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Research Article

Univalence of the Integral Operators Involving Produced Wright Function

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Abstract In this paper, we introduce and investigate an analytic function $\phi_{\lambda,\mu}(z)$ which produced from the Wright function. Here, some sufficient conditions for the integral operators involving produced Wright function to be univalent in the open unit disk are also given. The key tools in our proofs are the Becker's and the generalized version of the well-known Ahlfor's and Becker's univalence criteria.

Keywords Analytic function, Wright function, Alfor's univalence criteria, Becker's univalence criteria
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1. Introduction

Let A be the class of analytic functions $f(z)$ in the open unit disk $U=\{z\in\mathbb{C} : |z|<1\}$, normalized by $f(0)=0=f'(0)-1$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n ,\ a_n\in\mathbb{C} . \tag{1.1}$$

It is well-known that a function $f:\mathbb{C}\to\mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1=z_2$ if $f(z_1)=f(z_2)$ or $f(z_1)\neq f(z_2)$ if $z_1\neq z_2$. We denote by S the subclass of A consisting of functions which are also univalent in U.

In recent years there have been many studies (see for example [2 - 4]) on the univalence of the following integral operators:

$$G_p(z)=\left\{p\int_0^z t^{p-1}f'(t)dt\right\}^{1/p} ,\ z\in U , \tag{1.2}$$

$$G_{p,q}(z)=\left\{p\int_0^z t^{p-1}\left(\frac{f(t)}{t}\right)^q dt\right\}^{1/p} ,\ z\in U , \tag{1.3}$$

and

$$G_q(z)=\left\{q\int_0^z t^{q-1}\left(e^{f(t)}\right)^q dt\right\}^{1/q} ,\ z\in U , \tag{1.4}$$

where the function $f(z)$ belong to the class A and the parameter p,q are complex numbers such that the integrals in (1.2) – (1.4) exist.

Furthermore, Breaz *et al.* [5] have obtained various sufficient conditions for the univalence of the following integral operator:

$$G_{n,\alpha}(z)=\left\{\left[n(\alpha-1)+1\right]\int_0^z\left(\prod_{k=1}^n f_k(t)\right)^{\alpha-1}dt\right\}^{1/\left[n(\alpha-1)+1\right]}, \quad (1.5)$$

where n is a natural number, α is a real number and the functions $f_k\in A,\ k=1,\dots,n.$ By Baricz and Frasin [1] was obtained some sufficient conditions for the univalence of the integral operators of the type (1.3) – (1.5), when the function $f(z)$ is the normalized Bessel functions of the first kind.

In this paper, we investigate univalence of the integral operators type (1.3) - (1.5), when $f(z)$ is a function in A which produced from the Wright function.

For this end, we will give some preliminaries on the Wright function. It is well-known that the Wright function is defined by the following infinite series:

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\lambda n+\mu)}\frac{z^n}{n!}, \quad (1.6)$$

where $\lambda>-1,\ \mu,z\in\mathbb{C}$ and Γ is well-known Euler Gamma function. This series is absolutely convergent in $\mathbb{C}$, when $\lambda>-1$ and absolutely convergent in open unit disk for $\lambda=-1$. Furthermore, for $\lambda>-1$ the Wright function $W_{\lambda,\mu}(z)$ is an entire function. The Wright function was introduced by Wright in [10] and has appeared for the first time in the case $\lambda>0$ in connection wit his investigation in the asymptotic theory of partitions. Later on, it has found many other applications, first of all, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Furthermore, extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright function.

It is clear that Wright function $W_{\lambda,\mu}(z)$, defined by (1.6) does not belong to the class A. Thus, it is natural to consider the following function produced from Wright function:

$$\phi_{\lambda,\mu}(z):=\Gamma(\lambda+\mu)zW_{\lambda,\mu}{}'(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n+1)+\mu)}\frac{z^{n+1}}{n!},$$

which is easy to write

$$\phi_{\lambda,\mu}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)}\frac{z^n}{(n-1)!},\ \lambda>-1,\ \lambda+\mu>0,\ z\in U. \quad (1.7)$$

Throughout the study, we will say that the function $\phi_{\lambda,\mu}(z), z\in U$ is produced from Wright function. It is clear that $\phi_{\lambda,\mu}\in A$.

