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A class of sublinear operators and their commutators by with rough kernels on vanishing generalized Morrey spaces

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Abstract In this paper, we consider the boundedness of a class of sublinear operators and their commutators by with rough kernels associated with Calderón-Zygmund operator, Hard-Littlewood maximal operator, fractional integral operator, fractional maximal operator by with rough kernels both on vanishing generalized Morrey spaces and vanishing Morrey spaces, respectively.

Keywords Sublinear operator; Calderón-Zygmund operator; Hard-Littlewood maximal operator; fractional integral operator; fractional maximal operator; rough kernel; vanishing Morrey spaces; vanishing generalized Morrey space; commutator; BMO.

1. Introduction

To characterize the regularity of solutions to some partial differential equations(PDEs), Morrey [9] first introduced classical Morrey spaces $M_{p,\lambda}$ which naturally are generalizations of Lebesgue spaces.

We will say that a function $f \in M_{p,\lambda} = M_{p,\lambda}\left(\mathbb{R}^n\right)$ if

$$\sup_{x\in\mathbb{R}^n, r>0}\left[r^{-\lambda}\int_{B(x,r)}|f(y)|^p dy\right]^{1/p} < \infty. \tag{1.1}$$

Here, $1 < p < \infty$ and $0 < \lambda < n$ and the quantity of (1.1) is the (p,λ)-Morrey norm, denoted by $\|f\|_{M_{p,\lambda}}$.

We also refer to [1, 4] for the latest research on the theory of Morrey spaces associated with harmonic analysis. On the other hand, the study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $M_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VM_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [13] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r\to 0}\sup_{\substack{x\in\mathbb{R}^n\\ 0<t<r}}\left[t^{-\lambda}\int_{B(x,t)}|f(y)|^p dy\right]^{1/p} = 0,$$

where $1 < p < \infty$ and $0 < \lambda < n$ for brevity, so that

$$VM_{p,\lambda}(\mathbb{R}^n)=\left\{ f\in M_{p,\lambda}(\mathbb{R}^n):\lim_{r\to 0}\sup_{\substack{x\in\mathbb{R}^n\\ 0<t<r}} t^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,t))}=0\right\}.$$

Later in [14] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [7] and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. For the properties and applications of vanishing Morrey spaces, see also [2]. It is known that, there is no research regarding boundedness of the sublinear operators with rough kernel on vanishing Morrey spaces.

Let $\Omega\in L_s(S^{n-1})$ with $1<s\le\infty$ be homogeneous of degree zero and satisfies the cancellation condition

$$\int_{S^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.2}$$

where $x'=\frac{x}{|x|}$ for any $x\neq 0$. We define $s'=\frac{s}{s-1}$ for any $s>1$. Suppose that T_Ω represents a linear or a sublinear operator, which satisfies that for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$

$$|T_\Omega f(x)|\le c_0\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy, \tag{1.3}$$

where c_0 is independent of f and x. Similarly, we assume that $T_{\Omega,\alpha}$, $\alpha\in(0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$

$$|T_{\Omega,\alpha}f(x)|\le c_0\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy, \tag{1.4}$$

for some $\alpha\in(0,n)$, where c_0 is independent of f and x.

We point out that the condition (1.3) in the case $\Omega\equiv 1$ was first introduced by Soria and Weiss in [12]. The conditions (1.3) and (1.4) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund (C-Z) operators, Carleson's maximal operator, Hardy-Littlewood (H-L) maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means, fractional Marcinkiewicz operator, fractional maximal operator, fractional integral operator (Riesz potential) and so on (see [12] for details).

Let $f\in L_1^{loc}(\mathbb{R}^n)$. The C-Z singular integral operator $\overline{T}_\Omega$ and H-L maximal operator M_Ω by with rough kernels are defined by

$$\overline{T}_\Omega f(x)=p.v.\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy,$$

$$M_\Omega f(x)=\sup_{t>0}\frac{1}{|B(x,t)|}\int_{B(x,t)}|\Omega(x-y)||f(y)|dy$$

satisfy condition (1.3), where a homogeneous of degree zero function $\Omega(x')$ satisfies (1.2) on the unit sphere and belongs to $\Omega\in L_s(S^{n-1})$ with $1<s\le\infty$.

It is obvious that when $\Omega\equiv 1$, $\overline{T}_\Omega\equiv\overline{T}$ and $M_\Omega\equiv M$ are the standard C-Z singular intagral operator, briefly a C-Z operator and the H-L maximal operator, respectively.

On the other hand, in 1971, Muckenhoupt and Wheeden [10] defined the fractional integral operator with rough kernel $\overline{T}_{\Omega,\alpha}$ by

$$\overline{T}_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0 < \alpha < n$$

and a related fractional maximal operator with rough kernel $M_{\Omega,\alpha}$ is given by

$$M_{\Omega,\alpha} f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |\Omega(x-y)|\, |f(y)|\, dy \qquad 0 < \alpha < n,$$

where $\Omega \in L_s(S^{n-1})$ with $1 < s \le \infty$ is homogeneous of degree zero on $\mathbb{R}^n$ and also $\overline{T}_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ satisfy condition (1.4).

