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On solution of a nonlocal problem with dynamic boundary conditions for a loaded linear parabolic equation by straight-line methods

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Abstract

We consider a nonlocal problem with dynamic boundary conditions for a loaded linear parabolic equation. For this problem we prove the unique solvability in Sobolev's spaces and the maximum principle under some natural conditions. We suggest the numerical straight-lines method for the finding of the solution of the problem. The convergence of the straight-lines method to the exact solution is also proved.

Keywords: Nonlocal problem, loaded parabolic equation, dynamic boundary condition, straight lines method, numerical solution, maximum principle, rate of convergence.

1 Introduction

Let $N \geq 1$ be an integer, $T > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\Gamma \equiv \partial\Omega$. Denote $D \equiv D \equiv \Omega \times [0, T]$, $\Gamma_T \equiv \Gamma \times [0, T]$. In the domain D we consider the following initial - boundary value problem with dynamic boundary conditions for the unknown function $u(x, t)$, $((x, t) \in D)$

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$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(k_{ij}(x,t)\frac{\partial u}{\partial x_j})-b(x,t)u(x,t)+B[u]+f(x,t),\quad (x,t)\in D, \quad (1)$$

$$\frac{\partial u(x,t)}{\partial t}+\alpha(x,t)u(x,t)+\beta(x,t)u\circ e=\mu(x,t),\quad (x,t)\in\Gamma_T, \qquad (2)$$

$$u(x,0)=\varphi(x),\quad x\in\overline{\Omega}. \qquad (3)$$

Here $k_{ij}=k_{ij}(x,t)$, $b=b(x,t)$, $f(x,t)$, $\alpha=\alpha(x,t)$, $\beta=\beta(x,t)$, $\mu(x,t)$, $\varphi(x)$ are given functions and we suppose the ellipticity condition with some $\nu>0$

$$\nu\xi^2\le\sum_{i,j=1}^{N}k_{ij}\xi_i\xi_j\le\nu^{-1}\xi^2,\xi\in R^N, \qquad (4)$$

$$b(x,t)\ge\nu>0,\quad \alpha(x,t)+\beta(x,t)\ge\nu>0,\quad \beta(x,t)\le 0. \qquad (5)$$

The term $B[u]$ in equation (1) represents a given bounded generally nonlocal linear operator in the space of continuous functions $C(\overline{D})$ with the properties (B_0 denotes the norm of $B:C(\overline{D})\to C(\overline{D})$)

$$|B[u]|^{(0)}_{\overline{D}}\le B_0|u|^{(0)}_{\overline{D}},\ \text{and } B[u](x,t)\ge 0 \text{ in } \overline{D} \text{ if } u(x,t)\ge 0 \text{ in } \overline{D}, \qquad (6)$$

and for any positive constant $a>0$

$$-b(x,t)a+B[a](x,t)<0 \text{ in } \overline{D}, \qquad (7)$$

where $b(x,t)$ is the coefficient from (1) and $|u|^{(0)}_{\overline{D}}$ is the norm of u in $C(\overline{D})$

$$|u|^{(0)}_{\overline{D}}\equiv\max_{(x,t)\in\overline{D}}|u(x,t)|.$$

Note that due to linearity (7) is equivalent to

$$-b(x,t)+B[1](x,t)\le-\nu<0 \text{ in } \overline{D}. \qquad (8)$$

Finally, $u\circ e$ in boundary condition (2) means the composition of the unknown function $u(x,t)$ with some given diffeomorphism $e(x)$ of the surface Γ onto itself, that is

$$u \circ e \equiv u(e(x), t), \quad e(x) : \Gamma \to \Gamma. \tag{9}$$

It is worth noting that the linear operator $B[u]$ may be a nonlocal operator, for example,

$$B[u] = \int_D G(x, t; y, \tau) u(y, \tau) dy d\tau, \quad G(x, t; y, \tau) \geq 0 \text{ in } D \times D$$

or, for example,

$$B[u] = \sum_{k=1}^{m} b_k(x, t) u(x, t_k), \quad t_k \in [0, T], k = \overline{1, m}. \tag{10}$$

