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HYPER-TAUBERIAN ALGEBRAS DEFINED BY A BANACH ALGEBRA HOMOMORPHISM

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Let A and B be Banach algebras and $T : B \longrightarrow A$ be a continuous homomorphism. We consider left multipliers from $A \times_T B$ into its the first dual i.e., $A^* \times B^*$ and we show that $A \times_T B$ is a hyper-Tauberian algebra if and only if A and B are hyper-Tauberian algebras.

Keywords: Local operator, hyper-Tauberian algebra, Tauberian algebra

УДК 519.53

ГИПЕРТАУБЕРОВЫ АЛГЕБРЫ, ОПРЕДЕЛЕННЫЕ ГОМОМОРФИЗМОМ БАНАХОВОЙ АЛГЕБРЫ

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Пусть A и B – банаховы алгебры, а $T : B \longrightarrow A$ – непрерывный гомоморфизм. Мы рассматриваем левые мультипликаторы из $A \times_T B$ в его первое двойственное, т.е. $A^* \times B^*$, и показываем, что $A \times_T B$ является гипертауберовой алгеброй тогда и только тогда, когда A и B являются гипертауберовыми алгебрами.

Ключевые слова: локальный оператор, гипертауберова алгебра, тауберова алгебра.

Introduction

The notion of Hyper-Tauberian algebras is introduced by Samei [20]. These algebras are commutative Banach algebras that consist of all Tauberian algebras. Idea of definition of hyper-Tauberian algebras is related to the local derivation that this notion was introduced by Kadison [13].

Let $(A, \|\cdot\|_A)$ be a Banach algebra and $(X, \|\cdot\|_X)$ be a Banach space such that X is an A-bimodule. If the module actions maps i.e., $A \times X \longrightarrow X$ and $X \times A \longrightarrow X$ are continuous (in norm), then we say that X is a Banach A-bimodule. Now, let X be a Banach A-bimodule, then one can see that the first dual of X i.e., X^* is a Banach A-bimodule with the following module actions:

$$\langle x, a \cdot f\rangle = \langle x \cdot a, f\rangle \quad \text{and} \quad \langle x, f \cdot a\rangle = \langle a \cdot x, f\rangle,$$

for every $a \in A$, $x \in X$ and $f \in X^*$. A derivation from A into a Banach A-bimodule X is a linear map $D : A \longrightarrow X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for every $a, b \in A$. The set of all derivations from A into X is denoted by $\mathscr{Z}^1(A,X)$; which is a linear subspace of $\mathscr{B}(A,X)$, the space of all bounded linear maps from A into X. For a fixed $x \in X$, set $D_x : A \longrightarrow X$, $a \mapsto a \cdot x - x \cdot a$. Derivations of this form are called inner derivations, and an inner derivation D_x is implemented by x. The set of all inner derivations from A into X is a linear subspace $\mathscr{N}^1(A,X)$ of $\mathscr{Z}^1(A,X)$. We denote the first cohomology group of a Banach algebra A with coefficients in a Banach A-bimodule X by $\mathscr{H}^1(A,X)$, where it is equal to $\mathscr{Z}^1(A,X)/\mathscr{N}^1(A,X)$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. An operator $D : A \longrightarrow X$ is called a local derivation if, for every $a \in A$, there is a derivation $D_a : A \longrightarrow X$ such that $D(a) = D_a(a)$. Kadison proved that every bounded local derivation from a von Neumann algebra A into a dual Banach A-bimodule X belongs to $\mathscr{Z}^1(A,X)$ and Johnson proved the same result to a C^*-algebra A and Banach A-bimodule X [12].

The concept of amenability for Banach algebras was introduced by Johnson [10]. A Banach algebra A is called amenable if $\mathscr{H}^1(A,X^*) = \{0\}$ for any A-bimodule X. A Banach algebra A is called weakly amenable if $\mathscr{H}^1(A,A^*) = \{0\}$ i.e., every continuous derivation from A into A^* is inner. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [3] for commutative Banach algebras, and was extended to the noncommutative case by Johnson, see [11].