It is easily shown that

$$\phi_{1,\mu}(-z)=-\Gamma(\mu+1)z^{1-\mu/2}J_{\mu}\left(2\sqrt{z}\right)=-\frac{1}{4}J_{\mu}^{N}(4z),$$

where $J_{\mu}(z)$ is the Bessel function first kind and $J_{\mu}^{N}(z)$ is the normalized Bessel function.

It is well known that the Bessel function first kind $J_{\mu}(z)$ is defined as the particular solution of the second-order linear homogeneous differential equation (see, for example [8])

$$z^2w''(z)+zw'(z)+\left(z^2-\mu^2\right)w(z)=0.$$

This equation also is named Bessel differential equation. It is also well known that normalized Bessel function is defined as follows (see [8])

$$J_\mu^N(z) = 2^\mu \Gamma(\mu+1) z^{1-\mu/2} J_\mu\left(\sqrt{z}\right), \mu \neq -1, -2, \dots .$$

It can be easily seen that the function $\phi_{\lambda,\mu}(z)$ is satisfies the following homogeneous differential equations first and second order,

$$z\phi_{\lambda,\mu}{}'(z) - \phi_{\lambda,\mu}(z) - a(\lambda,\mu) z\phi_{\lambda,\lambda+\mu}(z) = 0, \tag{1.8}$$

$$z^2\phi_{\lambda,\mu}{}''(z) - 2z\phi_{\lambda,\mu}{}'(z) + 2\phi_{\lambda,\mu}(z) - b(\lambda,\mu)\phi_{\lambda,2\lambda+\mu}(z) = 0, \tag{1.9}$$

where

$$a(\lambda,\mu) = \Gamma(\lambda+\mu)/\Gamma(2\lambda+\mu) = 1/(\lambda+\mu)_\lambda,$$

$$b(\lambda,\mu) = \Gamma(\lambda+\mu)/\Gamma(3\lambda+\mu) = 1/(\lambda+\mu)_{2\lambda},$$

$(x)_r = \Gamma(r+x)/\Gamma(x), (x)_0 = 1$ is Pochhammer (or Appell) symbol defined in terms of the Euler Gamma function.

Note 1.1. We hope the function $\phi_{\lambda,\mu}(z)$ finds wide application area in mathematics and other appropriate fields.

Note that, the function

$$W_{\lambda,\mu}^{(1)}(z) := \Gamma(\mu) z W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \frac{z^{n+1}}{n!} =$$

$$z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{(n-1)!}, \ \lambda > -1, \ \mu > 0, \ z \in U$$

is named the normalized Wright function (for this, see for example [6,9]) .

In this paper, we give various sufficient conditions for the integral operators of type (1.2) - (1.4), when the functions $f_k(z), \ k = 1, 2, \dots, n$ and $f(z)$ are the produced Wright function $\phi_{\lambda,\mu}(z)$ to be univalent in the open unit disk U. Here, we would like to show that the univalence of integral operators which involve the function $\phi_{\lambda,\mu}(z)$ can be derived easily via some well-known univalence criteria.

In the introduction and preliminaries section of the paper, we provide the necessary information to prove our main results. In the third section we give the main results.

2. Preliminaries

In this section, we give the necessary lemmas, which shall need in our investigation.

Lemma 2.1 [7]. *If $f \in A$ and the following condition is satisfied:*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then the function $f(z)$ is univalent in U.