Nonlocal problems of the (1)- (3) type (sometimes in slightly different but similar forms) arise in different mathematical models in various physical, financial, biological, social and engineering applications. Without pretending on a more less complete survey of such applications we mention, for example, the papers [1–26]. In particular, equation (1) is loaded due to the presence of the nonlocal term $B[u]$. Boundary condition (2) contains the highest derivative u_t and the nonlocal term $u \circ e$; therefore, it is dynamical and nonlocal at the same time. Note that the term $u \circ e$ is an analogue of the Bitsadze- Samarskii conditions (the Bitsadze-Samarskii conditions use a diffeomorphism e of the boundary Γ onto some surface S which may not coincide with Γ). It is worth noting that the investigations of problems with dynamic boundary conditions is an active direction in the present theory of PDE including numerical approaches to such problems. However, the author is not aware of results regarding correctness of problems of the (1)-(3) type where a loaded equation has nonlocal dynamic boundary conditions or of respective numerical approaches. The numerical solutions of problems for loaded parabolic equations are covered in, for example, the papers [1–3, 5, 6, 16, 20, 25]. In particular, the paper [5] also discusses the application of the straight-line method to a different problem with a loaded parabolic equation. However, we consider the case of dynamic boundary conditions and prove the convergence of the straight-lines method.

The purpose of the present paper is twofold. In the next section 2 we first prove the unique solvability of problem (1)-(3) in Sobolev spaces $W_q^{2,1}(D)$ by the Fredholm theory and we prove a comparison principle for the problem.

It is worth noting that without conditions (5), (7) this Fredholm problem is incorrect in general (the more in the spaces of smooth functions) and may cause some inverse problems - see Remark 1. Main results of section 2 are presented in theorems 1- 4 below. Second, in the last section 3 we investigate the application of the straight-lines method in solving the one-dimensional setting of problem (1)-(3). Here we prove the convergence of this method to the exact solution and the main results are formulated in theorems 5-8.

In what follows we denote by the same symbols C or ν all absolute constants or constants depending only on fixed given data of the problem.

2 Correctness of problem (1)-(3) in spaces $W_q^{2,1}(D)$.

Let $\gamma \in (0,1)$ and let k be a nonnegative integer. We use the standard anisotropic Hölder spaces $C^{k+\gamma,\frac{k+\gamma}{2}}(\overline{D})$ of functions $u(x,t)$ in $\overline{D}$ with the smoothness in the x-variables up to the order $k+\gamma$ and with the smoothness in the t-variable up to the order $(k+\gamma)/2$. And we use also the Hölder spaces $C^{k+\gamma}(\overline{\Omega})$ and $C^{k+\gamma,\frac{k+\gamma}{2}}(\Gamma_T)$ in Ω and on Γ_T. Such spaces are sometimes designated also as $H^{k+\gamma,\frac{k+\gamma}{2}}(\overline{D})$, $H^{k+\gamma}(\overline{\Omega})$, and $H^{k+\gamma,\frac{k+\gamma}{2}}(\Gamma_T)$ and their definitions can be found in [27], for example. The norm in the spaces $C^{k+\gamma,\frac{k+\gamma}{2}}(\overline{D})$ and $C^{k+\gamma}(\overline{\Omega})$ are denoted by

$$|u|_{\overline{D}}^{(k+\gamma,\frac{k+\gamma}{2})} \equiv \|u\|_{C^{k+\gamma,\frac{k+\gamma}{2}}(\overline{D})}, \quad |u|_{\overline{\Omega}}^{(k+\gamma)} \equiv \|u\|_{C^{k+\gamma}(\overline{\Omega})}.$$

For $q \geq N = 1$ we use the standard anisotropic Sobolev space $W_q^{2,1}(D)$ of functions $u(x,t)$ in D with the standard norm

$$\|u\|_{q,D}^{(2,1)} \equiv \|u\|_{W_q^{2,1}(D)} \equiv \left[\int\limits_D \left(|u_t|^q + \sum_{i=1}^N |u_{x_i}|^q + \sum_{i,j=1}^N |u_{x_i x_j}|^q + |u|^q\right) dxdt\right]^{\frac{1}{q}}.$$

About the data of problem (1)-(3) we assume the following. Let the boundary Γ of the domain Ω be a surface of the class $C^{k+2+\gamma,\frac{k+2+\gamma}{2}}$ (see [27]) and let

$$f(x,t) \in C^{k+\gamma,\frac{k+\gamma}{2}}(\overline{D}), \quad \mu(x,t) \in C^{k+2+\gamma,\frac{k+2+\gamma}{2}}(\Gamma_T),$$

$$\varphi(x) \in C^{k+2+\gamma}(\overline{\Omega}), \quad e(x) \in C^{k+2+\gamma}(\Gamma), \tag{11}$$

$$k_{i,j}(x,t) \in C^{k+1+\gamma, \frac{k+1+\gamma}{2}}(\overline{D}), \quad b(x,t) \in C^{k+\gamma, \frac{k+\gamma}{2}}(\overline{D}),$$

$$\alpha(x,t), \beta(x,t) \in C^{k+2+\gamma, \frac{k+2+\gamma}{2}}(\Gamma_T). \tag{12}$$

Before we turn to the comparison and existence - uniqueness theorems we now discuss dynamic boundary condition (2).