Let A and B be Banach algebras such that A is a Banach B-bimodule with compatible actions and appropriate norm. The semidirect product of these Banach algebras is defined on $A \times B$ as follows:

$$(a,b)(a',b') = (aa' + a \cdot b' + b \cdot a', bb')$$

for $(a,b),(a',b') \in A \times B$. By the above defined product on $A \times B$, it becomes a Banach algebra with the ℓ^1-norm that we denote it by $A \ltimes B$. Moreover, if A and B are commutative such that A is a symmetric Banach B-bimodule (i.e., $a \cdot b = b \cdot a$ for every $a \in A$ and $b \in B$), then $A \ltimes B$ becomes a commutative Banach algebra.

Let A and B be Banach algebras and $\theta \in \sigma(B)$, where $\sigma(B)$ is the space of all continuous homomorphisms from B onto $\mathbb{C}$. Lau studied the Banach algebra $A \times_\theta B$ in [15], with the norm $\|(a,b)\| = \|a\|_A + \|b\|_B$ and with the following product:

$$(a,b)(a',b') = (aa' + \theta(b')a + \theta(b)a', bb'), \tag{1}$$

for all $(a,b),(a',b') \in A \times_\theta B$. Amenability and weak forms of amenability of $A \times_\theta B$ are studied in [7, 16]. Let $T : B \longrightarrow A$ be an algebra homomorphism, and A be a commutative Banach algebra. Following [5], we equip the Cartesian product space $A \times B$ with the following multiplication:

$$(a,b)(a',b') = (aa' + T(b)a' + T(b')a, bb'),$$

for all $(a,b),(a',b') \in A \times B$. By the above product, $A \times B$ becomes a Banach algebra; we denote it by $A \times_T B$. If A and B are Banach algebra and $\|T\| \leq 1$, then $A \times_T B$ is a Banach algebra with the following norm

$$\|(a,b)\| = \|a\|_A + \|b\|_B,$$

for $(a,b) \in A \times_T B$. Arens regularity and various notions of amenability of this new Banach algebra considered in [5]. With a slight difference in definition of the multiplication $\times_T$ from that given by Bhatt and Dabhi [5], we consider $A \times_T B$ with the following multiplication

$$(a,b)(a',b') = (aa' + T(b)a' + aT(b'), bb'), \qquad (a,a' \in A,\ b.b' \in B). \tag{2}$$

Note that if A is a commutative Banach algebra, then these multiplications coincide. Let A be a unital Banach algebra with unit e_A, $\theta \in \sigma(B)$, and define $T_0 : B \longrightarrow A$ by $T_0(b) = \theta(b)e_A$ $(b \in B)$. Then $A \times_{T_0} B$ coincides with the product (1). The Banach algebra $A \times_T B$ with the above multiplication that is a splitting of Banach algebra extension of Banach algebra B by A has been studied by many authors such as [1, 2, 6, 9, 17]. Splitting of Banach algebra extensions has important roles in studying of Banach algebras and they are good tools for giving counter examples for some concepts related to Banach algebras, for example see [4, 8, 23].

In this paper, we consider the Banach algebra $A \times_T B$ with the multiplication (2). We show that $A \times_T B$ is a Tauberian algebra if and only if A and B are Tauberian algebraa (Section 2) and in Section 3, we characterize left multipliers from $A \times_T B$ into its the first dual. Finally, we show that if A and B are hyper-Tauberian then $A \times_T B$ is a hyper-Tauberian and vice versa.

Tauberian algebra

In this section, we study on some basic properties of the Banach algebra $A \times_T B$. We identify $(A \times_T B)^*$ with $A^* \times B^*$ in the natural way

$$\langle (a,b),(f,g) \rangle = \langle a,f \rangle + \langle b,g \rangle$$

for all $(a,b) \in A \times_T B$ and $(f,g) \in A^* \times B^*$. Then by easy calculations, we have the following actions

$$(a,b) \cdot (f,g) = (a \cdot f + T(b) \cdot f, T^*(a \cdot f) + b \cdot g),$$

and

$$(f,g) \cdot (a,b) = (f \cdot a + f \cdot T(b), T^*(f \cdot a) + g \cdot b).$$

for all $(a,b) \in A \times_T B$ and $(f,g) \in (A \times_T B)^*$.