If $\alpha = 0$, then $M_{\Omega,0} \equiv M_{\Omega}$ and $\overline{T}_{\Omega,0} \equiv \overline{T}_{\Omega}$, respectively. It is obvious that when $\Omega \equiv 1$, $M_{1,\alpha} \equiv M_{\alpha}$ and $\overline{T}_{1,\alpha} \equiv \overline{T}_{\alpha}$ are the fractional maximal operator and the fractional integral operator (Riesz potential), respectively.

For a locally integrable function b on $\mathbb{R}^n$, suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_{\Omega,b} f(x)| \le c_0 \int_{\mathbb{R}^n} |b(x)-b(y)| \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy, \tag{1.5}$$

where c_0 is independent of f and x. Similarly, for a locally integrable function b on $\mathbb{R}^n$, suppose that the commutator operator $T_{\Omega,b,\alpha}$, $\alpha \in (0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_{\Omega,b,\alpha} f(x)| \le c_0 \int_{\mathbb{R}^n} |b(x)-b(y)| \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|dy, \tag{1.6}$$

where c_0 is independent of f and x.

On the other hand, for $b \in L_1^{loc}(\mathbb{R}^n)$, denote by B the multiplication operator defined by $\mathrm{B}f(x) = b(x)f(x)$ for any measurable function f. If $\overline{T}_{\Omega}$ is a linear operator on some measurable function space, then the commutator formed by B and $\overline{T}_{\Omega}$ is defined by

$$\overline{T}_{\Omega,b} f(x) = [b, \overline{T}_{\Omega}] f(x) := \left(\mathrm{B}\overline{T}_{\Omega} - \overline{T}_{\Omega}\mathrm{B}\right) f(x) = b(x)\overline{T}_{\Omega} f(x) - \overline{T}_{\Omega}(bf)(x).$$

In 1976, Coifman et al. [3] introduced the commutator generated by $\overline{T}_{\Omega}$ and a locally integrable function b as follows:

$$[b, \overline{T}_{\Omega}] f(x) \equiv b(x)\overline{T}_{\Omega} f(x) - \overline{T}_{\Omega}(bf)(x) = p.v. \int_{\mathbb{R}^n} [b(x)-b(y)] \frac{\Omega(x-y)}{|x-y|^n} f(y)dy. \tag{1.7}$$

Sometimes, the commutator defined by (1.7) is also called the commutator in Coifman-Rocherberg-Weiss's sense, which has its root in the complex analysis and harmonic analysis (see [3]) and corresponding the sublinear commutator of operator M_{Ω} is defined as follows

$$M_{\Omega,b}(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x)-b(y)|\, |\Omega(x-y)|\, |f(y)|\, dy.$$

And also, the operators $[b,\overline{T}_{\Omega}]$ and $M_{\Omega,b}$ satisfy condition (1.5). Let b be a locally integrable function on R^n, then for $0<\alpha<n$ and f is a suitable function, we define the commutators generated by fractional integral and maximal operators with rough kernel and b as follows, respectively:

$$[b,\overline{T}_{\Omega,\alpha}]f(x)\equiv b(x)\overline{T}_{\Omega,\alpha}f(x)-\overline{T}_{\Omega,\alpha}(bf)(x)=\int_{\mathrm{R}^n}[b(x)-b(y)]\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy,$$

$$M_{\Omega,b,\alpha}(f)(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int_{B(x,t)}|b(x)-b(y)||\Omega(x-y)||f(y)|dy$$

satisfy condition (1.6).

Remark 1 *Suppose that* $\overline{T}_{\Omega,\alpha}$, $\alpha\in(0,n)$ *represents a linear or a sublinear operator, when* Ω *satisfies the specified size conditions, the kernel of the operator* $\overline{T}_{\Omega,\alpha}$ *has no regularity, so the operator* $\overline{T}_{\Omega,\alpha}$ *is called a rough fractional integral operator. These include the commutator operator* $[b,\overline{T}_{\Omega,\alpha}]$. *This also applies to* $\alpha=0$. *In recent years, a variety of operators related to the fractional integrals, C-Z operators but lacking the smoothness required in the classical theory, have been studied (for example, see [5, 6]).*

It is worth noting that for a constant C, if $\overline{T}_{\Omega}$ is linear we have,

$$\begin{aligned}[b+C,\overline{T}_{\Omega}]f&=(b+C)\overline{T}_{\Omega}f-\overline{T}_{\Omega}((b+C)f)\\&=b\overline{T}_{\Omega}f+C\overline{T}_{\Omega}f-\overline{T}_{\Omega}(bf)-C\overline{T}_{\Omega}f\\&=[b,\overline{T}_{\Omega}]f.\end{aligned}$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b\in BMO$ (bounded mean oscillation space) has had the most historical significance.
Now, let us definition of BMO:

Definition 1 (BMO function) *Denote the bounded mean oscillation function space by*

$$BMO(\mathrm{R}^n)=\left\{f\in L_1^{loc}(\mathrm{R}^n):\|f\|_{BMO}:=\sup_{Ball:B\subset\mathrm{R}^n}\mathrm{M}_{f,B}<\infty\right\},$$

here and in the sequel

$$\mathrm{M}_{f,B}:=\frac{1}{|B|}\int_B|f(x)-f_B|dx,\qquad f_B=\frac{1}{|B|}\int_B f(y)dy.$$

