \xi \in \partial D, \delta > 0, where \tag{1}$$

$(A(\xi,\delta)) = \left\{z \in D : |(z-\xi, v_\xi)| < \delta, |z-\xi| < \sqrt{\delta}\right\}$,where $C > 0$ does not depend on ξ, δ. Then there is a constant $c' > 0$, so that for every $f, f \in H^p_D, p > 0$ the following result is valid

$$\int\limits_{D} |f(z)|^p d\mu(z) \leq c' \left(\int\limits_{\partial D} |f(\xi)|^p d\delta(\xi)\right) \tag{2}$$

and moreover the reverse assertion is also true if (2) is valid then (1) is valid.

It is natural to ask about similar type results for Hardy type mixed norm analytic spaces.

Namely

$$\widetilde{A}_\infty H^{\vec{p}} = \left\{ f \in H(D^m) : \left(\int\limits_{\partial D} \dots \left(\int\limits_{\partial D} |f(\vec{\xi})|^{p_1} d\delta(\xi_1)\right)^{\frac{p_2}{p_1}} \dots d\delta(\xi_m)\right)^{\frac{1}{p_m}} < \infty \right\},$$

and

$$A_\infty H^{\vec{p}} = \left\{ f \in H(D^m) : \left(\int\limits_{\Gamma} \dots \left(\int\limits_{\Gamma} \sup_{\vec{z} \in A_\alpha(\xi)} |f(\vec{z})|^{p_1} d\delta(\xi_1)\right)^{\frac{p_2}{p_1}} \dots d\delta(\xi_m)\right)^{\frac{1}{p_m}} < \infty \right\}$$

function spaces, where $0 < p_i < \infty, i = 1, \dots, m$.

Let $\left(V^r_\xi\right) = \left\{z \in \partial D : |(z-\xi, v_\xi)| < z; |z-\xi| < \sqrt{r}\right\}$, $(vol)(V^r_\xi) \asymp r^n, r > 0$. (see [4])

Theorem D_1 (see [4]) Let D be bounded in $\mathbb{C}^n$ with C^2 boundary.

Then for each $\alpha > 0$ we have the following $\sup\limits_{z \in A_\alpha(\xi)} |u(z)| \leq (C_\alpha)(Mu)(\xi)$; for every $u \in C(\bar{D})$, plurisubharmonic in D, where

$$[((Mf) \cdot (\xi))] = \left(\sup_{r>0}\right) \left(\frac{1}{\delta(V^r_\xi)}\right) \left(\int\limits_{V^r_\xi} (f(x)) d\delta(x)\right);$$

and $\| \sup\limits_{z \in A_\alpha(\xi)} |u(z)| \|_{L^p} \leq c_\alpha \|M(u)(\xi)\|_{L^p} \leq c\|u\|_{L^p}, 1 < p < \infty$.

Other estimates of similar type can be seen for example in [10-12].

It will be nice to extend these assertions to the case of mixed norm multifunctional function spaces on product domains. Finally at the end of this section we introduce new interesting scales of function spaces in bounded pseudoconvex domains with smooth boundary .

The separate study of properties of such spaces is a new interesting problem.

Let $\partial D_\varepsilon = \{z : \rho(z) = \varepsilon\}$, for defining ρ function of $\mathbb{D}$ domain.We denote by $H(D^m)$ the space of all Analytic functions on D^m. Then weighted Hardy spaces can be defined as follows on product of pseudoconvex domains with smooth boundary. (We consider model case of bidomains, the case of polydomains are similar).

$$\left(\widetilde{H}^{\vec{p}}_{\vec{\delta},\vec{k}}\right) = \left\{ f \in H(D^2) : \left(\sup_{z_1 \in (z,z_0)} \sum_{|\alpha| \le k_1} z_1^{\delta_1} \int_{\partial D_1} \sup_{z_2 \in (0,z_0)} \left(\sum_{|\alpha| \le k_2} (z_2^{\delta_2}) \int_{\partial D_2} |D^\alpha f|^{p_1} \, d\sigma_{z_2} \right)^{\frac{p_2}{p_1}} d\sigma_{z_1} \right)^{\frac{1}{p_2}} < \infty \right\}.$$

with natural extension to $m > 2$ case or another scales of weighted Hardy type spaces in bounded pseudoconvex domains with smooth boundary

$$\left(\widetilde{\widetilde{H}}^{\vec{p}}_{\vec{\delta},\vec{k}}\right) = \left\{ f \in H(D^2) : \left(\sup_{z_1,z_2 \in (0,z_0)} \right) \sum_{|\alpha| \le k_1} \sum_{|\alpha| \le k_2} (z_1^{\delta_1} z_2^{\delta_2}) \times \right.$$
$$\left. \times \left(\int_{\partial D_{z_1}} \left(\int_{\partial D_{z_2}} |D^\alpha f|^{p_1} \, d\sigma_{z_1} \right)^{\frac{p_2}{p_1}} d\sigma_{z_2} \right)^{\frac{1}{p_2}} < \infty \right\};$$

$\delta_j \ge 0, j = 1,2, 0 < p_i < \infty, i = 1,2$ (for $m = 1$ for all these were studied in [7]), $f = f(\xi_1, \xi_2)$ (we refer to [7] for D^α operators).

It is classical that Hardy spaces can be characterized via Lusin area integral $A_{\widetilde{\alpha}}(\xi)$ in the unit ball. The natural issue is to define spaces of Hardy type in product of bounded pseudoconvex domain via $A_{\widetilde{\alpha}_1}(\xi_1) \times \ldots \times A_{\widetilde{\alpha}_n}(\xi_n)$ sets, where $A_{\widetilde{\alpha}}(\xi)$ in pseudoconvex domain was specified above (Lusin region and Area integral). These Hardy type spaces we define below are new. Some similar type spaces in such domains studied very recently in [3, 7]. and now we define new scales of Hardy type spaces as follows.

$$H^{\vec{p}}_{\widetilde{\alpha}}(D^m) = \left\{ f \in H(D \times \ldots \times D) : \right.$$
$$\left. \left\| \left(\int_{\vec{A}_{\widetilde{\alpha}_1}(\xi_1)} \ldots \int_{\widetilde{A}_{\widetilde{\alpha}_m}(\xi_m)} (|f(\vec{z})|^p) \left(\prod_{j=1}^m \left(\delta(z_j)^{\alpha_j} dV(z_j) \right) \right) \right)^{\frac{1}{p}} \right\|_{L^q(d\delta_1 \ldots d\delta_m, \partial D \ldots \partial D)} < \infty \right\}$$

$0<p,q<\infty, \alpha_j>(-1), j=1,...,m, \delta(z)=dist(z,\partial D), z\in D, \vec{z}=(z_1,...,z_m)$
or

$$\overline{H}^{\vec{p}}_{\vec{\alpha}}(D^m)=f\in H(D^m):$$
$$:\left(\int\limits_{\partial D}\left(\int\limits_{A_\alpha(\xi)}...\int\limits_{\partial D}\left(\int\limits_{A_\alpha(\xi)}|f(\vec{z})|^{q_1}\delta_1^{\alpha_1}(z)dV(z_1)\right)^{\frac{q_2}{q_1}}...\delta^{\alpha_m}(z_m)dV(z_m)\right)\right)^{\frac{1}{q_m}}<\infty$$

$0<q_i<\infty, i=1,...,m, \alpha_j>-1, j=1,...,m.$

Defining various (mixed norm) spaces of Hardy type on products of bounded pseudoconvex domains with smooth boundary it is natural to show some relations between these scales of analytic function spaces and relations with other spaces in general. This is a very large topic some preliminary results will be provided in this paper. Such type spaces have close relations with so-called multifunctional analytic function spaces (see [2] for example for some recent results in this direction) and theorems A, B and theorems below.