Lemma 2.2 [7]. *Let $q \in \mathbb{C}$ and $a \in \mathbb{C}$ such that $\mathrm{Re}(q) \geq 1, \ a > 1$ and $2a|q| \leq 3\sqrt{3}$. If $f \in A$ satisfies the inequality $|zf'(z)| \leq a$ for all $z \in U$, then the function $G_q : U \to \mathbb{C}$ defined by (1.4) univalent in U.*

Lemma 2.3 [7]. *Let p and c be complex numbers such that $\mathrm{Re}(p) > 0$ and $|c| \leq 1, c \neq -1$. If the function $f \in A$ satisfies the inequality*

$$\left| c|z|^{2p} + \left(1-|z|^{2p}\right)\frac{zf''(z)}{pf'(z)} \right| \le 1$$

for all $z \in U$ *, the function* $G_p : U \to \mathbb{C}$ *defined by (1.2) is univalent in* U *.*

We shall need, also the following lemma.

Lemma 2.4. *Let* $\lambda \ge 1$ *and* $\lambda+\mu > x_0$ *where* $x_0 = 1.2581$ *is the numerical root of the equation*

$$2x-(x+1)e^{1/(x+1)}+1=0. \tag{2.1}$$

Then the following inequalities hold for all $z \in U$

$$\left|\frac{z\left(\phi_{\lambda,\mu}(z)\right)'}{\phi_{\lambda,\mu}(z)}-1\right| \le \frac{e^{1/(\lambda+\mu+1)}}{2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}}, \tag{2.2}$$

$$\left|z\left(\phi_{\lambda,\mu}(z)\right)'\right| \le \frac{1}{\lambda+\mu}\left\{(\lambda+\mu+2)e^{1/(\lambda+\mu+1)}-1\right\}. \tag{2.3}$$

Proof. By using the definition of the function $\phi_{\lambda,\mu}(z)$, we obtain

$$\left|\frac{z\left(\phi_{\lambda,\mu}(z)\right)'}{\phi_{\lambda,\mu}(z)}-1\right| = \left|\frac{z\left(\phi_{\lambda,\mu}(z)\right)'-\phi_{\lambda,\mu}(z)}{\phi_{\lambda,\mu}(z)}\right| \le \frac{\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)}\frac{1}{(n-2)!}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)}\frac{1}{(n-1)!}}. \tag{2.4}$$

Under hypothesis $\lambda \ge 1$ of the lemma the inequality $\Gamma(n-1+\lambda+\mu) \le \Gamma\left(\lambda(n-1)+\lambda+\mu\right)$, $n \in \mathbb{N}$ holds true. Since $\Gamma(n+x)=\Gamma(x)(x)_n$, where $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$, $(x)_0 = 1$ is Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function, we obtain

$$\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)} = \frac{\Gamma(\lambda+\mu)}{\Gamma\left(\lambda(n-1)+\lambda+\mu\right)} \le \frac{1}{(\lambda+\mu)_{n-1}}, n \in \mathbb{N} . \tag{2.5}$$

Also, the inequality

$$(\lambda+\mu)_{n-1} = (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-2) \ge (\lambda+\mu)(\lambda+\mu+1)^{n-2}$$

is true for all $n \in \mathbb{N}$, which is equivalent to

$$\frac{1}{(\lambda+\mu)_{n-1}} \le \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}, n \in \mathbb{N} . \tag{2.6}$$

Using (2.5) and (2.6), we get

$$\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)}\frac{1}{(n-2)!} \le \sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}\frac{1}{(n-2)!} = \frac{e^{1/(\lambda+\mu+1)}}{\lambda+\mu}. \tag{2.7}$$

Similarly, we have

$$\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)}\frac{1}{(n-1)!} \le \frac{\lambda+\mu+1}{\lambda+\mu}\left(e^{1/(\lambda+\mu+1)}-1\right). \tag{2.8}$$

With simple computer programming, we can see that $\varphi'(x) = 2-\frac{x}{x+1}e^{1/(x+1)} > 0$ for all $x > 0$, where $\varphi(x) = 2x+1-(x+1)e^{1/(x+1)}$. Hence, the function $\varphi(x), x > 0$ is strongly increasing function. Also, with simple computer programming, we can see that the numerical root of the equation (2.1) is $x_0 = 1.2581$. Therefore, if $x > x_0$, then $\varphi(x) > 0$ (see also figure 1).