Here and henceforth, $F\approx G$ means $F\gtrsim G\gtrsim F$; while $F\gtrsim G$ means $F\geq CG$ for a constant $C>0$; and p' and s' always denote the conjugate index of any $p>1$ and $s>1$, that is, $\frac{1}{p'}:=1-\frac{1}{p}$ and $\frac{1}{s'}:=1-\frac{1}{s}$ and also C stands for a positive constant that can change its value in each statement without explicit mention. Throughout the paper we assume that $x\in\mathrm{R}^n$ and $r>0$ and also let $B(x,r)$ denotes x-centred Euclidean ball with radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)|=v_n r^n$, where $v_n=|B(0,1)|$.

2. Background about vanishing generalized Morrey spaces

After studying Morrey spaces in detail, researchers have passed to the concept of generalized Morrey spaces. Firstly, motivated by the work of [9], Mizuhara [8] introduced generalized Morrey spaces $M_{p,\varphi}$ as follows:

Definition 2 (**Generalized Morrey space**; see [8]) *Let* $\varphi(x,r)$ *be a positive measurable function on* $\mathbb{R}^n \times (0,\infty)$. *If* $0 < p < \infty$, *then the generalized Morrey space* $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ *is defined by*

$$\left\{ f \in L_p^{loc}(\mathbb{R}^n) : \|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))} < \infty \right\}.$$

Obviously, the above definition recover the definition of $M_{p,\lambda}(\mathbb{R}^n)$ if we choose $\varphi(x,r) = r^{\frac{\lambda}{p}}$, that is

$$M_{p,\lambda}(\mathbb{R}^n) = M_{p,\varphi}(\mathbb{R}^n)|_{\varphi(x,r)=r^{\frac{\lambda}{p}}}.$$

Everywhere in the sequel we assume that $\inf_{x\in\mathbb{R}^n, r>0} \varphi(x,r) > 0$ which makes the above spaces non-trivial, since the spaces of bounded functions are contained in these spaces. We point out that $\varphi(x,r)$ is a measurable non-negative function and no monotonicity type condition is imposed on these spaces.

Recently, Gürbüz [5, 6] has proved the boundedness of the sublinear operators and their commutators by withrough kernels denoted by T_Ω, $T_{\Omega,\alpha}$, $T_{\Omega,b}$, $T_{\Omega,b,\alpha}$ on generalized Morrey spaces $M_{p,\varphi}$, respectively.

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

Now, recall that the concept of the vanishing generalized Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$ has been introduced in [11].

Definition 3 (**Vanishing generalized Morrey space**; see [11])*Let* $\varphi(x,r)$ *be a positive measurable function on* $\mathbb{R}^n \times (0,\infty)$ *and* $1 \le p < \infty$. *The vanishing generalized Morrey space* $VM_{p,\varphi}(\mathbb{R}^n)$ *is defined as the spaces of functions* $f \in L_p^{loc}(\mathbb{R}^n)$ *such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))} = 0. \tag{2.1}$$

Naturally, it is suitable to impose on $\varphi(x,t)$ with the following condition:

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^n} \frac{t^{\frac{n}{p}}}{\varphi(x,t)} = 0, \tag{2.2}$$

and

$$\inf_{t>1} \sup_{x\in\mathbb{R}^n} \frac{t^{\frac{n}{p}}}{\varphi(x,t)} > 0. \tag{2.3}$$

From (2.2) and (2.3), we easily know that the bounded functions with compact support belong to $VM_{p,\varphi}(\mathbb{R}^n)$.

The space $VM_{p,\varphi}(\mathbb{R}^n)$ is Banach space with respect to the norm (see, for example [11])

$$\|f\|_{VM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}. \tag{2.4}$$

The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ is closed subspaces of the Banach spaces $M_{p,\varphi}(\mathbb{R}^n)$, which may be shown by standard means.

Furthermore, we have the following embeddings:

$$VM_{p,\varphi} \subset M_{p,\varphi}, \qquad \|f\|_{M_{p,\varphi}} \le \|f\|_{VM_{p,\varphi}}.$$

The purpose of this paper is to consider the mapping properties for the operators T_Ω, $T_{\Omega,\alpha}$, $T_{\Omega,b}$, $T_{\Omega,b,\alpha}$ both on vanishing generalized Morrey spaces and vanishing Morrey spaces, respectively. Similar results still hold for the operators $\overline{T}_\Omega$, $\overline{T}_{\Omega,\alpha}$, $[b,\overline{T}_\Omega]$, $[b,\overline{T}_{\Omega,\alpha}]$, $M_{\Omega,b}$ and $M_{\Omega,b,\alpha}$, respectively. These operators T_Ω, $T_{\Omega,\alpha}$, $T_{\Omega,b}$, $T_{\Omega,b,\alpha}$ have not also been studied so far both on vanishing generalized Morrey spaces and vanishing Morrey spaces and this paper seems to be the first in this direction.