Below in this section we provide some extensions of theorems A and B to pseudoconvex domains with smooth boundary. Our results here are not sharp and some sharper versions still should be probably found.

We first give multifunctional result related with theorem D_1.

Theorem 1 Let $1<p_i<\infty; 0<q_i<\infty; i=1,...,m, 0<S<\infty;$ $\left(\frac{1}{S}\right)=\sum\limits_{i=1}^{m}\left(\frac{q_i}{p_i}\right).$

Let μ be Carleson measure in $\mathbb{D}$ bounded pseudoconvex domain. Then if there is a family Q_z so that $\int\limits_D\left(\sup\limits_{V_\xi^r}\frac{1}{\sigma(V)}\int\limits_{V_\xi^r}|f|d\sigma\right)^P d\mu(z)\le c\|f\|^P_{L^p(S)}$ for all $Q_z\subset V_\xi^r$, $z\in D, 1<p<\infty$.

Then $\left(\int\limits_D\left(\sup\limits_{V_\xi^r}\frac{1}{\sigma(V)}\int\limits_{V_\xi^r}|f_i|^{q_i}d\sigma\right)d\mu(z)\right)^S\le\tilde{c}\prod\limits_{i=1}^{m}\|f_i\|^{q_i}_{L^{p_i}};$ $f_i\in L^{p_i}(\partial D,\sigma), i=$
$=1,...,m, p_i>1, i=1,...,m.$

The proof follows directly from Hölder's inequality and the mentioned one functional result. (see [3] for various other similar multifunctional results).We omit easy details.

Similarly extending theorem C we have.

Theorem 1'

Let $0<p_i,q_i<\infty, i=1,...,m; \alpha>-1; 0<S<\infty$ $\left(\frac{1}{S}\right)=\sum\limits_{i=1}^{m}\left(\frac{q_i}{p_i}\right).$ $f_i\in H^{P_i}, i=1,...,m.$

Then we have that $\left\|\sup\limits_{A_\alpha(\xi)}\left(\prod\limits_{i=1}^{m}|f_i|^{q_i}\right)\right\|_{L^S}\le c\prod\limits_{i=1}^{m}\|f_i\|^{q_i}_{H^{P_i}};$

Extending partially theorems A, B, D we have the following theorem in bounded pseudoconvex domains with smooth boundary with smooth boundary.

Theorem 2 Let $\mu_j, i=1,...,m$ be positive Borel measures on $\mathbb{D}, 0<p_i,q_i<\infty, i=$
$=1,...,m,$ $\sum\limits_{i=1}^{m}\frac{q_i}{p_i}=1,$ $S>0, p>0, q>0, S+\frac{q}{p}=1.$ Then if

$$\left(\sup_a\right)\int\limits_{D^m}|f(z_1,...,z_m)|^q\times\delta(a)^S\cdot|K_{n+1}(z,a)|^S\prod_{i=1}^{m}\mu_i(z_i)\le c\|f\|^q_{A_\infty H^P}$$

or

$$\int\limits_{D^m}\prod_{i=1}^{m}|f_i(z_1...z_m)|^{q_i}\prod_{i=1}^{m}d\mu_i(z_i)\leq c_1\prod_{i=1}^{m}\|f\|^{q_i}_{A_\infty H^p},$$

then $\mu_i, i=1,...,m$, measures are Carleson measures.

The proof is a simple repetition of the unit disk case (see[3]), but use theorem C and some new results from the theory of pseudoconvex domains with smooth boundary namely estimate from below of Bergman kernel on Bergman ball and estimate from above of Bergman kernel on ∂D_r (see[16, 17]).

We for our proof we repeat arguments used in the ball use three facts, first, choose standart test function

$$f(\vec{w})=\prod_{i=1}^{m}f_i(w_i)=\prod_{i=1}^{m}(K_\tau)(z_i,w_i)\cdot\delta^V(z_i)),z_i\in D,w_i\in D,i=1,...,m,$$

where $\delta(z)=dist(z,\partial D)$ (see [2, 9]), $z\in D$, for some $\tau,\tau>0,v>0,v=v(\tau,n)$.

Then according to our maximal theorem C we have that

$$\|f\|_{A_\infty H^p(D^m)}\leq c\prod_{i=1}^{m}\|f_i\|_{H^p(D)}$$

Then it remains to use the following Lemma taking into account also that H^p Hardy spaces can be defined as

$$(H^P)(D)=\left\{f\in H(D):\left(\sup_{t>0}\right)\int\limits_{\partial D_t}|f(\xi)|^p d\sigma(\xi)<\infty\right\};$$

$0<p<\infty;\partial D_t=\{z:\rho(z)=t\}$, (see [10-12])

Lemma (see[16]). Let $\alpha,\beta>-1,s>0$, then $\forall_y\in D, 0<t<t_0$

$$\int\limits_{\{x:\rho(x)=t\}}|K_\alpha(x,y)|^s d\sigma(x)\asymp(\rho(y)+t)^{n-q};n<q\,and\,\rho(z)\asymp\delta(z),z\in D,q=\alpha s.$$

And it remains to use that

$$|K_\tau(z,a)|\geq|K_\tau(a,a)|;z\in B(a,r),a\in D,\tau>0,\tau\in N$$

for every Kobayashi ball (see [17]) and follow standard argument to get estimate from below.(see[17])

Combining all estimates we will have $\mu_j(B_D(a,r))\leq c\delta^n(a),a\in D,n\in N$. Hence μ_j is a Carleson measure $j=1,...,m$.

Remark 1.

The same results to our last theorem with almost the same proofs are valid for $A_\infty M^P$ space and even for more general versions namely for $A_\infty H^{\vec{P}}$ and $A_\infty M^{\vec{P}}$ spaces in one direction with mixed norm. The proof also and again is almost a simple repetition of the unit disk case (see [3]), but use theorem C and some new results from the theory of pseudoconvex domains with smooth boundary namely estimate from below of Bergman kernel on Bergman ball and estimate from above on ∂D_r (see [16, 17]).

Moreover similar results with verry similar proofs are valid in ubounded so-called tube domains over symmetric cones (see below).

Let further

$$M_\infty^* H^P = \left\{ f \in H(D^m) : \left(\sup_{\varepsilon_1>0}\right) \int\limits_{\partial D_{\varepsilon_1}} \dots \left(\sup_{\varepsilon_m>0}\right) \int\limits_{\partial D_{\varepsilon_m}} |f(\vec{z})|^p d\vec{\sigma}_\varepsilon < \infty \right\},$$

$$0 < p < \infty.$$

$$A_\infty^* H^P = \left\{ f \in H(D^m) : \left(\sup_{\substack{\varepsilon_1>0 \\ \varepsilon_m>0}}\right) \int\limits_{\partial D_{\varepsilon_1}} \dots \int\limits_{\partial D_{\varepsilon_m}} |f(\vec{z})|^p \dots d\vec{\sigma}_\varepsilon < \infty \right\},$$

$$0 < p < \infty.$$

Note our last theorem also can be similarly shown for these analytic Hardy type spaces (instead of $A_\infty H^P$ classes).

2. Some remarks on Poisson integrals on product domains and related problems

In this section we introduce new Poisson type integrals on product domains and discuss shortly some interesting questions and problems related with it.

Our main results in our next section are closely related with Hardy type spaces from one hand and with Poisson integrals from the other hand.

