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Research Article

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On the coefficient bounds of certain subclasses of analytic functions of complex order

Nizami MUSTAFA, Tarkan ÖZTÜRK

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Abstract In this paper, we introduce and examine a new subclass $S^*C(\beta,\tau)$ of analytic functions of complex order in the open unit disk. Here, we obtain upper bound estimates for the first three coefficients for the functions belonging to this class.

Keywords Analytic function; coefficient bound; starlike function; convex function; analytic function of complex order
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1. Introduction

Let A be the class of analytic functions $f(z)$ in the open unit disk $U=\{z\in\mathbb{C} : |z|<1\}$, normalized by $f(0)=0=f'(0)-1$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n, a_n\in\mathbb{C} . \tag{1.1}$$

Also, let us define by T the subclass of all functions $f(z)$ in A of the form

$$f(z)=z-a_2z^2-a_3z^3-\cdots-a_nz^n-\cdots=z-\sum_{n=2}^{\infty}a_nz^n, a_n\geq 0. \tag{1.2}$$

It is well-known that a function $f:\mathbb{C}\to\mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1=z_2$ if $f(z_1)=f(z_2)$ or $f(z_1)\neq f(z_2)$ if $z_1\neq z_2$. We denote by S the subclass of A consisting of functions which are also univalent in U.

Also, we will denote by S^* and C the subclasses of S that are, respectively, starlike and convex functions in the open unit disk U.

By definition (see for details, [3, 4], also [6])

$$S^*=\left\{f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U\right\}, \tag{1.3}$$

and

$$C=\left\{f\in S:\ \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in U\right\}. \tag{1.4}$$

It is easy to verify that $C \subset S^* \subset S$. For details on these classes, one could refer to the monograph by Goodman [4].

An interesting unification of the functions classes S^* and C is provided by the class $S^*C(\beta)$ of functions $f \in S$, which also satisfies the following condition:

$$\operatorname{Re}\left\{ \frac{zf'(z)) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} \right\} > 0, \beta \in [0,1],\ z \in U.$$

Also, we will denote $TS^*C(\beta) = T \cap S^*C(\beta)$. Note that the class $TS^*C(\beta)$ has been examined by Altıntaş et al [1, 2].

In special case, for $\beta = 0$ and $\beta = 1$, respectively, we have $S^*C(0) = S^*$ and $S^*C(1) = C$, in terms of the simpler classes S^* and C, defined by (1.2) and (1.3), respectively.

e define a subclass of analytic functions as follows.

Definition 1.1. A function $f \in S$ given by (1.1) is said to be in the class $S^*C(\beta,\tau)$, $\beta \geq 0, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$ if the following condition is satisfied

$$\operatorname{Re}\left\{ 1 + \frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} - 1 \right] \right\} > 0, z \in U. \tag{1.4}$$

In special case, we have $S^*C(\beta,1) = S^*C(\beta)$ for $\tau = 1$.

The object of the present paper is to obtained upper bound estimates for the first three coefficients for the functions belonging to the class $S^*C(\beta,\tau),\ \beta \geq 0,\ \tau \in \mathbb{C}^*$.

To prove our main results, we need require the following lemma.

Lemma 1.1 ([5]). *If $p \in \mathrm{P}$, then the estimates $|p_n| \leq 2, n = 1,2,3,\ldots$ are sharp, where P is the family of all functions p, analytic in U for which $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ $(z \in U)$, and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in U. \tag{1.5}$$

2. Upper bound estimates for the coefficients

In this section, we will obtain upper bound estimates for the first three coefficient of the functions belonging to the class $S^*C(\beta,\tau), \beta \geq 0,\ \tau \in \mathbb{C}^*$.

The following theorem is on upper bound estimates for the coefficient of the functions belonging to this class.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $S^*C(\beta,\tau), \beta \geq 0,\ \tau \in \mathbb{C}^*$. Then,*

$$|a_2| \leq \frac{2|\tau|}{1+\beta} \quad \text{and} \quad |a_3| \leq \frac{|\tau|(2|\tau|+1)}{1+2\beta}.$$

Also,

$$|a_4| \leq \frac{2|\tau|\left(2|\tau|^2 + 3|\tau| + 1\right)}{3(1+3\beta)}.$$

Proof. Let $f \in S^*C(\beta,\tau),\ \beta \geq 0, \tau \in \mathbb{C}^*$. It follows that

$$1+\frac{1}{\tau}\left[\frac{zf'(z)+\beta z^2 f''(z)}{\beta zf'(z)+(1-\beta)f(z)}-1\right]=p(z), z\in U, \tag{2.1}$$

where function $p(z)=1+\sum_{n=1}^{+\infty}p_n z^n$ is in the class P.