3. Main Results

Theorem 1 *Let* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *be homogeneous of degree zero, and* $1<p<\infty$. *Let* T_Ω *be a sublinear operator satisfying condition (1.3). Let for* $s'\le p$, *the pair* (φ_1,φ_2) *satisfies conditions (2.2)-(2.3) and*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\, t^{-\frac{n}{p}-1}\, dt < \infty \tag{3.1}$$

for every $\delta>0$, and

$$\int_r^\infty \frac{\varphi_1(x,t)}{t^{\frac{n}{p}+1}}\, dt \le C_0 \frac{\varphi_2(x,r)}{r^{\frac{n}{p}}}, \tag{3.2}$$

and for $1<p<s$ the pair (φ_1,φ_2) satisfies conditions (2.2)-(2.3) and also

$$c_{\delta'} := \int_{\delta'}^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\, t^{-\frac{n}{p}+\frac{n}{s}-1}\, dt < \infty \tag{3.3}$$

for every $\delta'>0$, and

$$\int_r^\infty \frac{\varphi_1(x,t)}{t^{\frac{n}{p}-\frac{n}{s}+1}}\, dt \le C_0 \frac{\varphi_2(x,r)}{r^{\frac{n}{p}-\frac{n}{s}}}, \tag{3.4}$$

where C_0 does not depend on $x\in\mathbb{R}^n$ and $r>0$.

Then the operator T_Ω is bounded from VM_{p,φ_1} to VM_{p,φ_2} for $p>1$. Moreover, we have for $p>1$

$$\|T_\Omega f\|_{VM_{p,\varphi_2}} \lesssim \|f\|_{VM_{p,\varphi_1}}.$$

Proof. Let $1<p<\infty$ and $s'\le p$. The estimation of the norm of the operator, that is, the boundedness in the vanishing generalized Morrey space follows from Lemma 2.1. in [5] and condition (3.2)

$$\|T_\Omega f\|_{VM_{p,\varphi_2}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\|T_\Omega f\|_{L_p(B(x,r))}$$

$$\lesssim \sup_{x\in\mathbb{R}^n,r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{p}} \int_r^\infty \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \sup_{x\in\mathbb{R}^n,r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{p}} \int_r^\infty \varphi_1(x,t)\left[\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right] \frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \|f\|_{VM_{p,\varphi_1}} \sup_{x\in\mathbb{R}^n,r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{p}} \int_r^\infty \varphi_1(x,t) \frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \|f\|_{VM_{p,\varphi_1}}.$$

So it is sufficient to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_1(x,r)^{-1}\|f\|_{L_p(B(x,r))} = 0 \Rightarrow \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1}\|T_\Omega f\|_{L_p(B(x,r))} = 0. \tag{3.5}$$

To show that $\sup_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1}\|T_\Omega f\|_{L_p(B(x,r))} < \varepsilon$ for small r, we split the right hand side of (2.1) in Lemma 2.1. in [5]:

$$\varphi_2(x,r)^{-1}\|T_\Omega f\|_{L_p(B(x,r))} \le C\left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\right], \tag{3.6}$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x,r) := \frac{r^{\frac{n}{p}}}{\varphi_2(x,r)} \left(\int_r^{\delta_0} \varphi_1(x,t) t^{-\frac{n}{p}-1} \left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))} \right) dt \right),$$

and

$$J_{\delta_0}(x,r) := \frac{r^{\frac{n}{p}}}{\varphi_2(x,r)} \left(\int_{\delta_0}^{\infty} \varphi_1(x,t) t^{-\frac{n}{p}-1} \left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))} \right) dt \right)$$

and $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (3.2) and (3.6). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \qquad 0 < r < \delta_0.$$

The estimation of the second term may be obtained by choosing r sufficiently small. Indeed, by (2.2) we have

$$J_{\delta_0}(x,r) \le c_{\delta_0}\|f\|_{VM_{p,\varphi}} \frac{r^{\frac{n}{p}}}{\varphi(x,r)},$$

where c_{δ_0} is the constant from (3.1). Then, by (2.2) it suffices to choose r small enough such that

$$\sup_{x\in\mathrm{R}^n}\frac{r^{\frac{n}{p}}}{\varphi(x,r)}\le\frac{\varepsilon}{2c_{\delta_0}\|f\|_{VM_{p,\varphi}}},$$

which completes the proof of (3.5).

For the case of $1<p<s$, we can also use the same method, so we omit the details, which completes the proof.

Remark 2 *Conditions (3.1) and (3.3) are not needed in the case when* $\varphi(x,r)$ *does not depend on* x, *since (3.1) follows from (3.2) and similarly, (3.3) follows from (3.4) in this case.*

Corollary 1 *Under the conditions of Theorem 1, the operators* M_Ω *and* $\overline{T}_\Omega$ *are bounded from* VM_{p,φ_1} *to* VM_{p,φ_2}.