For a finite complex Borel measure on S_n we define closely related with Hardy spaces

$$P[\mu](z) = \left(\int\limits_{S_n} P(z,\xi)\ d\mu(\xi) \right); z \in B_n; P(z,\xi) = \frac{(1-|z|)^n}{|(1-\xi z)^{2n}|}; z \in B_n, \xi \in \overline{B}_n, n \in N;$$

(see [6]) so-called Poisson type integral of Borel positive measure μ and then define an obvious natural extension

$$P[\mu](z_1,\dots,z_m) = \int\limits_{S_n} \left(\frac{\prod\limits_{j=1}^m (1-|z_j|^2)^{\frac{n}{m}} d\mu(\xi)}{\prod\limits_{j=1}^m |1-\langle z_j, \xi \rangle|^{\frac{2n}{m}}} \right); z_j \in B_n; j = 1,\dots,m.$$

Note $m=1$ we have standard Poisson integral. Then for $f \in L^1(S_n, d\sigma)$ we define $P[f]$ on B_n (see [6]). $P[f](z) = \int\limits_{S_n} (P(z,\xi))(f(\xi)) d\sigma(\xi); z \in B_n$, $d\sigma$ is a Lebegues measure on S_n and more generally we define

$$P_{\vec{p}}[f](z_1,\dots,z_m) =$$

$$= \left(\int\limits_{S_n} \dots \left(\int\limits_{S_n} \left[\prod_j^m P(z_j,\xi_j) \right] |f(\xi_1,\dots,\xi_m)|^{p_1} d\sigma(\xi_1) \right)^{\frac{p_2}{p_1}} \dots d\sigma(\xi_m) \right)^{\frac{1}{p_m}},$$

$z_j \in B; j = 1,...,m,$ (similarly $P_{\vec{p}}(\mu)$, $0 < p_j < \infty; j = 1,...,m, \vec{\mu} = (\mu_1,...\mu_m)$ where μ_j are positive Borel measures on $B_n, j = 1,...,m.$)

In the unit polydisk similar functions were studied in Rudins book (see [18]). Our objects are of more general form.

Note also that we have

$$\left(\sup_{z\varepsilon B_n}|f(z)|\right) \leqslant \left(\sup_{z\varepsilon B_n}\right)\left(\int_{S_n} P(z,\xi)|f(\xi)|^p d\sigma(\xi)\right)^{\frac{1}{p}} ;0 < p < \infty; \quad (1)$$

(see [6]) for all $f \in H^P, z \in B$.

Using ordinary induction and one variable result we have on product domains the following as an analogue of (1).

$$\left(\sup_{z_j\varepsilon B_n}\right)|f(z_1,...,z_m)| \leq \sup_{z_m\varepsilon B} \cdot$$

$$\cdot\left(\int_{S_n} P(z_m,\xi_m)...\left(\sup_{z_1\varepsilon B}\right)\left(\int_{S_n}|f(\xi_1,...,\xi_m)|^{P_1}\times P(z_1,\xi_1)\times d\sigma(\xi_1)\right)^{\frac{p_2}{p_1}}...d\sigma(\xi_m)\right)^{\frac{1}{p_m}},$$

$$0 < p_j < \infty, i = 0,...,m, f \in H^{\vec{P}}(B^m)$$

For $\xi\varepsilon S_n, \alpha > 1$, let $\Gamma_\alpha(\xi) = \left\{z\varepsilon B_n : |1-\langle z,\xi\rangle| < \alpha\left(1-|z|^2\right)\right\}$, $\Gamma_\alpha(\xi)$ regions (Luzin cone) fill, B_n as $\alpha \to \infty$. (see [6])

For $f\varepsilon C(B_n), \alpha > 1$ we define $(M_\alpha f)(\xi) = (\sup)\{|f(z)| : z\varepsilon\Gamma_\alpha(\xi)\}$, the same can be defined for product domains, we have $(M_\alpha P)[\mu](\xi) \leq c(M\mu)(\xi)$, where $M\mu = M(\mu)\ (\xi) = \left(\sup_{\delta>0}\right)\dfrac{|\mu|(Q(\xi,\delta))}{\sigma(Q(\xi,\delta))}$; $\xi\varepsilon S_n$ for every finite complex Borel measure μ on S_n (see [6]). For general version of $P[\mu]$ we can show easily more general result using induction and one variable result.

$$\left[\left(\sup_{z_j\varepsilon\Gamma_\alpha(\xi_j) j=1,...,m}\int_S\left(\int_S \frac{\prod_{j=1}^m (1-|z_j|)^{\frac{n}{p_{j-1}}}\times d\mu(\xi_1)}{\prod_{j=1}^m |1-\xi_j,z_j|^{\frac{2n}{p_{j-1}}}}\right)^{P_1}\times d\mu(\xi_2)\right)^{\frac{p_2}{p_1}}...d\mu(\xi_m)\right] \leq$$

$$\leq c\prod_{j=1}^m M^{\tau_j}(\mu_j)(\xi_j)$$

for some positive $\tau_j, j = 1,...,m, \tau_j = \tau_j(\vec{P}), p_j > 1, j = 1,...m$

Next we have that if μ is a finite complex Borel measure on S_n then (see [6]) $\left[\left(\lim_{z\longrightarrow\xi}\right)\right]P[\mu](\xi) = \left(\dfrac{d\mu}{d\sigma}\right)(\xi)$, $z \in D_\alpha(\xi)$, for almost every point $\xi\varepsilon S_n$.

We can consider the same problem for case of more general $(P[\vec{\mu}])$ Poisson integral on product domains, then also we can turn similarly to other questions on general Poisson integral.

For example, the following results can be seen in [6].
We as usual define similarly $H^P(B)$.

$$H^P = \{f \in H(B) : M_P(f,r) < \infty\}, 0 < p \leq \infty;$$

$where\, M_P^P(f,r) = \int\limits_{S_n} |f(r\xi)|^P d\sigma(\xi)$

Let

$$BMOA = f \in H^2 : \|f\|_{BMO}^2 =$$

$$= |f(0)|^2 + \sup\left(\frac{1}{\sigma(Q)}\right) \times \int\limits_Q |f - f_Q|^2 d\sigma(\xi) < \infty$$

where $f_Q = \frac{1}{\sigma(Q)} \int\limits_Q f(\xi) d\sigma(\xi)$, where $Q(w,r) = \left\{\xi \in S_n : |1 - (\xi, w)|^{\frac{1}{2}} < r\right\}$,
$w \in S_n,\ r > 0$.

<u>Theorem F</u> (see [6]).

Let $f \in H^2$. Then $f \in BMOA$ if and only if

$$\left(\sup_{a \varepsilon B_n}\right) \int\limits_{S_n} |f(\xi) - f(a)|^2 (P(a,\xi)) d\sigma(\xi) < \infty$$

The natural open problem to find an extension (or analogue) of this result, where $\prod\limits_{j=1}^m P(a_j, \xi_j)$ is involved, $a_j, \xi_j \in B, j = 1, ..., m$.

We turn to other problem.

Next let $\xi \in S_n, r > 0$. $(Q_r(\xi)) = \{z \in B_n : d(z,\xi) < r\}$;

$$d(z,w) = |1 - \langle z, w \rangle|^{\frac{1}{2}}, z, w \in \overline{B}_n.$$

We call $Q_r(\xi)$ Carleson tube at ξ.

We say μ is a Carleson measure (μ is a positive Borel measure on B) if

$$\mu(Q_r(\xi)) \leq cr^{2n}, \xi \in S_n, r > 0. (see[6])$$

<u>Theorem E</u>(see [6])

A positive Borel measure μ in B_n is a Carleson measure if

$$\left(\sup_{z \in B_n}\right) \left(\int\limits_{B_n} (P(z,\omega)) d\mu(w)\right) < \infty.$$

The natural problem again to find some extension of this classical result to general integrals with general Kernels $\widetilde{P}(z,\xi) = \left(\prod\limits_{j=1}^m P(\xi_j, z_j)\right), \xi_i, z_i \in B_m$,
$j = 1, ..., m$ and some related integrals on product domains.