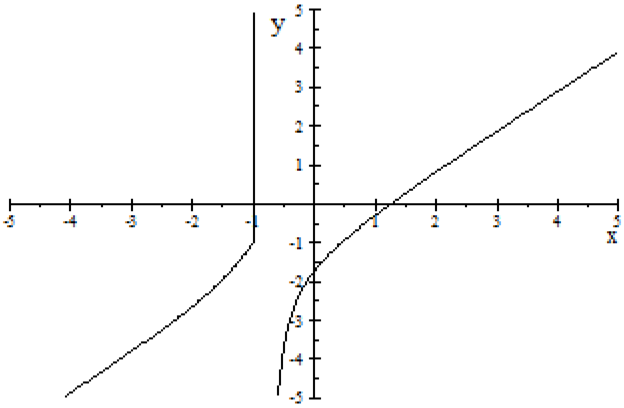

Figure 1: Graphic of the function $\varphi(x) = 2x + 1 - (x+1)e^{1/(x+1)}$.

Thus, if $\lambda + \mu > x_0$ from (2.4), (2.7) and (2.8), we immediately get that first assertion of the lemma holds.

Now, we will prove the second assertion of lemma. From the definition of the function $\phi_{\lambda,\mu}(z)$, we have

$$\left| z\left(\phi_{\lambda,\mu}(z)\right)' \right| \le 1 + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)} \frac{1}{(n-2)!} + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\mu)} \frac{1}{(n-1)!}.$$

Using (2.7) and (2.8), we obtain

$$\left| z\left(\phi_{\lambda,\mu}(z)\right)' \right| \le 1 + \frac{e^{1/(\lambda+\mu+1)}}{\lambda+\mu} + \frac{\lambda+\mu+1}{\lambda+\mu}\left(e^{1/(\lambda+\mu+1)} - 1\right),$$

which is same of the second assertion of lemma.
Thus, the proof of Lemma 2.4 is completed.

3. Univalence of certain integral operators

In this section our main aim is give sufficient univalence conditions for the integral operators of the type (1.2) - (1.4), when the function $f(z)$ is the function which defined by (1.7).

Firstly we consider the following integral operator

$$G_{\lambda,\mu}^{q}(z) = \int_0^z \left(\frac{\phi_{\lambda,\mu}(t)}{t} \right)^q dt,\ \lambda > -1,\ \lambda+\mu > 0,\ z \in U. \qquad (3.1)$$

On the univalence of this integral operator, we give the following theorem.

Theorem 3.1. *Let* $\lambda \ge 1$ *and* $\lambda + \mu > x_0$, *where* $x_0 = 1.2581$ *is the numerical root of the equation (2.1). Moreover, suppose that* q *is a complex number such that*

$$|q| \le \frac{\left(2(\lambda+\mu)+1\right) - (\lambda+\mu+1)e^{1/(\lambda+\mu+1)}}{e^{1/(\lambda+\mu+1)}}. \qquad (3.2)$$

Then, the integral operator $G_{\lambda,\mu}^{q} : U \to \mathbb{C}$ *defined by (3.1) is univalent in* U.

Proof. Since $\phi_{\lambda,\mu} \in A$, clearly $G_{\lambda,\mu}^{q} \in A$, i. e. $G_{\lambda,\mu}^{q}(0) = \left(G_{\lambda,\mu}^{q}(0)\right)' - 1 = 0$. On the other hand, it is easy to see that

$$\left(G_{\lambda,\mu}^{q}(z)\right)' = \left(\frac{\phi_{\lambda,\mu}(z)}{z}\right)^{q}$$

and

$$\frac{z\left(G_{\lambda,\mu}^{q}(z)\right)''}{\left(G_{\lambda,\mu}^{q}(z)\right)'} = q\left[\frac{z\left(\phi_{\lambda,\mu}(z)\right)'}{\phi_{\lambda,\mu}(z)} - 1\right].$$