Lemma 1. *[9, Theorem 2.2] Let A and B be Banach algebras, and let $T : B \longrightarrow A$ be an algebra homomorphism with norm at most 1. Let $F_1 = \{(\varphi, \varphi \circ T) : \varphi \in \sigma(A)\}$ and $F_2 = \{(0, \psi) : \psi \in \sigma(B)\}$. Then* *(i)* *if* $\sigma(A) = \emptyset$*, then* $F_1 = \emptyset$.

(ii) $\sigma(A \times_T B) = F_1 \cup F_2$.

(iii) F_1 *and* F_2 *are closed in* $\sigma(A \times_T B)$.

We recall the following definitions and notions from [20]. For a Banach algebra A and a Banach A-bimodule X the annihilator of A in X and the annihilator of X in A are the following sets

$$Ann_X(A) = \{x \in X : x \cdot a = 0 = a \cdot x, \text{for all } a \in A\},$$

and

$$Ann_A(X) = \{a \in A : x \cdot a = 0 = a \cdot x, \text{for all } x \in X\}.$$

For a commutative, semisimple and regular Banach algebra A the hull of a closed ideal I in A denoted by $h(I)$. The hull of I is the following set

$$\{t \in \sigma(A) : a(t) = 0 \text{ for all } a \in I\}.$$

For any element $x \in X$, $Ann_A(x)$ is a closed ideal in A and the hull of $Ann_A(x)$ is called the support of x in $\sigma(A)$, denoted by $supp_A\, x$ or $supp\, x$. Let $E \subseteq \sigma(A)$, we consider the following sets

$$I(E) = \{a \in A : a|_E = 0\},$$

$$I_0(E) = \{a \in A : a \text{ has a compact disjoint from } E\},$$

and

$$J(E) = \overline{\{a \in I(E) : supp\ a \text{ is compact}\}}.$$

The subset E of $\sigma(A)$ is called a set of synthesis for A if there is a unique closed ideal in A whose hull is E. We denote the set of all elements in A with the compact support by A_c. The Banach algebra A is called Tauberian algebra if A_c is dense in A [19]. Ideals of $A \times_T B$ are investigated in [9, Proposition 2.4] and we write it as follows:

Lemma 2. *Let A and B be Banach algebras and $T : B \longrightarrow A$ be a homomorphism with $\|T\| \leq 1$. Then ideals of $A \times_T B$ are one of the following form*

(i) A.

(ii) $I \times_T B$*, where* I *is a closed ideal of* A *and* $T(B) \subseteq I$.

(iii) $I \times_T J$*, where* I *is a closed ideal of* A *and* J *is a closed ideal of* B *such that* $T(J) \subseteq I$.

By Lemma , any subset E of $\sigma(A \times_T B)$ is a subset of F_1 or F_2. In other word, $E = \{(\varphi, \varphi \circ T) :\ \text{for some } \varphi \in \sigma(A)\}$ or $E = \{(0, \varphi \circ T) :\ \text{for some } \varphi \in \sigma(A)\}$ or $E = \{(0, \psi) :\ \text{for some } \psi \in \sigma(B)\}$.

Theorem 1. *Let A and B be commutative, semisimple, regular Banach algebras and $T : B \longrightarrow A$ be a homomorphism with $\|T\| \leq 1$. Then $A \times_T B$ is a Tauberian algebra if and only if A and B are Tauberian algebras.*

Proof. Let $A \times_T B$ be a Tauberian algebra. For every $a \in A$, $(a, 0) \in A \times_T B$. Then there is a net

$$(a_\alpha, b_\alpha) \subseteq (A \times_T B)_c = A_c \cup B_c \cup \{(a, b) : a \in A_c,\ b \in B_c\},$$

such that $(a_\alpha, b_\beta) \longrightarrow (a,0)$. This follows that $a_\alpha \longrightarrow a$ and consequently, $\overline{A_c} = A$. Similarly, one can show that B is a Tauberian algebra.