3. On Poisson type representation and decomposition of Hardy type spaces in the unit ball in $\mathbb{C}^n$

The problem we consider is well-known for functional spaces in $\mathbb{R}^n$ (the problem of equivalent norms) (see for example [6]). Let $X, (X_j)$ be a function space in a fixed product domain and(or) in $\mathbb{C}^n$ (normed or quazinormed) we wish to find equivalent expression for $\|f_1...f_m\|_X; m \in \mathbb{N}$ (Note these are closely connected with spaces on product domains since often if $f(z_1...z_m) = \prod_j^m f_j(z_j)$, then $\|f\|_X = \prod_{j=1}^m \|f_j\|_{X_j}$. These results also as we'll see extend some well-known assertions on atomic decomposition of A^p_α type spaces.

To study such group of functions it is natural ,for example, to ask about structure of each $\{f_j\}_{j=1}^m$ of this group.

This can be done for example if we turn to the following question find conditions on $\{f_1,...,f_m\}$; so that $\|f_1,...,f_m\|_X \asymp \prod_{i=1}^m \|f_j\|_{X_j}$ decomposition is valid. In this case we have if for some positive costant c; $\prod_{i=1}^m \|f_j\|_{X_j} \leq c\|f_1...f_m\|_X$; then we have each $f_j, f_j \in X_j; j = 1...m$ where X_j is a new normed (or quazinormed) function space in $\mathbb{D}$ domain and we can easily now provide properities of $\{f_j\}$ based on facts of already known one functional function space theory. (For example to use known theorems for each $f_j \in X_j, j = 1...m$ on atomic decompositions). This idea was used for Bergman spaces in the unit ball and then in bounded pseudoconvex domains with smooth boundary in recent papers [2]. We, in this paper, extend these results in various directions using modification of known proof.

We provide a complete proof of basic known case then show in details how to modify it to get new results. The old known proof is simple and very flexible as it turns out and we can easily get, as we can see below, various new interesting results from it directly. This remark is leading us to provide only some sketchy arguments sometimes below of proofs when we deal with new theorems, since the core of all proofs is basically the same in all our theorems. Here is the transparent proof of the classical case of the Bergman space A^p_α case in the unit ball $\mathbb{C}^n$. The case of Bergman space in more general pseudoconvex domain can be seen in [2].

We show now that $\|f_1...f_m\|_{A^p_\tau} \asymp \prod_{i=1}^m \|f_j\|_{A^p_{\alpha_j}}$ is valid under certain integral (A) condition (see below) if $p \leq 1$ and if $\tau = \tau(p, \alpha_1, ...\alpha_m, m)$;

Note from our discussion above the only interesting part is to show that

$$\prod_{i=1}^m \|f_j\|_{A^p_{\alpha_j}(B)} \leq c_1 \|f_1...f_m\|_{A^p_\tau(B)},$$

since the reverse follows directly from the uniform estimate (see [3])

$$|f(z)|(1-|z|)^{\frac{\alpha_j+n+1}{p}} \leq c\|f\|_{A^p_{\alpha_j}}; 0 \leq p < \infty; \alpha_j > -1; j = 1,...,m$$

and ordinary induction. This lead easily to the fact that τ can be calculated

$$\tau = (n+1)(m-1) + (\sum_{u=1}^m \alpha_j); \alpha_j > -1; 0 \leq p < \infty;$$

Note also similar very simple proof based only on various known uniform estimates can be used in all our proofs below for similar inequalities for various spaces. So we mainly concentrate on reverse estimates (see [6],[2]).

Note this argument also allows easily to obtain even more general version with $|f_1|^{p_1}...|f_m|^{p_m}$ instead of $|f_1|^p...|f_m|^p$ (which was discussed above where
$0 < p_j < \infty$,
$j = 1,...m$).

Let us now return to the proof of the reverse estimate.

Assuming that

$$f_1(w_1)...f_m(w_m) = c_b \int\limits_B \frac{(f_1(z)...f_m(z))(1-|z|)^{\alpha} dV(z)}{\prod\limits_j^m (1-<z,w_j>)^{\frac{n+1+\alpha}{m}}} \tag{A}$$

$$\alpha > \alpha_0, w_j \in B, j = 1,...,m.$$

Using Fubini's theorem and extended version of the following estimate (see [5])

$$\int\limits_B |f(z)|(1-|z|^2)^{\frac{n+1+\alpha}{p}-(n+1)} dV(z) \leq c\|f\|_{A^p_\alpha} \tag{$\tilde{A}$}$$

$$\alpha > -1, 0 < p \leq 1, f \in H(B_n);$$

$$f(z) = \frac{\tilde{f}(z)}{(1-wz)^v}; v > 0, w \in B, \tilde{f} \in H(B)$$

we get for $\tau = (n+1+\alpha)p-(n+1), \tau > -1$

$$\prod_{k=1}^m \int\limits_B |f_k(z_k)|^p \times (1-|z_k|)^{\alpha_k} d\delta(z_k) =$$

$$= \int\limits_B ... \int\limits_B \prod_{k=1}^m |f_k(z_k)|^p \prod_{k=1}^m (1-|z_k|^2)^{\alpha_k} d\delta(z_1)...d\delta(z_m) \leq$$

$$\leq c \int\limits_B ... \int\limits_B \int\limits_B \frac{|f_1(z)|^p ... |f_m(z)|^p \times (1-|z|^2)^{\tau}}{\prod\limits_{k=1}^m |1-<z,z_k>|^{\frac{n+1+\alpha}{m}p}} d\delta(z) \times$$

$$\times \prod_{k=1}^m (1-|z_k|^2)^{\alpha_k} \times d\delta(z_1)...d\delta(z_m) \leq c \int\limits_B \prod_{k=1}^m |f_k(z)|^p \times$$

$$\times (1-|z|^2)^{\tau} d\delta(z) \int\limits_B ... \int\limits_B \prod_{k=1}^m (1-|z_k^2|)^{\alpha_k} \times \frac{1}{\prod\limits_{k=1}^m |1-<z,z_k>|^{\frac{n+1+\alpha}{m}p}} d\delta(z_1)...d\delta(z_m);$$

where $\alpha_k > -1, k = 1,...,m, \tau = (n+1+\alpha)p-(n+1) > -1); \alpha > \alpha_0$;
$\alpha_0 = \alpha_0(n,m,\alpha_1,...,\alpha_m)$. Using the estimate

$$\int\limits_B \frac{((1-|z|)^2)^t dV(z)}{|1-<z,w>|^{n+1+t+S}} \leq \frac{c}{(1-|z|^2)^S};$$

$$t > -1, S > 0, z \in B;$$

We get finally from (A)

$$\prod_{k=1}^{m}\int_{B}|f_k(z)|^p(1-|z_k|^2)^{\alpha_k}dV(z_k) \le c\int_{B}\prod_{k=1}^{m}|f_k(z)|^p(1-|z|^2)^{\tau_1}dV(z) < \infty$$

where $0 < p \le 1, \tau_1 = (m-1)(n+1) + \sum\limits_{k=1}^{m}\alpha_k > -1.$

A carefull analysis of this proof shows that various extentions can be provided by small modification of these arguments.