Lemma 1. *Let conditions* (11), (12) *be satisfied. Then there exists the unique function* $\theta(x,t) \in C^{k+2+\gamma, \frac{k+2+\gamma}{2}}(\Gamma_T)$ *with*

$$|\theta|_{\Gamma_T}^{(k+2+\gamma, \frac{k+2+\gamma}{2})} \leq C \left(|\mu|_{\Gamma_T}^{(k+2+\gamma, \frac{k+2+\gamma}{2})} + |\varphi|_{\overline{\Omega}}^{(k+2+\gamma)} \right), \quad \theta(x,0) = \varphi(x)|_{\Gamma} \tag{13}$$

and condition (2) *is equivalent to the Dirichlet condition*

$$u(x,t)|_{\Gamma_T} = \theta(x,t). \tag{14}$$

Proof. Integrate condition (2) in t and write it in the form ($x \in \Gamma$)

$$\begin{aligned} u(x,t) = &\left(-\int_0^t \alpha(x,\tau) u(x,\tau) d\tau - \int_0^t \beta(x,\tau) u(e(x),\tau) d\tau \right) \\ &+ \left(\int_0^t \mu(x,\tau) d\tau + \varphi(x) \right) \equiv A[u] + \theta_0. \end{aligned} \tag{15}$$

Since coefficients α and β are smooth and since the change of variables $x \to e(x)$ is also smooth we see that

$$|A[u]|_{\Gamma_T}^{(k+2+\gamma, \frac{k+2+\gamma}{2})} \leq CT |u|_{\Gamma_T}^{(k+2+\gamma, \frac{k+2+\gamma}{2})}. \tag{16}$$

Analogously,

$$|\theta_0|_{\Gamma_T}^{(k+2+\gamma, \frac{k+2+\gamma}{2})} \leq T |\mu|_{\Gamma_T}^{(k+2+\gamma, \frac{k+2+\gamma}{2})} + |\varphi|_{\overline{\Omega}}^{(k+2+\gamma)}. \tag{17}$$

Let first T be so small that we have $CT \leq 1/2$ in (16). Then the operator $A[u]$ is a contraction on $C^{k+2+\gamma, \frac{k+2+\gamma}{2}}(\Gamma_T)$ and the equation $u = A[u] + \theta_0$

from (15) has the unique solution $\theta(x,t) \in C^{k+2+\gamma,\frac{k+2+\gamma}{2}}(\Gamma_T)$ which is equal to $\theta = (I-A)^{-1}\theta_0$. Besides, $|\theta|_{\Gamma_T}^{(k+2+\gamma,\frac{k+2+\gamma}{2})} \leq C|\theta_0|_{\Gamma_T}^{(k+2+\gamma,\frac{k+2+\gamma}{2})}$ and thus the first relation in (13) follows from (17). The second relation in (13) follows by construction from (15). Now if with this θ condition (14) is satisfied by some function $u(x,t)$, then, by the construction of θ, $u(x,t)$ satisfies (15). Differentiating this relation in t, we arrive at (2). Thus the lemma is proved for a sufficiently small $T > 0$.

Let now $T > 0$ be arbitrary. We can choose sufficiently small $0 < T_0 < T$ so that $CT_0 \leq 1/2$ in (16) and consider first the part Γ_{T_0} of Γ_T, $\Gamma_{T_0} \subset \Gamma_T$. As it is shown, we can find the corresponding function $\theta(x,t)$ on the time interval $[0,T_0]$. Then we consider the time interval $[T_0/2, 3T_0/2]$ (of length T_0) and the corresponding surface $\Gamma \times [T_0/2, 3T_0/2] \subset \Gamma_T$. Starting from the initial value of time $T_0/2$ we repeat our reasonings and obtain θ as the solution of (15) (with $T_0/2$ instead of 0) on the interval $[T_0/2, 3T_0/2]$. Moving now up to the t-axis by steps of length $T_0/2$, we obtain the desired function $\theta(x,t)$ on whole interval $[0,T]$. This proves the lemma. $\square$

We proceed with a maximum and a comparison principles for the problems (1)-(3) or (1), (14), (3).