Corollary 2 *Let* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *be homogeneous of degree zero satisfying condition (1.2). Let* $0<\lambda<n$, $1<p<\infty$. *Let* T_Ω *be a sublinear operator satisfying condition (1.3). Then for* $s'\le p$ *or* $p<s$, *we have*

$$\|T_\Omega f\|_{VM_{p,\lambda}}\lesssim\|f\|_{VM_{p,\lambda}}.$$

Proof. Let $1<p<\infty$ and $s'\le p$. By using $\varphi_1(x,r)=\varphi_2(x,r)=r^{\frac{\lambda}{p}}$ in the proof of Theorem 1 and condition (3.2), we get

$$\|T_\Omega f\|_{VM_{p,\lambda}}\lesssim\sup_{x\in\mathrm{R}^n,r>0}r^{-\frac{\lambda}{p}}r^{\frac{n}{p}}\int_r^\infty\varphi_1(x,t)t^{-\frac{n}{p}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt$$

$$\lesssim\|f\|_{VM_{p,\lambda}}\sup_{x\in\mathrm{R}^n,r>0}r^{\frac{n-\lambda}{p}}\int_r^\infty r^{\frac{\lambda}{p}}\frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim\|f\|_{VM_{p,\lambda}},$$

for the case of $p<s$, we can also use the same method, so we omit the details.

Corollary 3 *Under the conditions of Corollary 2, the operators* M_Ω *and* $\overline{T}_\Omega$ *are bounded on* $VM_{p,\lambda}(\mathrm{R}^n)$.

Theorem 2 *Let* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *be homogeneous of degree zero. Let* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Let* $T_{\Omega,\alpha}$ *be a sublinear operator satisfying condition (1.4). Let for* $s'\le p$ *the pair* (φ_1,φ_2) *satisfies conditions (2.2)-(2.3) and*

$$c_\delta:=\int_\delta^\infty\sup_{x\in\mathrm{R}^n}\varphi_1(x,t)\frac{dt}{t^{\frac{n}{q}+1}}<\infty \tag{3.7}$$

for every $\delta>0$, and

$$\int_r^\infty\varphi_1(x,t)\frac{dt}{t^{\frac{n}{q}+1}}\le C_0\frac{\varphi_2(x,r)}{r^{\frac{n}{q}}}, \tag{3.8}$$

and for $q<s$ the pair (φ_1,φ_2) satisfies conditions (2.2)-(2.3) and also

$$c_{\delta'} := \int_{\delta'}^{\infty} \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}} < \infty \tag{3.9}$$

for every $\delta' > 0$, and

$$\int_r^{\infty} \varphi_1(x,t) \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}} \le C_0 \frac{\varphi_2(x,r)}{r^{\frac{n}{q}-\frac{n}{s}}}, \tag{3.10}$$

where C_0 does not depend on $x\in\mathbb{R}^n$ and $r>0$.

Then the operator $T_{\Omega,\alpha}$ is bounded from VM_{p,φ_1} to VM_{q,φ_2} for $p>1$. Moreover, we have for $p>1$

$$\|T_{\Omega,\alpha} f\|_{VM_{q,\varphi_2}} \lesssim \|f\|_{VM_{p,\varphi_1}}.$$

Proof. Similar to the proof of Theorem 1, let $s' \le p$. The estimation of the norm of the operator follows from Lemma 3 in [6] and condition (3.8)

$$\|T_{\Omega,\alpha} f\|_{VM_{q,\varphi_2}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \|T_{\Omega,\alpha} f\|_{L_q(B(x,r))}$$

$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{q}} \int_r^{\infty} \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{q}} \int_r^{\infty} \varphi_1(x,t) \left[\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x,t))}\right] \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \|f\|_{VM_{p,\varphi_1}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{q}} \int_r^{\infty} \varphi_1(x,t) \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \|f\|_{VM_{p,\varphi_1}}.$$

Thus we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_1(x,r)^{-1} \|f\|_{L_p(B(x,r))} = 0 \Rightarrow \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1} \|T_{\Omega,\alpha} f\|_{L_q(B(x,r))} = 0. \tag{3.11}$$

To show that $\sup_{x\in\mathbb{R}^n} \frac{\|T_{\Omega,\alpha} f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} < \varepsilon$ for small r, we split the right-hand side of (2.1) in Lemma 3 in [6]:

$$\frac{r^{-\frac{n}{q}} \|T_{\Omega,\alpha} f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} \le C\left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\right],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x,r) := \frac{r^{\frac{n}{q}}}{\varphi_2(x,r)} \int_r^{\delta_0} \varphi_1(x,t) t^{-\frac{n}{q}-1} \left(\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x,t))}\right) dt,$$

and

$$J_{\delta_0}(x,r):=\frac{r^{\frac{n}{q}}}{\varphi_2(x,r)}\int_{\delta_0}^{\infty}\varphi_1(x,t)t^{-\frac{n}{q}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt,$$

and $r<\delta_0$ and the rest of the proof is the same as the proof of Theorem 1. Thus, we can prove that (3.11). For the case of $q<s$, we can also use the same method, so we omit the details, which completes the proof.

Remark 3 *Conditions (3.7) and (3.9) are not needed in the case when* $\varphi(x,r)$ *does not depend on* x, *since (3.7) follows from (3.8) and similarly, (3.9) follows from (3.10) in this case.*

Corollary 4 *Under the conditions of Theorem 2, the operators* $M_{\Omega,\alpha}$ *and* $\overline{T}_{\Omega,\alpha}$ *are bounded from* VM_{p,φ_1} *to* VM_{q,φ_2}.