In last sections also we provide some new results on atomic decomposition of Hardy multifunctional spaces using Poisson type integral formula in the unit ball and Henkin type representation formula in case of pseudoconvex domains with smooth boundary.Theorems on atomic decomposition of Hardy spaces (one functional case) in various domains attracted much attention in recent years (see [2,11,19,20]). We provide an extention to multifunctional case.

Though Poisson type and Henkin type formulas serve as base of all our proofs in last section in this direction, we mention some recent results ,where the same problems were solved in Bergman spaces via Bergman representation formula.(see [2]for such results)

The following result is valid in the unit ball though some extensions in pseudoconvex domains with smooth boundary were provided also in recent papers [2]. We provide Bloch spaces version, the similar result is valid for Bergman space. This result will be needed for us later. We denote by ∇ the gradient of f (see [2, 6]).

<u>Theorem C'</u> (see [2]) Let $f_k \in H(B), k = 1,\dots,m, m \in N$. Let $(f_1,\dots,f_m)_{\widetilde{B}} =$
$= ||\nabla(f_1,\dots,f_m)\left(\log\frac{1}{1-|z|^2}\right)^{1-m}(1-|z|^2)|| < \infty.$
Let also for all $i = 1,\dots,m,$

$$f_1(z_1)\dots\nabla f_i(z_i)\dots f_m(z_m) =$$

$$= c_{\vec{\alpha}}\left(\int_{B}\frac{\nabla(f_1,\dots,f_m)(z)dV_{\alpha_j}(z)}{\prod\limits_{j=1}^{m}(1-\langle z,z_k\rangle)^{\frac{n+1+\alpha_i}{m}}}\right)^m$$

for some positive $\alpha_k, k = 1,\dots,m,$ where $z_j \in B, j = 1,\dots,m; b > n$. Then there exists a sequence $\{a_j\}$ in B such that every function f_k can be represented as $(f_k)(z) = \sum\limits_{K}\left(c_j^{(K)}\right)\left(\frac{1-|a_j|^2}{1-\langle z_1,a_j\rangle}\right)^b(K)$, $k = 1,\dots,m$ where $\left\{C_j^{(K)}\right\} \in l^{\infty}$; $\{a_j\}\,\varepsilon B$, $j = 1,\dots,m$ and the series converges in weak topology of B. Conversely if f_k has the form (K) then $(f_1,\dots,f_m)_{\widetilde{B}} < \infty$.

This theorem, as we see , gives immediately atomic decomposition of multifunctional Bloch $\tilde{B}$ class of functions (see [2]). One of main intension of this paper is to try to find complete analogues of this result for multifunctional Hardy - type spaces in pseudoconvex domains with smooth boundary and in convex domain of finite type in $\mathbb{C}$.

In this section we prove our main results first for unit ball case.

A classical result from the theory of H^p spaces says (in the unit ball in C^n) that $\| \int\limits_S P(z,\xi)f(\xi)d\delta(\xi)\|_{H^p} \le \widetilde{C}\|f\|_{H^p}; p \ge 1$ where P is classical Poisson Kernel, S is a unit sphere; $S = \partial B_n$.

We find first complete extension of this result to the case of several functions (products of functions) and then to much more general D domains in C^n simultaneously (for $p = 1$) extending classical Poisson type Kernel to product domains $D \times \ldots \times D$.

First we give a detailed proof of the unit ball extensions then turn to convex domains of finite type and to bounded pseudoconvex domains with smooth boundary in C^n. This gives solution of atomic decomposition problem for multifunction Hardy space.

In the theory of Hardy spaces the role of invariant Poisson Kernel is fundamental and very vital.

Some extensions of known results on Poisson integrals and Poisson type integrals in the unit ball and in the Polydisk can be seen in [13,14]. This paper provide another list of such type results concerning Poisson type integrals in the unit ball and bounded pseudoconvex domains with smooth boundary. For our proof we need (see [6]) $\int\limits_S \frac{d\sigma(\xi)}{|1-r\xi|^\alpha} \le \frac{c}{(1-r)^{\alpha-n}}; \alpha > n, r \in I = (0,1), (A_0)$. Note first that from ordinary induction and known uniform estimates in the ball

$$\left(\sup_{z\in B}\right) |f(z)|(1-|z|^2)^{\frac{n}{p_i}} \le c\|f\|_{H^{P_i}};$$

$0 < p_i < \infty$, $i = 1, \ldots, k+1 k \in N$, we can get the following estimate (even for H^p spaces in various domains tube,pseudoconvex ect).

$$\left(\sup_{r<1}\right)\left(\int\limits_S \prod_{i=1}^m |f_i(r\xi)|^{P_i} d\sigma(\xi)\right)\cdot(1-r)^{n(m-1)} \le c\prod_{i=1}^m \|f_i\|_{H^{P_j}} \qquad (A)$$

Assume further

$$\prod_{i=1}^m \|f_i\|_{H^{P_i}} \le c_1 \sup_r \prod_{i=1}^m \left(\int\limits_S |f_i(r\xi)|^{P_i} d\sigma(\xi)\right)^{\frac{1}{p_i}} \qquad (A')$$

(the reverse is obvious this assumption will be dropped below) for $r \in (0,1); p_i > 1, i = 1, \ldots, m$. As we just showed the same type result is valid in A^p_α Bergman space. Moreover for Bergman classes (even in various domains) under certain integral condition (see [2]);

$$[f_1(w_1)\ldots f_m(w_m)] = (c_\alpha)\int\limits_B \frac{\left(\prod\limits_{i=1}^m f_i(z)\right) dV_\alpha(z)}{\prod\limits_{i=1}^m (1-\bar{z}w_j)^{\frac{n+1+\alpha}{m}}}, w_j \in B, j = 1, \ldots, m \qquad (C);$$

where $\alpha > \alpha_0; \alpha_0$ is large enough the reverse is also true, so we have

$$\prod_{i=1}^m \|f_i\|_{A^p_{\alpha_k}(B)} \asymp \|\prod_{i=1}^m |f_i|^P\|_{L^1_{\tau_1}} \qquad (B);$$

$\tau_1 = (m-1)(n+1) + \left(\sum\limits_{k=1}^m \alpha_k\right)$; $\alpha_k > -1$ (see [2]); (The proof of this theorem even for pseudoconvex domains with smooth boundary can be seen in [2])

We now wish to show (B) type sharp result for weighted Hardy spaces (and not only in the unit ball B) using Poisson type integrals and it is properties. So we have to show only the reverse to (A) and (A') using some version of (C) for Poisson (not Bergman) type integrals.

We have the following result which is valid even in bounded pseudoconvex domains with smooth boundary (and in particular in the unit ball).

Theorem 3 Let $1<p<\infty$; $f_i \in H^P; i=1,...,m$; then if

$$\prod_{i=1}^{m}(f_i(z_i)) = c_\alpha \left(\int_{S^n} \frac{\left(\prod_{i\geq 1}^{m}[f_i(|z|\xi)]\right) d\sigma(\xi)}{\left|\prod_{j=1}^{m}(1-\overline{\xi}z_j)^\alpha\right|} \right)\cdot\left((1-|z|)^{-n+\alpha m}\right); \qquad (G);$$

(Special integral condition) $z_i \in B, i=1,\dots,m; \alpha>n; |z_j|=|z|$; we have

$$\left(\sup_{r<1}\right)\left((1-r)^{(m-1)n}\right)\left(\int_{S^n}\prod_{i=1}^{m}|f_i(r\xi)|^P d\sigma(\xi)\right) \leq$$

$$\leq c\left(\sup_{r<1}\right)\prod_{k=1}^{m}\|(f_k)(r\xi)\|^P_{L^P(S)}; (G_1).$$

and reverse is also true for $p=1$.