The equation (2.1), we can write as follows:

$$\sum_{n=2}^{+\infty}(n-1)\left[1+(n-1)\beta\right]a_n z^n=\tau\left\{z+\sum_{n=2}^{+\infty}\left[1+(n-1)\beta\right]a_n z^n\right\}\sum_{n=1}^{+\infty}p_n z^n.$$

Therefore,

$$\begin{gathered}(1+\beta)a_2 z^2+2(1+2\beta)a_3 z^3+3(1+3\beta)a_4 z^4+\ldots=\\ \tau\left\{p_1 z^2+\left[(1+\beta)a_2 p_1+p_2\right]z^3+\left[(1+2\beta)a_3 p_1+(1+\beta)a_2 p_2+p_3\right]z^4+\ldots\right\}.\end{gathered} \tag{2.2}$$

Comparing the coefficients of the like power of z in both sides of (2.2), we have

$$(1+\beta)a_2=\tau p_1, \tag{2.3}$$

$$2(1+2\beta)a_3=\tau\left[(1+\beta)a_2 p_1+p_2\right], \tag{2.4}$$

$$3(1+3\beta)a_4=\tau\left[(1+2\beta)a_3 p_1+(1+\beta)a_2 p_2+p_3\right]. \tag{2.5}$$

From these, we get

$$a_2=\frac{\tau}{1+\beta}p_1, \tag{2.6}$$

$$a_3=\frac{\tau}{2(1+2\beta)}p_2+\frac{\tau^2}{2(1+2\beta)}p_1^2, \tag{2.7}$$

$$a_4=\frac{\tau}{3(1+3\beta)}p_3+\frac{\tau^2}{2(1+3\beta)}p_1 p_2+\frac{\tau^3}{6(1+3\beta)}p_1^3. \tag{2.8}$$

Since $|p_1|\le 2$, from (2.6), we obtain

$$|a_2|\le\frac{2|\tau|}{1+\beta}. \tag{2.9}$$

Using triangle inequality and applying the inequalities $|p_n|\le 2, n=1,2$, from (2.7) we obtain

$$|a_3|\le\frac{|\tau|(2|\tau|+1)}{1+2\beta}. \tag{2.10}$$

Similarly, from (2.8), we have

$$|a_4|\le\frac{2|\tau|\left(2|\tau|^2+3|\tau|+1\right)}{3(1+3\beta)}. \tag{2.11}$$

Thus, from (2.9)-(2.11) the proof of Theorem2.1 is completed.

Setting $\tau=1$ in Theorem 2.1, we can readily deduce the following result.

Corollary 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*C(\beta), \beta\ge 0$. *Then,*

$$|a_2|\le\frac{2}{1+\beta}, \quad |a_3|\le\frac{3}{1+2\beta}, \quad |a_4|\le\frac{4}{1+3\beta}.$$

Setting $\beta=0$ and $\beta=1$ in Corollary 2.1, we can readily deduce the following results, respectively.

Corollary 2.2. *Let the function* $f(z)$ *given by (1.1) be in the class* S^*. *Then,*

$$|a_n| \le n,\ n = 2,3,4.$$

Corollary 2.3. *Let the function* $f(z)$ *given by (1.1) be in the class* C. *Then,*

$$|a_n| \le 1,\ n = 2,3,4.$$

Note 2.1. As you can see, Corollary 2.2 confirmed that the Bieberbach's Conjecture (see for example [3]) $|a_n| \le n$ has been provided for $n = 2,3,4$.

Remark 2.1. Using this work, we can be examined $|a_3 - \mu a_2^2|$ the Fekete - Szegö problem for the coefficients of the function class $S^*C(\beta,\tau)$. Also, using this work we can be find $H_2(2) = a_2 a_4 - a_3^2$ second Hankel determinant for the functions belonging in the class $S^*C(\beta,\tau)$. Hence, we find upper bound estimate for the $|a_2 a_4 - a_3^2|$.

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References

[1]. Altıntaş, O., Irmak, H. and Srivastava , H.M., Fractional calculus and certain starlike functions with negative coefficients. Computers & Mathematics with Applications, 30(1995), No.2, 9-16

[2]. Altıntaş , O., Özkan , Ö. and Srivastava , H.M., Neighbourhoods of a Certain Family of Multivalent Functions with Negative Coefficients. Computers & Mathematics with Applications, 47 (2004), 1667-1672

[3]. Duren, P. L., Univalent Functions, Grundlehren der Mathematischen Wissenshaften. Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[4]. Goodman, A. W., Univalent Functions. Volume I, Polygonal, Washington, 1983.

[5]. Pommerenke CH. Univalent Functions. Vandenhoeck and Rupercht, Göttingen, 1975.

[6]. Srivastava, H. M. and Owa, S., Editors, Current Topics in Analytic Funtion Theory. World Scientific, Singapore, 1992.