Theorem 1. *Let conditions* (4)-(7), (11), (12) *be satisfied. Let also*

$$f(x,t) \leq 0 \text{ in } D, \quad \mu(x,t) \leq 0 \text{ (or } \theta(x,t) \leq 0\text{) on } \Gamma_T, \quad \varphi(x) \leq 0 \text{ in } \Omega. \tag{18}$$

Let $u(x,t)$ be a solution to (1)- (3) *from the space $C(\overline{D})$ and it's derivatives in x up to the second order are continuous in the open domain D and the derivative u_t is continuous in the open domain D including Γ_T.*

Then $u(x,t) \leq 0$ in $\overline{D}$.

Proof. The simple proof is based on standard for the maximum principle arguments by contradiction. We consider only boundary condition (2) since the difference for simpler condition (14) is evident.

Let there are points in $\overline{D}$, where $u(x,t)$ is positive. Since $u(x,t) \in C(\overline{D})$, there exists $(x_0,t_0) \in \overline{D}$, where $u(x,t)$ attains it's positive maximum over $\overline{D}$. This can not happen at the bottom $\Omega \times \{0\}$ of $\overline{D}$ because of the last condition in (18). This is also not possible for points on Γ_T since at such point must be $u_t(x_0,t_0) \geq 0$ and

$$\frac{\partial u(x_0,t_0)}{\partial t} + \alpha(x_0,t_0)u(x_0,t_0) + \beta(x_0,t_0)u(e(x_0),t_0) \geq$$

$$\geq [\alpha(x_0,t_0)+\beta(x_0,t_0)]\,u(x_0,t_0)>0,$$

since $u(x_0,t_0) \geq u(e(x_0),t_0)$, $\beta(x_0,t_0) \leq 0$, and $[\alpha(x_0,t_0)+\beta(x_0,t_0)] > 0$. But this contradicts to the second condition in (18).

Analogously, let (x_0,t_0) be an inner point of $\overline{D}$. Consider the right hand side of equation (1) at this point. We have

$$\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(k_{ij}(x_0,t_0)\frac{\partial u}{\partial x_j})-b(x_0,t_0)u(x_0,t_0)+B[u](x_0,t_0)+f(x_0,t_0)=$$

$$=\sum_{i,j=1}^{N}k_{ij}(x_0,t_0)\frac{\partial^2 u(x_0,t_0)}{\partial x_i\partial x_j}+\sum_{i,j=1}^{N}\frac{\partial k_{ij}(x_0,t_0)}{\partial x_i}\frac{\partial u}{\partial x_j}(x_0,t_0)+$$

$$+\{-b(x_0,t_0)u(x_0,t_0)+B[u(x_0,t_0)](x_0,t_0)\}+B[u-u(x_0,t_0)](x_0,t_0)+f(x_0,t_0)$$

$$\equiv I_1+I_2+I_3+I_4+I_5. \tag{19}$$

The term $I_1 \leq 0$ since at the maximum point (x_0,t_0) the matrix of the second derivatives $\left\{\frac{\partial^2 u(x_0,t_0)}{\partial x_i\partial x_j}\right\}$ is negatively defined. The second term I_2 contains the first derivatives and so it vanishes at the maximum point. The term $I_3 < 0$ is strictly negative because of (7) with $a = u(x_0,t_0)$. Since $u(x,t)-u(x_0,t_0) \leq 0$ and due to the second condition in (6) the term $I_4 \leq 0$ is nonpositive and also $I_5 = f(x_0,t_0) \leq 0$. Thus, the right hand side of equation (1) at this point is strictly negative. But on the other hand the left hand side of equation (1) at this point must be nonnegative $u_t(x_0,t_0) \geq 0$. This contradiction finishes the proof of the theorem. $\square$

As a corollary we have the following comparison theorem.

Theorem 2. *Let under the conditions of Theorem 1 $u_i(x,t)$, $i=1,2$ be solutions to problem* (1)-(3) *(or* (1), (14), (3)*) with data $f_i(x,t)$, $\mu_i(x,t)$ ($\theta_i(x,t)$), $\varphi_i(x)$, $i=1,2$, correspondingly. If*

$$f_1(x,t)\leq f_2(x,t),\quad \mu_1(x,t)\leq\mu_2(x,t)\ (\theta_1(x,t)\leq\theta_2(x,t)),\quad \varphi_1(x)\leq\varphi_2(x),$$

then $u_1(x,t) \leq u_2(x,t)$.