Let A and B be Tauberian algebras and let $(a,b) \in A \times_T B$. Then there are nets $(a_\alpha)_{\alpha \in I} \subseteq A_c$ and $(b_\beta)_{\beta \in J} \subseteq B_c$ such that $a_\alpha \longrightarrow a$ and $b_\beta \longrightarrow b$. We define an indexing directed set $\Gamma = I \times \prod_{\alpha \in I} J$ equipped with the product ordering, and for each $(\alpha, f) \in \Gamma$, we define $c_\gamma = c_{\alpha, f(\alpha)} = c_{\alpha,\beta}$. Then by the Theorem on iterated limits [14],

$$\lim_{\gamma \in \Gamma} c_\gamma = \lim_{\alpha \in I} \lim_{\beta \in J} c_{\alpha,\beta}.$$

Now, set $c_\gamma = c_{\alpha,\beta} = (a_\alpha, b_\beta) \in (A \times_T B)_c$. By the above arguments, we conclude that $c_\gamma \longrightarrow (a,b)$. Thus, $A \times_T B$ is a Tauberian algebra. □

Left multipliers from $A \times_T B$ into $(A \times_T B)^*$

Let A be a Banach algebra and X be a left (right) Banach A-module. A linear mapping $T : A \longrightarrow X$ is called a left (right) multiplier if $T(ab) = a \cdot T(b)$ $(T(ab) = T(a) \cdot b)$. In this section we characterize left multipliers from $A \times_T B$ into $(A \times_T B)^*$ and a reason for investigating of left multipliers related to the next section.

Theorem 2. *Let A and B be Banach algebras and $T : A \longrightarrow B$ be a homomorphism with $\|T\| \leq 1$. If $F : A \times_T B \longrightarrow A^* \times B^*$ is a left multiplier, then*

(i) there are coordinate maps F_1 and F_2 such that $F = (F_1, F_2)$ and F_1 and F_2 are left multipliers on A and B, respectively.

(ii) $F_2(aa', 0) = T^*(a \cdot F_1(a', 0))$ and $F_1(0, bb') = T(b) \cdot F_1(0, b')$ for every $a, a' \in A$ and $b, b' \in B$.

(iii) if $Ann_A(A^*) \neq A$ or A is without of order, then $F_1(T(b), 0) = F_1(0, b)$ for every $b \in B$. Similarly, if $Ann_B(B^*) \neq B$ or B is without of order, then $T^*(F_1(a,0)) = F_2(a,0)$ for every $a \in A$.

Proof. Let $F = (F_1, F_2)$, where F_1 and F_2 are coordinate maps related to F and it is easy to check that they are linear and continuous.

(i)-(ii) For every $a, a' \in A$, we have

$$\begin{aligned} (F_1(aa',0), F_2(aa',0)) &= F(aa',0) = F((a,0)(a',0)) = (a,0) \cdot F(a',0) \\ &= (a,0) \cdot (F_1(a',0), F_2(a',0)) \\ &= (a \cdot F_1(a',0), T^*(a \cdot F_1(a',0))). \end{aligned} \tag{3}$$

Therefore $F_1(aa',0) = a \cdot F_1(a',0)$ and $F_2(aa',0) = T^*(a \cdot F_1(a',0))$ for every $a, a' \in A$. This means that F_1 on A is a left multiplier. For every $b, b' \in B$,

$$\begin{aligned} (F_1(0,bb'), F_2(0,bb')) &= F(0,bb') = F((b,0)(0,b')) = (0,b) \cdot F(0,b') \\ &= (0,b) \cdot (F_1(0,b'), F_2(0,b')) \\ &= (T(b) \cdot F_1(0,b'), b \cdot F_2(0,b')). \end{aligned} \tag{4}$$

The above relations show that $F_1(0,bb') = T(b) \cdot F_1(0,b')$ and $F_2(0,bb') = b \cdot F_2(0,b')$ for every $b, b' \in B$. This shows that F_2 is a left multiplier on B.