Corollary 5 *Let* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *be homogeneous of degree zero. Let* $0<\alpha,\lambda<n$, $1<p<\frac{n-\lambda}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ *and* $\frac{\lambda}{p}=\frac{\mu}{q}$. *Let* $T_{\Omega,\alpha}$ *be a sublinear operator satisfying condition (1.4). Then for* $s'\le p$ *or* $q<s$, *we have*

$$\|T_{\Omega,\alpha}f\|_{VM_{q,\mu}}\lesssim\|f\|_{VM_{p,\lambda}}.$$

Proof. Let $s'\le p$. By using $\varphi_1(x,r)=r^{\frac{\lambda}{p}}$ and $\varphi_2(x,r)=r^{\frac{\mu}{q}}$ in the proof of Theorem 2 and condition (3.8), it follows that

$$\|T_{\Omega,\alpha}f\|_{VM_{q,\mu}}\lesssim\sup_{x\in\mathbb{R}^n,r>0}r^{-\frac{\mu}{q}}r^{\frac{n}{q}}\int_r^{\infty}\varphi_1(x,t)t^{-\frac{n}{q}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt$$

$$\lesssim\|f\|_{VM_{p,\lambda}}\sup_{x\in\mathbb{R}^n,r>0}r^{\frac{n-\mu}{q}}\int_r^{\infty}r^{\frac{\lambda}{p}}\frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim\|f\|_{VM_{p,\lambda}},$$

for the case of $q<s$, we can also use the same method, so we omit the details, which completes the proof.

Corollary 6 *Under the conditions of Corollary 5, the operators* $M_{\Omega,\alpha}$ *and* $\overline{T}_{\Omega,\alpha}$ *are bounded from* $VM_{p,\lambda}$ *to* $VM_{q,\mu}$.

Now below, we obtain the boundedness of operators both $T_{\Omega,b}$ and $T_{\Omega,b,\alpha}$ on the vanishing generalized Morrey spaces $VM_{p,\varphi}$.

Theorem 3 *Let* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *be homogeneous of degree zero. Let* $1<p<\infty$ *and* $b\in BMO(\mathbb{R}^n)$. *Let* $T_{\Omega,b}$ *is a sublinear operator satisfying condition (1.5). Let for* $s'\le p$ *the pair* (φ_1,φ_2) *satisfies conditions (2.2)-(2.3) and*

$$c_\delta:=\int_{\delta}^{\infty}\left(1+\ln\frac{t}{r}\right)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)t^{-\frac{n}{p}-1}dt<\infty \tag{3.12}$$

for every $\delta>0$, and

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\varphi_1(x,t)}{t^{\frac{n}{p}+1}}dt \le C_0\frac{\varphi_2(x,r)}{r^{\frac{n}{p}}}, \tag{3.13}$$

and for $p<s$ the pair (φ_1,φ_2) satisfies conditions (2.2)-(2.3) and also

$$c_{\delta'} := \int_{\delta'}^{\infty}\left(1+\ln\frac{t}{r}\right)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)t^{-\frac{n}{p}+\frac{n}{s}-1}dt<\infty \tag{3.14}$$

for every $\delta'>0$, and

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\varphi_1(x,t)}{t^{\frac{n}{p}-\frac{n}{s}+1}}dt \le C_0\frac{\varphi_2(x,r)}{r^{\frac{n}{p}-\frac{n}{s}}}, \tag{3.15}$$

where C_0 does not depend on $x\in\mathbb{R}^n$ and $r>0$.

Then the operator $T_{\Omega,b}$ is bounded from VM_{p,φ_1} to VM_{p,φ_2}. Moreover,

$$\left\|T_{\Omega,b}f\right\|_{VM_{p,\varphi_2}} \lesssim \|b\|_{BMO}\|f\|_{VM_{p,\varphi_1}}.$$

Proof. The proof follows more or less the same lines as for Theorem 1, but now the arguments are different due to the necessity to introduce the logarithmic factor into the assumptions. Let $s'\le p$. The estimation of the norm of the operator, that is, the boundedness in the vanishing generalized Morrey space follows from Lemma 2.2. in [5] and condition (3.13)

$$\left\|T_{\Omega,b}f\right\|_{VM_{p,\varphi_2}} = \sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}\left\|T_{\Omega,b}f\right\|_{L_p(B(x,r))}$$

$$\lesssim \|b\|_{BMO}\sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}r^{\frac{n}{p}}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x,t))}\frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \|b\|_{BMO}\sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}r^{\frac{n}{p}}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\left[\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right]\frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \|b\|_{BMO}\|f\|_{VM_{p,\varphi_1}}\sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}r^{\frac{n}{p}}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \|b\|_{BMO}\|f\|_{VM_{p,\varphi_1}}.$$

So we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\varphi_1(x,r)^{-1}\|f\|_{L_p(B(x,r))}=0 \Rightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\varphi_2(x,r)^{-1}\left\|T_{\Omega,b}f\right\|_{L_p(B(x,r))}=0. \tag{3.16}$$