The proof directly follows from the previous theorem if we note that the difference $u_1(x,t) - u_2(x,t)$ is a solution to linear problem (1)-(3) (or (1), (14), (3)) with the corresponding data.

Now we give some estimate for the maximum modulus norm $|u|_{\overline{D}}^{(0)}$ of a solution.

Theorem 3. *Under the conditions of Theorem 1*

$$|u|_{\overline{D}}^{(0)} \leq C\left(|f|_{\overline{D}}^{(0)} + |\mu|_{\Gamma_T}^{(0)} + |\varphi|_{\overline{\Omega}}^{(0)}\right), \tag{20}$$

for problem (1)-(3) *or*

$$|u|_{\overline{D}}^{(0)} \leq C\left(|f|_{\overline{D}}^{(0)} + |\theta|_{\Gamma_T}^{(0)} + |\varphi|_{\overline{\Omega}}^{(0)}\right) \tag{21}$$

for problem (1), (14), (3), *where the constants C do not depend on f, μ, θ, φ.*

Proof. We consider only (20) since (21) is completely analogous and more simple.

Consider the function $F(x,t) = K(R-x^2)$, where $K = |f|_{\overline{D}}^{(0)} + |\mu|_{\Gamma_T}^{(0)} + |\varphi|_{\overline{\Omega}}^{(0)}$ and the constant $R > 0$ is sufficiently large and will be chosen later. Note that since Ω is bounded there exists $r > 0$ with $x^2 \leq r$ for $(x,t) \in D$ and consequently $F(x,t) \geq K(R-r) > 0$ if we choose big R. Moreover, for any $\varepsilon \in (0, 1/2)$ we can choose $R = R(r)$ so large that

$$R - r \geq (1-\varepsilon)R \text{ and so } F(x,t) \geq K(1-\varepsilon)R. \tag{22}$$

Let $u(x,t)$ be a solution to (1)-(3). Denote the difference $v(x,t) = u(x,t) - F(x,t)$. Since problem (1)-(3) is linear, it is directly verified that the function $v(x,t)$ satisfies problem (1)- (3) with $\widehat{f}$, $\widehat{\mu}$, and $\widehat{\varphi}$ instead of f, μ, and φ, where

$$\widehat{\varphi}(x) = \varphi(x) - K(R - x^2), \tag{23}$$

$$\widehat{\mu}(x,t) = \mu(x,t) - \left[\alpha F(x,t) + \beta F(e(x),t)\right], \tag{24}$$

$$\widehat{f}(x,t) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(k_{ij}(x,t)\frac{\partial F}{\partial x_j}) - b(x,t)F(x,t) + B[F] + f(x,t). \tag{25}$$

For the function $\widehat{\varphi}(x)$ we have

$$\widehat{\varphi}(x) \leq \varphi(x) - |\varphi|_{\Omega}^{(0)} R/2 \leq |\varphi|_{\Omega}^{(0)}(1 - R/2) \leq 0, \tag{26}$$

if $R = R(r)$ is sufficiently large. Considering $\widehat{\mu}(x,t)$ and taking into account (5), we see that

$$\widehat{\mu}(x,t) \leq \mu(x,t) - K\left[\alpha(R - x^2) + \beta(R - e(x)^2)\right] \leq$$
$$\leq |\mu|_{\Gamma_T}^{(0)} - K(\alpha+\beta)R + rK(|\alpha|_{\Gamma_T}^{(0)} + |\beta|_{\Gamma_T}^{(0)}) \leq$$
$$\leq |\mu|_{\Gamma_T}^{(0)}\left\{1 - \nu R + r(|\alpha|_{\Gamma_T}^{(0)} + |\beta|_{\Gamma_T}^{(0)})\right\} \leq 0 \tag{27}$$

if $R = R(r, \alpha, \beta)$ is chosen sufficiently large.