Remark 2

Our Estimates may have many applications in the unit ball. (Note (G) almost vanishes for $m=1$, $\alpha=2n$ case and for $\alpha=n$, $m=1$ case it vanishes).(see [6])

Remark 3

(G_1) is also valid for $\prod_{i=1}^{m}|f_i|^{p_i}$ with same proof $1<p_i<\infty, i=1,...,m$.

Remark 3'

Theorem 3 gives an analogue of theorem C' for weighted multifunctional Hardy spaces for $p=1$.

Proof of theorem 3

Note first using Holders inequality we have (see [6])

$$|f(z)|^P \leq c\int_S P(\xi,z)|f(\xi)|^P d\sigma(\xi); 1<p<\infty, f\in H^P.$$

Note first it remains to apply this to $\left(\prod_i^m f_i\right)$ to get approximately (G). Note for $p=1$ case we get immediately reverse to (G$_1$),$indeed we get directly from (G) and$(A$_0$), and then estimates (A), (A') the estimate (G_1) and the reverse estimate for all $p>1$ easily.

Concerning second estimate (A') we use standart arguments. We first get $G(f_1,...,f_m) = c\left(\int\limits_S |f_1(\xi)|^{P_1} d\xi_1\right) \times ... \times \left(\int\limits_S |f_m(\xi)|^{P_m} d\xi_m\right);\ p_i > 1, i =$ $= 1,...,m;\ f_i \in H^{P_i}, i = 1,...,m$ then apply to $(f_R)(z) = f(Rz); R < 1$ function to get that $G_R(f_1,...,f_m) \le c\prod\limits_{i=1}^{m}\int\limits_S |f_i(R\xi_i)|^{P_i} d\xi_i \le \tilde{c}\sup\limits_{R<1}\prod\limits_{i=1}^{m}\int\limits_S |f_i(R\xi)|^{P_i} d\xi$ and it is not difficult to show finally that $\lim\limits_{R\to 1} G_R(f_1,...,f_m) = G(f_1,...,f_m)$.(see[6]) And, hence, obviously $\left(\sup\limits_{R<1}\right)\left(\prod\limits_{i=1}^{m}\int\limits_S |f_i(R\xi)|^{P_i} d\xi\right) \asymp$ $c\prod\limits_{i=1}^{m}\|f_i\|_{H^{P_i}}^{P_i};$

Theorem is proved.

For same theorem in bounded pseudoconvex domains with smooth boundary in C^n , we have to use only some estimates from [1] and equalities taken from [2].(see below). Complete analogue of theorem 3 can be also valid in harmonic function spaces, matrix ball, in minimall ball, in tubular domains ect. with the same proof.

Remark 3"

Let $B(z,r)$ be the Bergman ball in B

$$X_{p,q,r}(B) = \left\{ f \in H(B) : \sum_{k\ge 0}\left(\int\limits_{D(a_k,r)} |f(z)|^q (1-|z|)^{\tau} dV(z)\right)^{\frac{p}{q}} < \infty \right\}$$

$$Y_{p,q,r}(B) = \left\{ f \in H(B) : \int\limits_B \left(\int\limits_{B(w,r)} |f(z)|^q (1-|z|)^{\tau} dV(z)\right)^{\frac{p}{q}} dV(w) < \infty \right\};$$

$$0 < p,q < \infty, \tau > -1;$$

be analytic Herz type spaces.

In the unit ball we have that for Herz type $X_{p,q,\tau}$ and $Y_{p,q,\tilde{\tau}}$ spaces.

$$\sum_{K\ge 0}\left(\int\limits_{D(a_k,r)} |f_1,...,f_m|^q (1-|z|)^{\tau} dV(z)\right)^{\frac{p}{q}} \le$$

$$\le c \times \|f_1\|_{X_{p,q,\alpha_1}} \times ... \times \|f_m\|_{X_{p,q,\alpha_m}}$$

for some fixed τ, α_1, α_m just use induction and an estimate

$$\left(\sup_{z\in D} |f_i(z)|(1-|z|)\right)^{\tau_0} \le c\|f_i\|_{X^{p,q,\alpha_1}},\ for\, some\, \tau_0, i = 1,...,m.$$

Similarly based on similar uniform estimate for some $\tilde{\tau}, \alpha_j, j = 1,...,m.$ we have

$$\int\limits_D \left(\int\limits_{B(w,r)} |f_1(z)...f_m(z)|^q (1-|z|)^{\tilde{\tau}} dV(z)\right)^{\frac{p}{q}} dV(w) \le$$

$$\le c\|f_1\|_{Y_{p,q,\alpha_1}} \times ... \times \|f_m\|_{Y_{p,q,\alpha_m}};$$

We now can use Bergman representation formula (C) to show the reverse to these estimates, namely for example that $\prod_{i=1}^{m}\|f_i\|_{Y_{p,q,\alpha_i}} \le \tilde{c}\|f_1...f_m\|_{Y_{p,q,\tilde{\tau}}}$; for some $\alpha_i, i=1,...,m, \tilde{\tau} > 0, f_i \in H(D), i=1,...,m$. See [23] where we obtained such type sharp results.

We refer to recent paper [21] for some other new sharp results for Herz spaces in higher dimension.

4. On some new results concerning Poisson type integrals and Hardy spaces in bounded pseudoconvex domains

In this section we now want to extend our results of third section in the unit ball to bounded pseudoconvex domains $D=\{z:\rho<0\}$ with smooth boundary.

For any bounded domain $0<p\le\infty$ with smooth boundary we have by induction for $\tilde{f}=f_1...f_{m+1}$; as in case of unit ball. Note $\partial D_\varepsilon=\{z:\rho(z)=\varepsilon\}$.

$$\left(\sup_{\varepsilon>0}\right)\left(\int_{\partial D_\varepsilon}|\tilde{f}_1|^p d\sigma(\xi)\right)\times(\varepsilon^{\tau})\le((\tau=nm-n))\le$$

$$\le\left[\sup_{\varepsilon}\left(\sup_{\partial D_\varepsilon}|f_1|^p(\varepsilon)^n\right)\right]\cdot\left(\int_{\partial D_\varepsilon}|f_2(\xi)...f_m(\xi)|^p d\xi(\varepsilon^{\tau-n})\right)\le M$$

Hence

$$M\le\left[\left(\sup_{\xi\in D}\right)\right]|f_1(\xi)|^p\delta^n_{(\xi)}\times\left(\int_{\partial D_\varepsilon}|f_2...f_m|^p d\xi\right)\left((\varepsilon^{\tau-n})\right)\le$$

$$\le\mathbb{C}\left(\|f_1\|^p_{H^p}\right)\cdot\left(\int_{\partial D_\varepsilon}|f_2...f_m|^p d\xi\right)(\varepsilon^{\tau-n})\le\mathbb{C}_1\|f_1\|^p_{H^p}...(\|f_m\|^p_{H^p});$$

The same result is valid also for more general case when we look $\left[\prod_{i=1}^{m}(|f_i|^{p_i})\right]$; $o<p_i<\infty, i=1,...,m$; with same proof.

Since $|f(z)|\le\mathbb{C}\|f\|_{H^p}(d(z,\partial\Lambda)^{\frac{-n}{p}}); f\in H^p; z\in D$; for $0<p\le\infty$ and for any bounded domain in $\mathbb{C}^n$ with smooth $\mathbb{C}^2$ boundary (see [10]),

where $H^p=\left\{f\in H(D):\left(\sup_{\varepsilon>0}\right)\left(\int_{\partial D_\varepsilon}|f|^p d\sigma(\xi)\right)<\infty\right\}$; for $0\le p<\infty$

Note weighted Hardy spaces with quazinorms $\left(\sup_{\varepsilon>0}\right)\left(\left(\int_{\partial D_\varepsilon}|f|^p d\sigma(\xi)\right)\varepsilon^{\tau}\right)<\infty$; $0<p\le\infty, \tau>0$ were studied in [5,7] ;

Let us show the reverse to this using Poisson type integral formulas in D see [9] and estimates for Bergman - Siege Kernel.