To show that $\sup_{x\in\mathbb{R}^n}\varphi_2(x,r)^{-1}\left\|T_{\Omega,b}f\right\|_{L_p(B(x,r))}<\varepsilon$ for small r, we split the right-hand side of the first inequality in Lemma 2.2. in [5]:

$$\varphi_2(x,r)^{-1}\|T_{\Omega}f\|_{L_p(B(x,r))} \le C\left[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)\right], \tag{3.17}$$

where $\delta_0>0$ (we may take $\delta_0<1$), and

$$I_{\delta_0}(x,r):=\|b\|_{BMO}\frac{r^{\frac{n}{p}}}{\varphi_2(x,r)}\left(\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)t^{-\frac{n}{p}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt\right),$$

and

$$J_{\delta_0}(x,r):=\|b\|_{BMO}\frac{r^{\frac{n}{p}}}{\varphi_2(x,r)}\left(\int_{\delta_0}^{\infty}\left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)t^{-\frac{n}{p}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt\right)$$

and $r<\delta_0$. Now we choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}<\frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (3.13) and (3.17). This allows to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\|b\|_{BMO}\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\qquad 0<r<\delta_0.$$

The estimation of the second term may be obtained by choosing r sufficiently small. Indeed, by (2.2) we have

$$J_{\delta_0}(x,r)\le\|b\|_{BMO}c_{\delta_0}\|f\|_{VM_{p,\varphi}}\frac{r^{\frac{n}{p}}}{\varphi(x,r)},$$

where c_{δ_0} is the constant from (3.12). Then, by (2.2) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n}\frac{r^{\frac{n}{p}}}{\varphi(x,r)}\le\frac{\varepsilon}{2\|b\|_{BMO}c_{\delta_0}\|f\|_{VM_{p,\varphi}}},$$

which completes the proof of (3.16).
For the case of $p<s$, we can also use the same method, so we omit the details.

Remark 4 *Conditions (3.12) and (3.14) are not needed in the case when $\varphi(x,r)$ does not depend on x, since (3.12) follows from (3.13) and similarly, (3.14) follows from (3.15) in this case.*

Corollary 7 *Under the conditions of Theorem 3, the operators $M_{\Omega,b}$ and $[b,\overline{T}_\Omega]$ are bounded from VM_{p,φ_1} to VM_{p,φ_2}.*

Corollary 8 *Let $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, be homogeneous of degree zero satisfying condition (1.2). Let $0<\lambda<n$, $1<p<\infty$. Let $1<p<\infty$ and $b\in BMO(\mathbb{R}^n)$. Let $T_{\Omega,b}$ be a sublinear operator satisfying condition (1.5). Then for $s'\le p$ or $p<s$, we have*

$$\|T_{\Omega,b}f\|_{VM_{p,\lambda}}\lesssim\|b\|_{BMO}\|f\|_{VM_{p,\lambda}}.$$

Proof. Let $1<p<\infty$, $b\in BMO(\mathbb{R}^n)$ and $s'\le p$. By using $\varphi_1(x,r)=\varphi_2(x,r)=r^{\frac{\lambda}{p}}$ in the proof of Theorem 3 and condition (3.13), we get

$$\|T_{\Omega,b}f\|_{VM_{p,\lambda}} \lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} r^{\frac{n}{p}} \int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)t^{-\frac{n}{p}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt$$

$$\lesssim \|b\|_{BMO}\|f\|_{VM_{p,\lambda}} \sup_{x\in\mathbb{R}^n, r>0} r^{\frac{n-\lambda}{p}} \int_r^\infty \left(1+\ln\frac{t}{r}\right) r^{\frac{\lambda}{p}} \frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim \|b\|_{BMO}\|f\|_{VM_{p,\lambda}},$$

for the case of $p < s$, we can also use the same method, so we omit the details.

Corollary 9 *Under the conditions of Corollary 8, the operators* $M_{\Omega,b}$ *and* $[b,\overline{T}_\Omega]$ *are bounded on* $VM_{p,\lambda}(\mathbb{R}^n)$.

Theorem 4 *Let* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *be homogeneous of degree zero. Let* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Let* $T_{\Omega,b,\alpha}$ *be a sublinear operator satisfying condition (1.6). Let for* $s' \le p$ *the pair* (φ_1,φ_2) *satisfies conditions (2.2)-(2.3) and*

$$c_\delta := \int_\delta^\infty \left(1+\ln\frac{t}{r}\right)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\frac{dt}{t^{\frac{n}{q}+1}} < \infty \tag{3.18}$$

for every $\delta > 0$, and

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\frac{dt}{t^{\frac{n}{q}+1}} \le C_0\frac{\varphi_2(x,r)}{r^{\frac{n}{q}}}, \tag{3.19}$$

and for $q < s$ the pair (φ_1,φ_2) satisfies conditions (2.2)-(2.3) and also

$$c_{\delta'} := \int_{\delta'}^\infty \left(1+\ln\frac{t}{r}\right)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}} < \infty \tag{3.20}$$

for every $\delta' > 0$, and

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}} \le C_0\frac{\varphi_2(x,r)}{r^{\frac{n}{q}-\frac{n}{s}}}, \tag{3.21}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator $T_{\Omega,b,\alpha}$ is bounded from VM_{p,φ_1} to VM_{q,φ_2}. Moreover,

$$\|T_{\Omega,b,\alpha}f\|_{VM_{q,\varphi_2}} \lesssim \|b\|_{BMO}\|f\|_{VM_{p,\varphi_1}}.$$