Further, for the first term in $\widehat{f}(x,t)$ we have

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(k_{ij}(x,t)\frac{\partial F}{\partial x_j}) = -2K\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(k_{ij}(x,t)\cdot x_j) \leq KC(k_{ij})r.$$

Consequently, taking into account (8),

$$\widehat{f}(x,t) \leq KC(k_{ij})r + K\left\{-b(R-x^2) + B[R-x^2]\right\} + f =$$
$$= KC(k_{ij})r + KR\{-b(x,t) + B[1]\} + br - B[x^2] + f \leq$$
$$\leq -K\{vR - C(k_{ij}, b, B, r)\} + f \leq |f|_{\overline{D}}^{(0)}\{1 + C(k_{ij}, b, B, r) - vR\} \leq 0 \tag{28}$$

if $R = R(k_{ij}, b, B, r)$ is chosen sufficiently large.

Thus, taking $R = R(k_{ij}, b, B, \alpha, \beta, \Omega)$ sufficiently large, we obtain (26)-(28). On the base of Theorem 1 we conclude that $v(x,t) = u(x,t) - F(x,t) \leq 0$ in $\overline{D}$, that is

$$u(x,t) \leq F(x,t) \leq C(k_{ij}, b, B, \alpha, \beta, \Omega)(|f|_{\overline{D}}^{(0)} + |\mu|_{\Gamma_T}^{(0)} + |\varphi|_{\Omega}^{(0)}).$$

Considering now in absolutely the same way the function $-u(x,t)$ instead of $u(x,t)$, we obtain for this function exactly the above inequality as well. This means (20) and completes the proof of the theorem. □

Formulate now the existence - uniqueness theorem.

Theorem 4. *Let conditions* (4), (11), (12), (6), (8) *be satisfied. Let in addition the operator* $B[u]$ *satisfy*

$$|B[u]|_{\overline{D}}^{(k+\gamma,\frac{k+\gamma}{2})} \leq C|u|_{\overline{D}}^{(k+\gamma,\frac{k+\gamma}{2})}, \quad k \geq 0, \tag{29}$$

Then problem (1)-(3) *has the unique solution* $u(x,t) \in W_q^{2,1}(D) \cap C^{\gamma,\gamma/2}(\overline{D})$ *and*

$$\|u\|_{q,D}^{(2,1)} + |u|_{\overline{D}}^{(\gamma,\frac{\gamma}{2})} \leq C\left(|f|_{\overline{D}}^{(\gamma,\frac{\gamma}{2})} + |\mu|_{\Gamma_T}^{(2+\gamma,\frac{2+\gamma}{2})} + |\varphi|_{\Omega}^{(2+\gamma)}\right), \tag{30}$$

where the constant C *does not depend on* f, μ, φ. *Moreover, for any subdomain* D' *of domain* D *with* $\overline{D'} \subset D$ *the solution* $u(x,t)$ *belongs to the class* $C^{k+2+\gamma,(k+2+\gamma)/2}(D')$ *and*

$$|u|_{\overline{D'}}^{(k+2+\gamma,\frac{k+2+\gamma}{2})} \leq C_{D'}\left(|f|_{\overline{D}}^{(k+\gamma,\frac{k+\gamma}{2})} + |\mu|_{\Gamma_T}^{(k+2+\gamma,\frac{k+2+\gamma}{2})} + |\varphi|_{\Omega}^{(k+2+\gamma)}\right), \tag{31}$$

where the constant $C_{D'}$ *does not depend on* f, μ, φ.

Remark 1. *Although all the data of the problem are smooth, the solution of the problem is not generally smooth in whole closed cylinder* $\overline{D}$. *The reason is that we can not insure necessary compatibility conditions at* $\Gamma \times \{t = 0\}$. *Such conditions of the first order, for example, look like (we substitute* u_t *from the equation in the boundary conditions at* $t = 0$*)*

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(k_{ij}(x,0)\frac{\partial \varphi}{\partial x_j}) - b(x,0)\varphi(x) + B[u](x,0) + f(x,0) +$$

$$+\alpha(x,0)\varphi(x) + \beta(x,0)\varphi \circ e = \mu(x,0).$$

This condition directly follows from the requirement of smoothness of the solution up to $\Gamma \times \{t = 0\}$. *But it contains nonlocal operator* $B[u]$ *and the last can not be directly calculated from the data. Thus, such condition may play a role of an additional condition on the unknown solution. Generally, such requirements may lead to some statements of inverse problems for* (1)-(3), *see, for example, [2, 5, 20]. We do not consider this issue in the present paper.*

