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Research Article

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Metrizability of Fuzzy Real Line

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Abstract Zadeh [1], 1965, introduced the concept of fuzzy sets by defining them in terms of mapping from a set into the unit interval on the real line. Fuzzy sets were introduced to provide means to describe situations mathematically which give rise to ill-defined classes, i.e., collections of objects for which there is no precise criteria for membership.

Keywords Metrizability, Fuzzy Real Line

Zadeh [1], 1965, introduced the concept of fuzzy sets by defining them in terms of mapping from a set into the unit interval on the real line. Fuzzy sets were introduced to provide means to describe situations mathematically which give rise to ill-defined classes, i.e., collections of objects for which there is no precise criteria for membership. Collections of this type have vague or "fuzzy" boundaries; there are objects for which it is impossible to determine whether or not they belong to the collection. The classical mathematical theories, by which certain types of certainty can be expressed, are the classical set theory and the probability theory. In terms of set theory, uncertainty is expressed by any given set of possible alternatives in situations where only one of the alternatives may actually happen. Uncertainty expressed in terms of sets of alternatives results from the nonspecificity inherent in each set. Probability theory expresses uncertainty in terms of a classical measure on subsets of a given set of alternatives. The set theory, introduced by Zadeh, presents the notion that membership in a given subset is a matter of degree rather than that of totally in or totally out. With fuzzy set theory, one obtains a logic in which statements may be true or false to different degrees rather than the bivalent situation of being true or false; consequently, certain laws of bivalent logic do not hold, e.g. the law of the excluded middle and the law of contradiction. This results in an enriched scientific methodology. Chang [2], introduced the notion of a fuzzy topology of a set in 1968, and our work is based on the study of the properties of fuzzy topological spaces.

Throughout this paper, L will be a Hutton algebra, i.e. complete and completely distributive lattice which has an order-reversing involution $':L\to L'$, the smallest element 0 and the largest element 1 ($0\neq 1$). We assume that a reader is familiar with the usual notions and basic concepts of L- topology and the lattice theory. For all undefined basic concepts are given in [1-6]. Let $\lambda:\boldsymbol{R}\to L$ be a monotone decreasing mapping.

Let $md_{\boldsymbol{R}}(L)=\left\{\lambda\in L^{\boldsymbol{R}}:\bigvee_{t\in\boldsymbol{R}}\lambda(t)=1,\ \bigwedge_{t\in\boldsymbol{R}}\lambda(t)=0\right\}$.

For $\forall\lambda\in md_{\boldsymbol{R}}(L)$ and $\forall t\in\boldsymbol{R}$ we put $\lambda(t-)=\bigwedge_{s<t}\lambda(s)$ and $\lambda(t+)=\bigvee_{t<s}\lambda(s)$. We define an equivalence relation "~" on $md_{\boldsymbol{R}}(L)$ as follows: for $\lambda,\mu\in md_{\boldsymbol{R}}(L)$ $\lambda\sim\mu\Leftrightarrow\forall t\in\boldsymbol{R}$, $\lambda(t-)=\mu(t-)$, $\lambda(t+)=\mu(t+)$.

Put $[\lambda]=\{\mu\in md_{\boldsymbol{R}}(L):\mu\sim\lambda\}$ and $R[L]=\{[\lambda]:\lambda\in md_{\boldsymbol{R}}(L)\}$.

1. Auxiliary lemmas.

Lemma 1.1 If $\lambda(t_0-)\neq\mu(t_0-)$ for some $t_0\in\boldsymbol{R}$, then for $\forall\varepsilon>0$ $\exists s_0\in(t_0-\varepsilon,t_0):\lambda(t_0-)\nleq\mu(s_0)$ or $\mu(t_0-)\nleq\lambda(s_0)$.