Then, by using first assertion of Lemma 2.4, we obtain

$$\left|\frac{z\left(G_{\lambda,\mu}^{q}(z)\right)''}{\left(G_{\lambda,\mu}^{q}(z)\right)'}\right| \le \frac{|q|\,e^{1/(\lambda+\mu+1)}}{2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}} \tag{3.3}$$

for all $z \in U$ and $\lambda+\mu > x_0$, where $x_0 = 1.2581$ is the numerical root of the equation (2.1). From this, we write the following inequality

$$\left(1-|z|^2\right)\left|\frac{z\left(G_{\lambda,\mu}^{q}(z)\right)''}{\left(G_{\lambda,\mu}^{q}(z)\right)'}\right| \le \left(1-|z|^2\right)\frac{|q|\,e^{1/(\lambda+\mu+1)}}{2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}} \tag{3.4}$$

for all $z \in U$ and $\lambda+\mu > x_0$.

Since $0 \le |z| < 1$ and the following inequality

$$\frac{|q|\,e^{1/(\lambda+\mu+1)}}{2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}} < 1$$

is equivalent to the condition (3.2), then from (3.4), in view of Lemma 2.1, we arrive at the desired result of the theorem.

Thus the proof of Theorem 3.1 is completed.

By setting $q = 1$ in Theorem 3.1, we arrive at the following corollary.

Corollary 3.1. *Let $\lambda \ge 1$ and $\lambda+\mu > x_1$ where $x_1 = 2.4898$ is the numerical root of the equation*

$$2x-(x+2)e^{\frac{1}{x+1}}+1=0. \tag{3.4}$$

Then, the integral operator $G_{\lambda,\mu} : U \to \mathbb{C}$ defined by

$$G_{\lambda,\mu}(z) = \int_0^z \frac{\phi_{\lambda,\mu}(t)}{t}\,dt,\ z \in U$$

univalent in U.

Note 3.1. Note that, the equations (3.4) is solved with simple computer programming.

Taking $\lambda = 1, \mu \equiv \mu+1$ in Theorem 3.1, we immediately obtain the following result.

Corollary 3.2. *The function $G_{\mu}^{q} : U \to \mathbb{C}$ defined by*

$$G_{\mu}^{q}(z) = \int_0^z \left(\frac{J_{\mu}^{N}(-t)}{t}\right)^{q} dt,\ z \in U$$

is univalent in U if $\mu > x_0 - 2$, where $x_0 = 1.2581$ is the numerical root of the equation (2.1) and q is a complex number such that

$$|q| \le \frac{(2\mu+5)-(\mu+3)e^{1/(\mu+3)}}{e^{1/(\mu+3)}}.$$

By taking $q=1$ in Corollary 3.2, we have the following corollary.

Corollary 3.3. *Let* $\mu > x_1 - 2$, *where* $x_1 = 2.4898$ *is the root of the equation (3.4).*

Then the function $G_\mu : U \to \mathbb{C}$ *defined*

$$G_\mu(z) = \int_0^z \frac{J_\mu^N(-t)}{t}\,dt,\ z \in U$$

is univalent in U.

Remark 3.1. Note that, recently the function $G_{\lambda,\mu}^q : U \to \mathbb{C}$ defined by (3.1) was investigated by Prajapat [9] and obtained some sufficient conditions for the univalence of this integral operator, when $\phi_{\lambda,\mu}(z)$ is normalized Wright function.

Now, we consider the following integral operator

$$G_{\lambda,\mu}^{p,q}(z) = \left\{ p\int_0^z t^{p-1}\left(\frac{\phi_{\lambda,\mu}(t)}{t}\right)^q dt \right\}^{1/p},\ \lambda > -1,\ \lambda+\mu > 0,\ z \in U. \qquad (3.5)$$

On the univalence of the function $G_{\lambda,\mu}^{p,q}(z), z \in U$ we give the following theorem.