Let us show the reverse estimate following unit ball case. We will use the following integral representations and facts.(see [9])

$$(\delta(\xi))^t \left(\int\limits_{\partial D_\xi} \frac{d\sigma(z)}{|\tilde{\phi}(\xi,z)|^{n+t+\frac{1}{2}}} \right) \leq \left(\frac{C}{\delta(\xi)^{-\frac{1}{2}}} \right); t > t_0$$

And $(f(z)) = \int\limits_{\partial D_\varepsilon} [(f(\xi))] (\eta_0(z,\xi)) |\phi(z,\xi)|^{-m}; z \in D$ (see [9]), where $\eta_0 \in C^\infty$

Put now for $m > 1, \tilde{m} = n + \tau, \tau > 0$

$$[(f_1(z_1))...(f_m(z_m))] = \int\limits_{\partial D_\xi} \rho(\xi)^{-n+\tilde{m}m} (f_1(\xi)...f_m(\xi)) \times (\eta_0(z,\xi)) \times$$

$$\times \prod_{j=1}^{m} \left(|\Phi(z_j,\xi)|^{-\tilde{m}} \right) \leq \tilde{D}; (K)$$

Using (K) we have for $\tilde{m} = (n+\tau)$, $\tau > 0$ that (see [16]) $s > 0, \alpha > 1$ $\int\limits_{x:(r(x)=t)} |K_\alpha(x,y)|^S d\sigma(x) \leq [r(y)+t]^{-n-q}, n < q; q = \alpha S; t \in (0,\varepsilon)$; and hence

$$\prod_{i=1}^{m} \left(\int\limits_{\partial D\varepsilon} |f_j(z_j)| d\sigma(z_j) \right) \leq c \left(\sup_{\varepsilon>0} \right) \left(\int\limits_{\partial D_\varepsilon} \left(\prod_{i=1}^{m} f_i(\xi) \right) \times (\delta(\xi))^{-n+nm} \right);$$

The idea we use in section 3 is the same, we have $\delta(\xi) \asymp \rho(\xi) = \varepsilon$; if $\xi \in \partial D_\varepsilon$.

So we get what we need if (K) holds we have that

$$\left(\sup_{\varepsilon>0} \right) \left(\prod_{j=1}^{m} \right) \left(\int\limits_{\partial D_\varepsilon} |f_j(z_j)| d\sigma(z_j) \right) \leq \tilde{\mathbb{C}} \sup_{\varepsilon>0} \left(\int\limits_{\partial D_\varepsilon} \prod_{j=1}^{m} |f_j(\xi)| d\sigma(\xi)| \right) \left(\varepsilon^{nm-n} \right)$$

Let $\{a_k\}$ be fixed r-lattice in pseudoconvex domain $D = \{r < 0\}$. We now define new weighted Hardy type analytic function space as A space of all analytic functions with quazinorms.

$$\|f\|_{H_1} = \sup_r \sum_{k \geq 0} \left(\int\limits_{\partial D_r} |f(\xi)|^P |K_\tau(\xi,a_k)|^S d\sigma_\xi (\delta(a_k))^{\tilde{k}} \right)^{\frac{q}{p}} (r^\alpha) < \infty$$

$$\alpha \geq 0; \tau, s \geq 0; 0 < p,q < \infty; \tilde{k} > 0, \tilde{k} > k_0.$$

Note if $q = p$ we have that

$$\|f\|_{H_1} \leq c \left(\sup_{r>0} \right) \left(\int\limits_{\partial D_r} |f(\xi)|^P d\sigma_\xi \right) \left(r^{\tilde{s}} \right);$$

for some $\tilde{s} > 0$; since by Forelly - Rudin (see [2]) estimate we have

$$\left(\delta(a_k)^{\tilde{k}} \right) \sum_{k \geq 0} |K_\tau(\xi,a_k)|^S \leq \tilde{c} \sum_{k \geq 0} \left(\delta_{(a_k)}^{\tilde{k}-(n+1)} \right) \int\limits_{B(a_k,\rho)} |K_\tau(w,\xi)|^S dw \leq cr(\xi)^v = \tilde{c} r^v;$$

for some parameter v.

For $\tilde{k} = s = 0$ we have here known weighted Hardy space.

It will be nice to extend our result to this spaces.

5. On Poisson type integrals and Hardy spaces in convex domains of finite type

In this section we provide some direct extensions of results of third section to convex domains of finite type.

Since $A^p_{-1} = H^p, 0 < p < \infty$ and for A^p_α Bergman class results of third section are known (see [2]) our theorens and estimates can be considered as extensions of these results for A^p_α spaces.

Note also for A^p_α classes such type estimates were used in [2] to get new results for atomic decomposition of multifunctional Bergman spaces, similar direct corollaries we also have from results of this paper for some weighted multifunctional Hardy spaces. For one functional such results we refer to [10,19,20].

For example, the natural question is can be decompose each $(f_j), j = 1,...,m$ into atoms as in [10,19,20] if

$$\left(\sup_{\varepsilon>0}\right)\left(\int_{\partial D_\varepsilon}\left|\prod_{i=1}^m f_i(\xi)\right| d\sigma(\xi)\right)\left(\varepsilon^{nm-n}\right) < \infty, m > 1, n \in \mathbb{N}, f_i \text{ is analytic.}$$

In convex domains or bounded pseudoconvex domains with smooth boundary our theorem of forth, fifth section provides a positive answer under certain integral condition (Poisson type integral representation).

The proof of convex in $\mathbb{C}^n$ domains is based on following simple observations.

Let D be convex bounded domain of finite type.

We show that in particular if

$$[f_1(z_1)...f_m(z_m)] = \int_X (Q(\xi,\vec{z}))\,(\tilde{f}(\xi))d\sigma(\xi); f_i \in H^p, i = 1,...,m \qquad (D)$$

where X is a subdomain bounded in $\mathbb{D}$; $\mathbb{D} \subset \mathbb{C}^n; z_j \in D; (\tilde{f}) = \prod_{j=1}^m (f_j); Q(\xi,\vec{z})$ is a Poisson type Kernel in $\mathbb{D}$, $Q(\xi,\vec{z}) = Q(\xi, Z_1,...,Z_m), z_j \in D, j = 1...m$, then $\|\int_X (Q_\gamma(\xi,\vec{z}))(\tilde{f}(\xi))d\sigma(\xi)\|_{Y(D)} \leq \mathbb{C}\|\tilde{f}\|_{Y(D)}$ where Y is a weighted Hardy space in $\mathbb{D}$ for $p = 1$. Moreover we show the reverse of this also.

We now discuss the case of convex domains in $\mathbb{C}^n$ of finite type based mainly on results from [15]. We refer to [15] for details concerning Q, K_t, S functions and some details on convex domains of finite type.

Note first that $f_t(z_t) = f_t(z). Wereferforz_t$ map to same paper.

We repeat arguments of third section again.