Proof. First of all, due to Lemma 1 we can replace condition (2) by Dirichlet condition (14). Further, making the change of the unknown function, $u \to v$, $u(x,t) = v(x,t) + \varphi(x)$, we can reduce the general situation to the case $\varphi(x) \equiv 0$. After such change of the unknown the righthand side f in (1) and θ in (14) are also changed but their properties in (11) are preserved. Moreover, due to the second relation in (13), new boundary condition $\theta(x,t)$ will satisfy $\theta(x,0) \equiv 0$. Now we can extend the function $\theta(x,t)$ from Γ_T to the whole domain $\overline{D}$ (the way of such extension is described in, for example, [27]) up to the function $\Theta(x,t)$ of the class $C^{k+2+\gamma,(k+2+\gamma)/2}(\overline{D})$. If we make one more change of the unknown function, $u \to v$, $u(x,t) = v(x,t) + \Theta(x,t)$, we reduce the original problem to problem (1), (14), (3) with $\varphi(x) \equiv 0$ and $\theta(x,t) \equiv 0$.

We choose $q > N+1$ so big that according to the Sovolev embedding $W_q^{2,1}(D) \subset W_q^{1,1}(D) \subset C^{\gamma,\gamma/2}(\overline{D})$ that is for a function $u \in W_q^{2,1}(D)$ we have

$$|u|_{\overline{D}}^{(\gamma,\gamma/2)} \le C \, \|u\|_{q,D}^{(2,1)} . \tag{32}$$

Denote further by $\widetilde{W}_q^{2,1}(D)$ the proper subspace of $W_q^{2,1}(D)$ which consists of functions that vanish at Γ_T and at $\{t=0\}$ that is satisfy zero conditions (14) and (3). We are going to apply the well known Fredholm theory for operator equations so we consider on $\widetilde{W}_q^{2,1}(D)$ the equation

$$Lu - Bu = f, \quad f \in L_q(D), \quad u \in \widetilde{W}_q^{2,1}(D), \tag{33}$$

where

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(k_{ij}(x,t)\frac{\partial u}{\partial x_j}) + b(x,t)u(x,t), \quad Bu \equiv B[u]. \tag{34}$$

Note that the operator $B[u] : \widetilde{W}_q^{2,1}(D) \to L_q(D)$ is well defined. Really, from \ (6) and (32) it follows that for $u \in \widetilde{W}_q^{2,1}(D)$

$$\|B[u]\|_{q,D} \le |D|^{1/q} |B[u]|_{\overline{D}_T}^{(0)} \le C B_0 |u|_{\overline{D}}^{(0)} \le C \, \|u\|_{q,D}^{(2,1)} ,$$

where $|D|$ is the measure of D and $\|B[u]\|_{q,D}$ denote the norm of $B[u]$ in the space $L_q(D)$. It is evident that equation (33) is exactly rewritten equation (1) with the operator $\lambda B[u]$ instead of $B[u]$ and the condition $u \in \widetilde{W}_q^{2,1}(D)$ guarantees the necessary boundary and initial conditions.

Recall now that the embedding $C^{\gamma,\gamma/2}(\overline{D}_T) \subset\subset L_q(D)$ is compact and on the base of (32) and (29) we conclude that the operator $B[u] : \widetilde{W}_q^{2,1}(D) \to C^{\gamma,\gamma/2}(\overline{D}) \to L_q(D)$ is a compact operator. Besides, the operator L is an invertible operator as an operator $L : \widetilde{W}_q^{2,1}(D) \to L_q(D)$. This follows directly from Theorem 9.1 in [27], where it is proved that for any $f \in L_q(D)$ the equation $Lu = f$ has the unique solution $u(x,t) \in \widetilde{W}_q^{2,1}(D)$ and

$$\|u\|_{q,D}^{(2,1)} \leq C \, \|f\|_{q,D} \, . \tag{35}$$

These two facts mean that equation (33) is a Fredholm equation and it's solvability and invertibility of the operator $L-B$ are equivalent to the unique solvability for $f \equiv 0$ of equation (33). Let us show that (33) has the zero solution only for $f \equiv 0$.