(iii) By (1) and (2) we have

$$\begin{aligned}
F((a,b)(a',b')) &= F(aa'+aT(b')+T(b)a',bb') \\
&= (F_1(aa'+aT(b')+T(b)a',bb'),F_2(aa'+aT(b')+T(b)a',bb')) \\
&= (F_1(aa',0),0)+(F_1(aT(b'),0),0)+(F_1(T(b)a',0),0) \\
&\quad +(F_1(0,bb'),0)+(0,F_2(aa',0))+(0,F_2(aT(b'),0)) \\
&\quad +(0,F_2(T(b)a',0))+(0,F_2(0,bb')) \\
&= (a\cdot F_1(a',0),0)+(a\cdot F_1(T(b'),0),0)+(T(b)\cdot F_1(a',0),0) \\
&\quad (T(b)\cdot F_1(0,b'),0)+(0,T^*(a\cdot F_1(a',0))) \\
&\quad +(0,T^*(a\cdot F_1(T(b'),0)))+(0,T^*(T(b)\cdot F_1(a',0))) \\
&\quad +(0,b\cdot F_2(0,b'))
\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}
(a,b)\cdot F(a',b') &= (a,b)\cdot F((a',0)+(0,b'))=(a,b)\cdot F(a',0)+(a,b)\cdot F(0,b') \\
&= (a,b)\cdot(F_1(a',0),F_2(a',0))+(a,b)\cdot(F_1(0,b'),F_2(0,b')) \\
&= (a\cdot F_1(a',0)+T(b)\cdot F_1(a',0),T^*(a\cdot F_1(a',0))+b\cdot F_2(a',0)) \\
&\quad +(a\cdot F_1(0,b')+T(b)\cdot F_1(0,b'),T^*(a\cdot F_1(0,b'))+b\cdot F_2(0,b')) \\
&= (a\cdot F_1(a',0),0)+(T(b)\cdot F_1(a',0),0)+(0,T^*(a\cdot F_1(a',0))) \\
&\quad +(0,b\cdot F_2(a',0))+(a\cdot F_1(0,b'),0)+(T(b)\cdot F_1(0,b'),0) \\
&\quad +(0,T^*(a\cdot F_1(0,b')))+(0,b\cdot F_2(0,b'))
\end{aligned} \tag{6}$$

The relations (5) and (6) imply that

$$\begin{cases} a\cdot F_1(T(b'),0)=a\cdot F_1(0,b') \\ \\ T^*(T(b)\cdot F_1(a',0))=b\cdot F_2(a',0). \end{cases} \tag{7}$$

For every $x\in B^*$, $b\in B$ and $a'\in A$,

$$\begin{aligned}
\langle x,b\cdot F_2(a',0)\rangle &= \langle x,T^*(T(b)\cdot F_1(a',0))\rangle=\langle T(x),T(b)\cdot F_1(a',0)\rangle \\
&= \langle T(x)T(b),F_1(a',0)\rangle=\langle T(xb),F_1(a',0)\rangle \\
&= \langle xb,T^*(F_1(a',0))\rangle=\langle x,b\cdot T^*(F_1(a',0))\rangle.
\end{aligned} \tag{8}$$

Then we can write (7) as follows:

$$\begin{cases} a\cdot F_1(T(b'),0)=a\cdot F_1(0,b') \\ \\ b\cdot T^*(F_1(a',0))=b\cdot F_2(a',0). \end{cases} \tag{9}$$

Now if one of the assumptions in (iii) holds, we conclude the desire. □

We now consider the converse of the above Theorem as follows:

Theorem 3. *Let A and B be Banach algebras and $T:A\longrightarrow B$ be a homomorphism with $\|T\|\le 1$. If $F_A:A\longrightarrow A^*$ and $F_B:B\longrightarrow B^*$ are left multipliers, then $F:A\times_T B\longrightarrow A^*\times B^*$ defined as*

$$F(a,b)=(F_A(a+T(b)),T^*\circ F_A(a+T(b))+F_B(b)) \tag{10}$$

for every $(a,b) \in A \times_T B$, *is a left multiplier.*

Proof. For every $(a,b),(a',b') \in A \times_T B$, we have

$$\begin{aligned}(a,b)\cdot F(a',b') &= (a,b)\cdot(F_A(a'+T(b')),T^*\circ F_A(a'+T(b'))+F_B(b')) \\ &= (a\cdot F_A(a'+T(b'))+T(b)\cdot F_A(a'+T(b')),b\cdot F_B(b') \\ &\quad +b\cdot T^*\circ F_A(a'+T(b'))+T^*(a\cdot F_A(a'+T(b')))). \end{aligned} \tag{11}$$

Also, for all $a,a' \in A$ and $b,b',x \in B$, we have

$$\begin{aligned}\langle x,T^*\circ F_A(T(b)a'+T(bb'))\rangle &= \langle T(x),F_A(T(b)a'+T(bb'))\rangle \\ &= \langle T(x),T(b)\cdot F_A(a'+T(b'))\rangle \\ &= \langle xb,T^*(F_A(a'+T(b')))\rangle \\ &= \langle x,b\cdot T^*(F_A(a'+T(b')))\rangle. \end{aligned} \tag{12}$$

Then by (12) we have the following

$$\begin{aligned}F((a,b)(a',b')) &= (F_A(aa'+T(b)a'+aT(b')+T(bb')),T^*\circ F_A(aa'+aT(b')) \\ &\quad +T^*\circ F_A(T(b)a'+T(bb'))+F_B(bb')) \\ &= (a\cdot F_A(a'+T(b'))+T(b)\cdot F_A(a'+T(b')),b'\cdot F_B(b') \\ &\quad +b\cdot T^*\circ F_A(a'+T(b'))+T^*(a\cdot F_A(a'+T(b')))). \end{aligned} \tag{13}$$

Then the relations (11) and (13) imply that F is a left multiplier. □

Hyper-Tauberian algebra

Let X and Y be left (right) Banach A-modules. An operator $T : X \longrightarrow Y$ is called local with respect to the left (right) A-module action if *supp* $T(x) \subseteq$ *supp* x, for every $x \in X$. If A is a Tauberian algebra and X is a left (right) Banach A-module, then a bounded operator $T : A \longrightarrow X$ is called local if *supp* $T(a) \subseteq$ *supp* a, for every $a \in A_c$ [20, Proposition 2].

In this section we assume that all Banach algebras are commutative, semisimple and regular. The Banach algebra A is a said to be a hyper-Tauberian algebra if every bounded local operator $T : A \longrightarrow A^*$ is a multiplier. Hyper-Tauberian algebras are defined by Samei in [20]. He proved that every hyper-Tauberian algebra is Tauberian algebra and is weakly amenable [20, Theorem 5]. In light of Lemma , we have the following Lemma.

Lemma 3. *Let A and B be Banach algebras and $T : B \longrightarrow A$ be a homomorphism with $\|T\| \leq 1$. Then*

(i) *supp* $(a,0) = \{t \in \sigma(A) : a(t) \neq 0\}$.

(ii) *supp* $(0,b) = \{s \in \sigma(B) : b(s) \neq 0\}$.

(iii) *supp* $(a,b) = \{(t,s) \in \sigma(A \times_T B) : a(t)+b(s) \neq 0\}$.

Theorem 4. *Let A and B be Banach algebras and $T : B \longrightarrow A$ be a homomorphism with $\|T\| \leq 1$. Then $A \times_T B$ is a hyper-Tauberian algebra if and only if A and B are hyper-Tauberian algebras.*

Proof. Let A and B be hyper-Tauberian algebras. Since $\frac{A\times_T B}{A}\cong B$, so $\frac{A\times_T B}{A}$ is a hyper-Tauberian algebra. Then by [20, Theorem 9], $A\times_T B$ is hyper-Tauberian.