Proof. Suppose that $\exists\varepsilon>0$: for $\forall s\in(t_0-\varepsilon,t_0)$ $\lambda(t_0-)\leq\mu(s)$ and $\mu(t_0-)\leq\lambda(s)$ Then $\lambda(t_0-)\leq\underset{t_0-\varepsilon<s<t_0}{\wedge}\mu(s)$ and $\mu(t_0-)=\underset{t_0-\varepsilon<s<t_0}{\wedge}\lambda(s)$. Since λ and μ are monotone decreasing, we have

$$\lambda(t_0-)\leq\underset{t_0-\varepsilon<s<t_0}{\wedge}\mu(s)=\underset{s<t_0}{\wedge}\mu(s)=\mu(t_0-),$$

$$\mu(t_0-)\leq\underset{t_0-\varepsilon<s<t_0}{\wedge}\lambda(s)=\underset{s<t_0}{\wedge}\lambda(s)=\lambda(t_0-).$$

Hence, the relation $\lambda(t_0-)\neq\mu(t_0-)$ is contrary to the hypothesis. ∇

Lemma 1.2 If $\lambda(t_0-)\neq\mu(t_0-)$ for some $t_0\in\boldsymbol{R}$, then for $\forall\varepsilon>0$, $\exists s_0\in(t_0-\varepsilon,t_0)$ $\lambda(s_0+)\neq\mu(s_0+)$.

Proof. By Lemma 1.1 $\exists s_0\in(t_0-\varepsilon,t_0)$: $\lambda(t_0-)\nleq\mu(s_0)$ or $\mu(t_0-)\nleq\lambda(s_0)$ Suppose that $\lambda(t_0-)\nleq\mu(s_0)$. Since μ is monotone decreasing, we have $\mu(s_0+)\leq\mu(s_0)$. Hence, $\lambda(t_0-)\nleq\mu(s_0+)$. It is easy to see that

$$\{\lambda(s):s_0<s<t_0\}\subset\{\lambda(s):s<t_0\}.$$

Therefore, $\underset{s_0<s<t_0}{\vee}\lambda(s)\geq\underset{s<t_0}{\wedge}\lambda(s)=\lambda(t_0-)$, i.e. $\lambda(t_0-)\leq\underset{s_0<s<t_0}{\vee}\lambda(s)$.

Since λ is monotone decreasing we have

$$\lambda(t_0)\leq\underset{s_0<s<t_0}{\vee}\lambda(s)=\underset{s_0<s}{\vee}\lambda(s)=\lambda(s_0+),$$

i.e. $\lambda(t_0-)\leq\lambda(s_0+)$. Hence, since $\lambda(t_0-)\nleq\mu(s_0+)$, we have $\lambda(s_0-)\nleq\mu(s_0+)$. Therefore, $\lambda(s_0+)\neq\mu(s_0+)$. The following lemma is proved similarly. ∇

Lemma 1.3 If $\lambda(t_0+)\neq\mu(t_0+)$ for some $t_0\in\boldsymbol{R}$, then for $\forall\varepsilon>0$, $\exists s_0\in(t_0,t_0+\varepsilon)$: $\lambda(s_0-)\neq\mu(s_0-)$.

2. Metrics on $R[L]$.

Theorem 2.1 The map $d:R[L]\times R[L]\to\boldsymbol{R}_*=[-\infty,+\infty]$, defined as

$$d([\lambda],[\mu])=\sup_{\substack{k\in L,\\ k>0}}\left|\underset{\lambda(t)\geq k}{\vee}t-\underset{\mu(t)\geq k}{\vee}t\right|$$ is a metric on $R[L]$.

Proof (i) we will prove that $[\lambda]=[\mu]\Rightarrow d([\lambda],[\mu])=0$. For $\forall k\in L$ we put

$$t_1:=\underset{\lambda(t)\geq k}{\vee}t,\ t_2:=\underset{\mu(t)\geq k}{\vee}t \text{ and}$$

$$A:=\{t:\lambda(t)\geq k\},\ B:=\{t:\mu(t)\geq k\}.$$

If $\forall \varepsilon > 0,$ then it is clear that $\exists t' \in B: t_2 - \varepsilon < t'$. Therefore, $\mu(t') \geq k$ and $t_2 - \varepsilon < t'$. Since $[\lambda] = [\mu]$ and μ is monotone decreasing, then $\lambda(t'-) = \mu(t'-)$ and $\mu(t') \leq \mu(t'-)$. Therefore,