Note also

$$\int_{\xi\in\partial D} |K_t(z,\xi)|^s d\sigma(\xi) \leq \tilde{c}(t^{-n(s-1)}); \qquad (1)$$

and (see [15])

$$\int_{\xi\in\partial D} \frac{(Q(z,\xi))\wedge(\overline{\partial^T}(Q(z,\xi))^{n-1})}{(S(z,\xi))^{\alpha n}} d\sigma(\xi) \leq \mathbb{C}_1\left(t^{n-\alpha n}\right); \tag{2}$$

the second estimate (we can put this as additional assumption) can be obtained from the carefull analysis of the proof of the first estimate (see also[15] for the proof of similar estimate)and note now that following integral representation solves our problem (see [15]).

$$(f_1(z_1))...(f_m(z_m)) = ct^{\alpha m-n} \int_{\xi\in\partial D} (\tilde{f}_t(\xi)) \prod_{i=1}^{m} \frac{Q(z_j,\xi)\wedge(\overline{\partial}^T Q(z_j,\xi))^{n-1}}{S(z_1,\xi)^{\alpha},...,S(z_m,\xi)^{\alpha}} \tag{3}$$

where $\alpha = n+\tau, \tau > 0, z_j \in D, j = 1...m.$

The rest is the repetition of the proof of the unit ball case we provided above, we omit easy details here .

So now if (3) holds then we have as in the unit ball that

$$\left(\sup_{t>0}\right)\left(\int_{\partial D} |f_1(Z_t)|d\sigma ... \int_{\partial D} |f_m(Z_t)|d\sigma\right) \leq \mathbb{C}\left(\sup_{t<0} t^{nm-n} \int_{\xi\in D} |\tilde{f}_t(\xi)|d\sigma(\xi)\right);$$

For bounded convex domain of finite type the reverse is also valid.(see for this the previous section based on known uniform estimate for Hardy spaces in such domains

see for example [11,12]).

Namely we have that

$$\sup_{t<0}(t^{nm-n}) \int_{\partial D} |\tilde{f}_t(\xi)|d\sigma(\xi) \leq c\prod_{i=1}^{m} \|f_i\|_{H^1} \tag{4}$$

Note similar to (4) is valid also for $\prod_{i=1}^{m} \|f_i\|_{H^{p_i}}, p_i > 1, \tilde{f} = \prod_{i=1}^{m} f_i$. this can be easily shown using same uniform estimate(see also previous section)

6. Some remarks on Hardy type spaces in tubular domains over symmetric cones

Finnaly we add some remarks on embedding theorems for Hardy type classes in unbounded Tubular domains in higher dimension. Let T_Λ be a tubular domain over symmetric cone, let $T_\Lambda^m = T_\Lambda \times \cdots \times T_\Lambda$, $H(T_\Lambda^m)$ be the space of all analytic functions on (T_Λ^m).

Let $B(z,r)$ be the Bergman ball in T_Λ.

Let $\Delta^\alpha(\gamma)$ be determinant function. We refer to [22] for basic details on function theory in tubular domains over symmetric cones.

Let $B_\nu(z,w)$ be the Bergman Kernel (weighted) in T_Λ

$$B_\nu(z,w) = \frac{1}{\Delta^\nu\left(\frac{z-\overline{w}}{i}\right)}; z,w \in T_\Lambda$$

Similar (not sharp) embedding theorems (see secton 1) are valid with very similar proofs (we mean only necessity condition on measure) in tubular domain over symmetric cones. For example, if $0 < p < \infty$ we define new Hardy type classes in tube.

Let

$$H^p(T_\Lambda^m) = \left\{ f \in H(T_\Lambda^m) : \right.$$

$$\left. : \left(\sup_{\gamma_m \in \Lambda}\right) \int_{R^n} \left(\sup_{\gamma_1 \in \Lambda}\right) \int_{R^n} |f(x_1 + i\gamma_1, ..., x_m + i\gamma_m)|^p dx_1, ..., dx_m \leq \infty \right\};$$

and

$$\widetilde{H}^p(T_\Lambda^m) = \left\{ f \in H(T_\Lambda^m) : \right.$$

$$\left. : \left(\sup_{\substack{\gamma_1 \in \Lambda \\ \gamma_m \in \Lambda}}\right) \int_{R^n} ... \int_{R^n} |f(x_1 + i\gamma_1, ..., x_m + i\gamma_m)|^p dx_1, ..., dx_m < \infty \right\};$$

We have the following assertion for Carleson measure (that is for those μ for which $\mu(B(z,r)) \leq c(\Delta^{\frac{n}{r}}(I_m z)); z \in T_\Lambda$ (see[22])).

Proposition B Let $\mu_1, ..., \mu_m$ be positive Borel measures on T_Λ. Let $0 < p_i$, $q_i < \infty, i = 1, ..., m; \sum\limits_{i=1}^{m} \frac{q_i}{p_i} = 1$. Then if

$$\int_{T_\Lambda^m} \prod_{i=1}^{m} |f_i|(z_1, ..., z_m)|^{q_i} \prod_{i=1}^{m} d\mu_i(z_i) \leq c \prod_{i=1}^{m} \|f_i\|_{\tilde{H}^{p_i}}^{q_i}; \, (or \, H^p),$$

then μ_i measures are Carleson.

The proof of our assertion in tubular domain is based on two standard facts.

First the important estimate from below of Bergman kernel on Bergman balls (see [22]) and then an estimate of test function (Bergman type Kernel) on our Hardy type spaces.(see[22])

We must simply repeat our arguments provided already by us in the unit ball case.,again using standard weighted Bergman kernel as test function.

Indeed, we have from one hand that

$$\Delta^\tau(Imz) \leq |B_\nu(z,z) \leq c|B_\nu(z,w)|; \nu > 0; z \in B(w,r), w, z \in T_\Lambda, \nu > \nu_0;$$

for some τ, and from the other hand

$$(J_\alpha)(y) = \int_{R^n} |\Delta^\alpha \left(\frac{x+iy}{i}\right)|dx = (c_\alpha) \times |\Delta^{\alpha + \frac{n}{r}}(y)|; y \in \Lambda; \alpha \in R$$

and also that Δ^α is monotone on Λ (see [22]) which gives an estimate from above of a test function in Hardy space norm on product domains immediately.

This new interesting result can be probably sharp and can be similarly easily extended also to $H^{\vec{p}}(T_\Lambda^m)$ and $\widetilde{H}^{\vec{p}}(T_\Lambda^m)$new Hardy type spaces with mixed norm (see previous sections for same type spaces in other domains and for arguments needed for such an extension).

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О КЛАССАХ ТИПА ХАРДИ В НЕКОТОРЫХ ОБЛАСТЯХ В $\mathbb{C}^n$ И СВЯЗАННЫЕ С НИМИ ПРОБЛЕМЫ

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Введены несколько новых шкал пространств типа Харди со смешанной нормой в единичном шаре., в ограниченных псевдовыпуклых областях и в трубчатых областях над симметрическими конусами в $\mathbb{C}^n$. В этих пространствах обобщающих известное пространство Харди обсуждаются различные задачи. Для пространств такого типа в единичном шаре приводятся в частности точные многофункциональные теоремы вложения типа Карлесона, приводятся также некоторые многофункциональные максимальные теоремы. В трубчатых и в псевдовыпуклых областях получены некоторые прямые аналоги и частичные обобщения этих теорем вложения. При одном дополнительном интегральном условии получены теоремы декомпозиции для весовых мультифункциональных пространств Харди в областях указанного типа,обобщающие ранее известные теоремы такого рода в случае обычных однофункциональных весовых пространств Харди. Ранее первым автором теоремы такого типа были получены в многофункциональных пространствах Бергмана. Наконец вводится прямое обобщение интеграла типа Пуассона в произведении единичных шаров в $\mathbb{C}^n$ и обсуждаются некоторые задачи и обобщения известных результатов связанные с ним.

Ключевые слова: псевдовыпуклые, выпуклые, трубчатые области, теоремы вложения, лассы типа Харди, интеграл типа Пуассона

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