Theorem 3.2. *Let* $\lambda \ge 1$ *and* $\lambda + \mu > x_0$, *where* $x_0 = 1.2581$ *is the numerical root of the equation (2.1). Moreover, suppose that* p, q and c *are complex number such that* $\mathrm{Re}(p) > 0,\ |c| < 1$ *and the following condition is satisfied*

$$|c| \le 1 - \frac{|q|e^{1/(\lambda+\mu+1)}}{|p|\left[2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}\right]}. \qquad (3.6)$$

Then, the integral operator $G_{\lambda,\mu}^{p,q} : U \to \mathbb{C}$ *defined by (3.5) is univalent in* U.

Proof. The integral operator (3.5), we can rewrite as follows

$$G_{\lambda,\mu}^{p,q}(z) = \left\{ p\int_0^z t^{p-1}\left(G_{\lambda,\mu}^q(t)\right)' dt \right\}^{1/p}, \qquad (3.7)$$

where the function $G_{\lambda,\mu}^q : U \to \mathbb{C}$ defined by (3.1).

Under hypothesis of the theorem. second assertion of Lemma 2.4, we obtain

$$\left| c|z|^{2p} + \left(1-|z|^{2p}\right)\frac{z\left(G_{\lambda,\mu}^q(z)\right)''}{p\left(G_{\lambda,\mu}^q(z)\right)'} \right| \le |c| + \frac{|q|e^{1/(\lambda+\mu+1)}}{|p|\left[2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}\right]}.$$

On the other hand, since (3.6) is satisfied, the expression in the right side of the above inequality is bounded by 1. In that case, according to Lemma 2.3, the function $G_{\lambda,\mu}^{p,q}(z)$ defined by (3.7); that is the function $G_{\lambda,\mu}^{p,q}(z)$ defined by (3.5) is univalent in U.

Thus, the proof of Theorem 3.2 is completed.

By setting $q=1$ in Theorem 3.2, we arrive at the following corollary.

Corollary 3.4. *Let* $\lambda \geq 1$ *and* $\lambda + \mu > x_0$, *where* $x_0 = 1.2581$ *is the numerical root of the equation (2.1). Moreover, suppose that* p and c *be complex number such that* $\operatorname{Re}(p) > 0$, $|c| < 1$ *and the following condition is satisfied*

$$|c| \leq 1 - \frac{e^{1/(\lambda+\mu+1)}}{|p|\left[2(\lambda+\mu)+1-(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}\right]}.$$

Then, the integral operator $G^p_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$G^p_{\lambda,\mu}(z) = \left\{ p\int_0^z t^{p-2}\phi_{\lambda,\mu}(t)\,dt \right\}^{1/p}, \; z \in U \tag{3.8}$$

is univalent in U.

Remark 3.2. Note that, recently the function $G^p_{\lambda,\mu} : U \to \mathbb{C}$ defined by (3.8) was investigated by Prajapat [9] and obtained some sufficient conditions for the univalence of this integral operator, when $\phi_{\lambda,\mu}(z)$ is normalized Wright function.

We now consider the integral operator of type (1.4), when the function $f(z)$ is the produced Wright function $\phi_{\lambda,\mu}(z)$.

Let

$$H^q_{\lambda,\mu}(z) = \left\{ q\int_0^z t^{q-1}\left(e^{\phi_{\lambda,\mu}(t)}\right)^q dt \right\}^{1/q}, \lambda > -1, \lambda+\mu > 0, z \in U. \tag{3.9}$$

On the univalence of the function $H^q_{\lambda,\mu}(z), z \in U$, we give the following theorem.

Theorem 3.3. *Let* $\lambda \geq 1$ *and* $\lambda + \mu > 0$. *Moreover, suppose that* q *is a complex number such that* $\operatorname{Re}(q) \geq 1$ *and the following condition is satisfied*

$$|q| \leq \frac{3\sqrt{3}(\lambda+\mu)}{2\left[(\lambda+\mu+2)e^{1/(\lambda+\mu+1)} - 1\right]}. \tag{3.10}$$

Then, the integral operator $H^q_{\lambda,\mu} : U \to \mathbb{C}$ *defined by (3.9) is univalent in* U.