Proof. Similar to the proof of Theorem 3, let $s' \le p$. The estimation of the norm of the operator follows from Lemma 4 in [6] and condition (3.19)

$$\|T_{\Omega,b,\alpha}f\|_{VM_{q,\varphi_2}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1}\|T_{\Omega,b,\alpha}f\|_{L_q(B(x,r))}$$

$$\lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{q}} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{q}} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \varphi_1(x,t) \left[\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x,t))}\right] \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \|b\|_{BMO} \|f\|_{VM_{p,\varphi_1}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{\frac{n}{q}} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \varphi_1(x,t) \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \|b\|_{BMO} \|f\|_{VM_{p,\varphi_1}}.$$

Thus we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_1(x,r)^{-1} \|f\|_{L_p(B(x,r))} = 0 \Rightarrow \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1} \|T_{\Omega,b,\alpha} f\|_{L_q(B(x,r))} = 0. \tag{3.22}$$

To show that $\sup_{x\in\mathbb{R}^n} \dfrac{\|T_{\Omega,b,\alpha} f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} < \varepsilon$ for small r, we split the right-hand side of the first inequality in Lemma 4 in [6]:

$$\frac{r^{-\frac{n}{q}} \|T_{\Omega,b,\alpha} f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} \le C\left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\right],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x,r) := \|b\|_{BMO} \frac{r^{\frac{n}{q}}}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1+\ln\frac{t}{r}\right) \varphi_1(x,t) t^{-\frac{n}{q}-1} \left(\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x,t))}\right) dt,$$

and

$$J_{\delta_0}(x,r) := \|b\|_{BMO} \frac{r^{\frac{n}{q}}}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} \left(1+\ln\frac{t}{r}\right) \varphi_1(x,t) t^{-\frac{n}{q}-1} \left(\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x,t))}\right) dt,$$

and $r < \delta_0$ and and the rest of the proof is the same as the proof of Theorem 3. Thus, we can prove that (3.22).

For the case of $q < s$, we can also use the same method, so we omit the details, which completes the proof.

Remark 5 *Conditions (3.18) and (3.20) are not needed in the case when* $\varphi(x,r)$ *does not depend on* x*, since (3.18) follows from (3.19) and similarly, (3.20) follows from (3.21) in this case.*

Corollary 10 *Under the conditions of Theorem 4, the operators* $M_{\Omega,b,\alpha}$ *and* $[b,\overline{T}_{\Omega,\alpha}]$ *are bounded from* VM_{p,φ_1} *to* VM_{q,φ_2}.

Corollary 11 *Let* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *be homogeneous of degree zero. Let* $0 < \alpha, \lambda < n$, $1 < p < \dfrac{n-\lambda}{\alpha}$, $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\alpha}{n}$, *and* $\dfrac{\lambda}{p} = \dfrac{\mu}{q}$ *and* $b \in BMO(\mathbb{R}^n)$. *Let* $T_{\Omega,b,\alpha}$ *be a sublinear operator satisfying condition (1.6). Then for* $s' \le p$ *or* $q < s$, *we have*

$$\left\|T_{\Omega,\mathrm{b},\alpha}f\right\|_{VM_{q,\mu}} \lesssim \|b\|_{BMO}\|f\|_{VM_{p,\lambda}}.$$

Proof. Similar to the proof of Corollary 8, Let $s' \le p$. By using $\varphi_1(x,r) = r^{\frac{\lambda}{p}}$ and $\varphi_2(x,r) = r^{\frac{\mu}{q}}$ in the proof of Theorem 4 and condition (3.19), it follows that

$$\begin{aligned}\left\|T_{\Omega,\mathrm{b},\alpha}f\right\|_{VM_{q,\mu}} &\lesssim \|b\|_{BMO} \sup_{x\in\mathrm{R}^n,r>0} r^{-\frac{\mu}{q}} r^{\frac{n}{q}} \int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)t^{-\frac{n}{q}-1}\left(\varphi_1(x,t)^{-1}\|f\|_{L_p(B(x,t))}\right)dt \\ &\lesssim \|b\|_{BMO}\|f\|_{VM_{p,\lambda}} \sup_{x\in\mathrm{R}^n,r>0} r^{\frac{n-\mu}{q}} \int_r^\infty \left(1+\ln\frac{t}{r}\right) r^{\frac{\lambda}{p}} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\lesssim \|b\|_{BMO}\|f\|_{VM_{p,\lambda}},\end{aligned}$$

for the case of $q < s$, we can also use the same method, so we omit the details, which completes the proof.

Corollary 12 *Under the conditions of Corollary 11, the operators* $M_{\Omega,b,\alpha}$ *and* $[b,\overline{T}_{\Omega,\alpha}]$ *are bounded from* $VM_{p,\lambda}$ *to* $VM_{q,\mu}$.

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