$$k \leq \mu(t') \leq \mu(t'-) = \lambda(t'-),$$

i.e. $k \leq \lambda(t'-)$. It is clear that $\exists t^* : t_2 - \varepsilon < t^* < t'$. Hence, from

$$k \leq \lambda(t'-) = \underset{s<t}{\wedge} \lambda(s) \leq \lambda(t^*),$$

we have $t_2 - \varepsilon < t^*$ and $k \leq \lambda(t^*)$ i.e. $t_2 - \varepsilon < t^*$ and $t^* \in A$. Because of the arbitrariness of k we have $d([\lambda],[\mu]) = 0$. Similarly, $t_1 \leq t_2$, i.e. $t_1 = t_2$. Now, we will show that $d([\lambda],[\mu]) = 0 \Rightarrow [\lambda] = [\mu]$. If $[\lambda] \neq [\mu]$, then for some $\exists t_0 \in □ : \lambda(t_0 -) \neq \mu(t_0 -)$ or $\lambda(t_0 +) \neq \mu(t_0 +)$. By Lemmas 1.2 and 1.3 we can assume that $\lambda(t_0 -) \neq \mu(t_0 -)$. Then for $a := \mu(t_0 -) = \underset{s<t_0}{\wedge} \mu(s)$ we have

$$a \not\leq \underset{s<t_0}{\wedge} \lambda(s) \text{ or } a \not\geq \underset{s<t_0}{\wedge} \lambda(s)$$

without loss of generality we can assume that $a \not\leq \underset{s<t_0}{\wedge} \lambda(s)$. Then $\exists s_0 :\ s_0 < t_0$ and $a \not\leq \lambda(s_0)$. (1)

Let $A := \{t : \lambda(t) \geq a\}$ and $B := \{t : \mu(t) \geq a\}$. Then $\vee B \geq t_0$. Really, since for $\forall \varepsilon > 0$, $\exists s' : t_0 - \varepsilon < s' < t_0$, then $\mu(s') \geq \underset{s<t_0}{\wedge} \mu(s) = a$, i.e. $\mu(s') \geq a$. Hence, $s' \in B$. Because of the arbitrariness of ε we have $t_0 \leq \vee B$. According to the hypothesis $\vee B = \underset{\mu(t)\geq k}{\vee} t = \underset{\lambda(t)\geq k}{\vee} t \geq t_0$. Then, for $\varepsilon := t_0 - s_0 > 0, \exists t' \in A : t_0 - \varepsilon < t'$, i.e. $a \leq \lambda(t')$ and $s_0 < t'$. Since λ is monotone decreasing, we have $a \leq \lambda(t')$ and $\lambda(t') < \lambda(s_0)$ i.e. $a \leq \lambda(s_0)$. This contradicts (1).

(ii) It is clear that $d([\lambda],[\mu]) = d([\mu],[\lambda])$.

(iii) We will show that $d([\lambda],[\eta]) \leq d([\lambda],[\mu]) + d([\mu],[\eta])$.

Let $\forall k_0 \in L (k_0 \neq 0)$.

Let

$$t_1(k_0) := \underset{\lambda(t)\geq k_0}{\vee} t,\ t_2(k_0) := \underset{\mu(t)\geq k_0}{\vee} t \text{ and } t_3(k_0) := \underset{\delta(t)\geq k_0}{\vee} t.$$

Since

$$|t_1(k_0) - t_3(k_0)| \leq |t_1(k_0) - t_2(k_0)| + |t_2(k_0) - t_3(k_0)|, \text{ then}$$

$$\sup_{\substack{k\in L,\\ k>0}} |t_1(k_0) - t_3(k_0)| \leq \sup_{\substack{k\in L,\\ k>0}} |t_1(k_0) - t_2(k_0)| \leq \sup_{\substack{k\in L,\\ k>0}} |t_2(k_0) - t_3(k_0)|$$

i.e. $d([\lambda],[\eta]) \leq d([\lambda],[\mu]) + d([\mu],[\eta])$. ∇

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