Let $A\times_T B$ be hyper-Tauberian. First, we show that A is hyper-Tauberian. Let $F: A\longrightarrow A^*$ be a bounded local operator. Consider the projection map $\pi_A: A\times_T B\longrightarrow A$ defined by $\pi_A(a,b)=a+T(b)$, for all $(a,b)\in A\times_T B$. Then $\pi_A^*\circ F\circ\pi_A: A\times_T B\longrightarrow A^*\times B^*$ is a bounded local operator. Because by Lemma 3, we have

$$\begin{aligned}
supp\ \pi_A^*\circ F\circ\pi_A(a,b) &= supp\ \pi_A^*\circ F(a+T(b))\subseteq supp\ \pi_A^*(a+T(b))\\
&= supp(a+T(b),0)\\
&= \{t\in\sigma(A):(a+T(b))(t)\neq 0\}\\
&= \{(t,t\circ T)\in\sigma(A\times_T B):(a+T(b))(t)\neq 0\}\\
&\subseteq supp(a,b).
\end{aligned}$$

This means that $\pi_A^*\circ F\circ\pi_A$ is local and therefore it is a multiplier. Then

$$\begin{aligned}
\pi_A^*\circ F\circ\pi_A((a,b)(a',b')) &= (a,b)\cdot\pi_A^*\circ F\circ\pi_A((a',b'))\\
&= (a,b)\cdot\pi_A^*\circ F(a'+T(b'))=(a,b)\cdot(F(a'+T(b')),0)\\
&= (a\cdot F(a'+T(b'))+T(b)\cdot F(a'+T(b')),T^*(a\cdot F(a'+T(b'))))\\
&= (a\cdot F(a')+a\cdot F(T(b'))+T(b)\cdot F(a')\\
&\quad +T(b)\cdot F(T(b')),T^*(a\cdot F(a'+T(b')))). \qquad (14)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\pi_A^*\circ F\circ\pi_A((a,b)(a',b')) &= \pi_A^*\circ F\circ\pi_A(aa'+T(b)a'+aT(b'),bb')\\
&= \pi_A^*\circ F(aa'+T(b)a'+aT(b')+T(bb'))\\
&= (F(aa'+T(b)a'+aT(b')+T(bb')),0)\\
&= (F(aa')+F(T(b)a')+F(aT(b'))+F(T(bb'))),0). \qquad (15)
\end{aligned}$$

By taking $b=b'=0$ and using relations (14) and (15), we conclude that F is a multiplier. This shows that A is hyper-Tauberian.

Finally, we prove that B is hyper-Tauberian. Define $F: A\times_T B\longrightarrow\frac{A\times_T B}{A}\cong B$ by $F(a,b)=(a,b)+A$, for every $(a,b)\in A\times_T B$. Clearly, F is a bounded and onto homomorphism. Since $A\times_T B$ is hyper-Tauberian algebra, by [20, Theorem 12], B is hyper-Tauberian. □

Amenability of $A\times_T B$ studied in [6]. Authors in [6, 18] proved that weak amenability of $A\times_T B$ implies weak amenability of A and B, but converse is not true in general. Samei in [20] proved that every hyper-Tauberian algebra is weakly amenable. By this fact and above Theorem we have the following result.

Corollary. *Let A and B be hyper-Tauberian algebras and $T: B\longrightarrow A$ be a homomorphism with $\|T\|\leq 1$. Then $A\times_T B$ is weakly amenable if and only if A and B are weakly amenable.*

Example. Let G and H be locally compact abelian groups and $A(G)$ and $A(H)$ be Fourier algebras on them. Define $T: A(G)\longrightarrow A(H)$ by $T(\omega)(h)=\omega(\tau(h))$ for every $h\in H$, where $\tau: H\longrightarrow G$ is continuous. According to [22], if T is an isometry, then τ is of the form $\tau(h)=g\phi(h)$ for every $g\in G$ and $\phi: H\longrightarrow G$ is a group homomorphism. Thus, $A(G)\times_T A(H)$ is a hyper-Tauberian algebra, by [20, Proposition 18] and Theorem

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