Proof. From (2.3), we have

$$\left| z\left(\phi_{\lambda,\mu}(z)\right)' \right| \leq \frac{1}{\lambda+\mu}\left\{(\lambda+\mu+2)e^{1/(\lambda+\mu+1)} - 1\right\} \tag{3.11}$$

for all $z \in U$.

It is easily shown that

$$\frac{1}{\lambda+\mu}\left\{(\lambda+\mu+2)e^{1/(\lambda+\mu+1)} - 1\right\} = 1 + \frac{1}{\lambda+\mu}\left\{(\lambda+\mu+2)e^{1/(\lambda+\mu+1)} - (\lambda+\mu+1)\right\}.$$

Moreover, it can easily be seen that $(x+2)e^{1/(x+1)} - (x+1) > 0$ from the computer account for all $x > 0$. Taking

$$a = \frac{1}{\lambda+\mu}\left\{(\lambda+\mu+2)e^{1/(\lambda+\mu+1)} - 1\right\},$$

we can easily see that $a > 1$ for all $\lambda + \mu > 0$ and $2a|q| \le 3\sqrt{3}$ under hypothesis of the theorem. Moreover, from (3.11) it is clear that $\left| z\left(\phi_{\lambda,\mu}(z)\right)' \right| \le a$. Thus, all hypothesis of Lemma 2.2 is satisfied. With this, the proof of Theorem 3.3 is completed.

By setting $q = 1$ in Theorem 3.3, we have the following result.

Corollary 3.5. *Let* $\lambda \ge 1$ *and* $\lambda + \mu > x_2$, *where* $x_2 = 1.6692$ *is the numerical root of the equation*

$$3\sqrt{3}x - 2(x+2)e^{1/(x+1)} + 2 = 0. \tag{3.12}$$

Then, the integral operator $H_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$H_{\lambda,\mu}(z) = \int_0^z e^{\phi_{\lambda,\mu}(t)} dt, z \in U$$

is univalent in U.

Note 3.2. Note that, the above equation (3.12) is solved with simple computer programming.

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References

[1]. Baricz, A., Frasin, B.A. (2010). Univalence of integral operators involving Bessel functions. *Appl. Math. Lett.* 23(4): 371-376.

[2]. Becker, J. (1972). Löwnersche differentialgleichung und quasikoform fortsetzbare schlichte funktionen. *J. Reine Angw Math.* 255: 23-43 (in German).

[3]. Blezu, D. (2006). On Univalence Criteria, *General Mathematics*, 14(1): 87-93.

[4]. Breaz, D., Günay, H.Ö. (2008). On the univalence criterion of a general integral operator. *J. Inequal. Appl.* p. 8 Art. ID 702715.

[5]. Breaz, D., Breaz, N., Srivastava, H.M. (2009). An extension of the univalent condition for a family of integral oprator. *Appl. Math. Lett.* 22: 41-44.

[6]. Mustafa, N. (2017). Univalence of certain integral operators involving normalized Wright functions. *Commun. Fac. Sci. Univ. Ank Ser. A 1 Math. Stat*. 66(1): 19-28.

[7]. Pescar, V. (1996). A new generalization of Ahlfor's and Becker's criterion of univalence. *Bull. Math. Soc. (Second Series),* 9: 53-54.

[8]. Prajapat, J.K. (2011). Certain geometric properties of normalized Bessel functions. *Appl. Math. Lett*. 24: 2133-2139.

[9]. Prajapat, J.K. (2015). Certain geometric properties of the Wright function. *Integral Transforms Spec. Funct.* 26(3): 203-212.

[10]. Wright, E.M. (1933). On the coefficients of power series having exponential singularities. *J. London Math. Soc.* 8: 71-79.