Let some $u \in \widetilde{W}_q^{2,1}(D)$ satisfy equation (33) with $f \equiv 0$. Write this assumption as

$$Lu = B[u].$$

Since in this case $u, B[u] \in C^{\gamma,\gamma/2}(\overline{D})$, we conclude on the base of well known local estimates for parabolic equations (see [27]) that the solution u is in fact smooth inside D. Thus, the solution u satisfies all the conditions of the comparison theorems 1-3. Applying now to equation (33) with $f = 0$ Theorem 3, we see that the solution $u \equiv 0$. This means that equation (33) has the unique solution $u \in \widetilde{W}_q^{2,1}(D)$ for any $f \in L_q(D)$ and estimate (30) is valid. Estimate (31) now follows from local estimates for parabolic equations (see [27]) because all the data of the problem are smooth. This completes the proof of the theorem. □

3 Application of straight-lines method

In this section we investigate the application of straight-lines method for finding the solution of some particular one-dimensional statement of problem (1)-(3).

3.1 Problem statement and application of straight-lines method

Let us formulate this one-dimensional problem.

Let $l, T > 0$, m is a positive integer. It is required to find the continuous in the closed domain $\overline{D} = \{0 \le x \le l, 0 \le t \le T\}$ function $u(x,t)$ according to the conditions

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(k(x,t)\frac{\partial u}{\partial x}) - bu(x,t) + \sum_{k=1}^{m} b_k u(x,t_k) + f(x,t), \quad (x,t) \in D, \tag{36}$$

$$\frac{\partial u(0,t)}{\partial t} + \alpha_1 u(0,t) + \beta_1 u(l,t) = \mu_1(t), \quad 0 \le t \le T, \tag{37}$$

$$\frac{\partial u(0,t)}{\partial t} + \alpha_2 u(0,t) + \beta_2 u(l,t) = \mu_2(t), \quad 0 \le t \le T, \tag{38}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l. \tag{39}$$

Here $k(x,t) > 0$, $f(x,t)$, $\mu_1(t)$, $\mu_2(t)$, $\varphi(x)$ are given continuous functions, $k(x,t)$ is continuously differentiable with respect to x, b, b_k, $k = 1,2,\ldots,m$, α_1, β_1, α_2, β_2 are given constants, $t_1, t_2, \ldots, t_m \in (0,T]$ are given fixed points. To apply the straight-lines method we make the problem discrete in x. Divide the interval $[0,l]$ into N identical parts by the points $x_n = nh$, $n = 0,1,\ldots,N$, $Nh = l$, and consider equation (36) on straight lines $x = x_n$, $n = 1,\ldots,N-1$. We associate to problem (36)-(39) the following problem for unknown functions $y_n(t)$, $n = 0,1,\ldots,N$,

$$\begin{aligned} \frac{dy_n(t)}{dt} = \frac{1}{h}\Bigg[&\frac{k(x_{n+1},t)+k(x_n,t)}{2}\frac{y_{n+1}(t)-y_n(t)}{h} \\ &- \frac{k(x_n,t)+k(x_{n-1},t)}{2}\frac{y_n(t)-y_{n-1}(t)}{h}\Bigg] \\ &+ by_n(t) + \sum_{k=1}^{m} b_k y_n(t_k) + f_n(t), \\ &n = 1,\ldots,N-1, \quad 0 < t \le T, \end{aligned} \tag{40}$$

$$\frac{dy_0(t)}{dt} + \alpha_1 y_0(t) + \beta_1 y_N(t) = \mu_1(t), \quad 0 \le t \le T, \tag{41}$$

$$\frac{dy_N(t)}{dt} + \alpha_2 y_0(t) + \beta_2 y_N(t) = \mu_2(t), \quad 0 \le t \le T, \tag{42}$$

$$y_n(0) = \varphi(x_n), \quad n = 0, 1, \ldots, N. \tag{43}$$

Here by $y_n(t)$ we denote the approximate value of the solution $u(x,t)$ to problem (36)-(39) on the straight line $x = x_n$, $f_n(t) = f(x_n, t)$. We show below that problem (40)- (43) approximates original problem (36)-(39) to within $O(h^2)$ provided that the solution of (36)-(39) $u(x,t)$ and the coefficient $k(x,t)$ have in the domain $D = \{0 < x < l, 0 < t \le T\}$ bounded partial derivatives in the variable x up to fourth and third orders respectively. To solve problem (40)-(43) it is first necessary to solve the problem

$$\frac{dy_0(t)}{dt} + \alpha_1 y_0(t) + \beta_1 y_N(t) = \mu_1(t), \quad 0 \le t \le T, \tag{44}$$

$$\frac{dy_N(t)}{dt} + \alpha_2 y_0(t) + \beta_2 y_N(t) = \mu_2(t), \quad 0 \le t \le T, \tag{45}$$

$$y_0(0) = \varphi(0), \quad y_N(0) = \varphi(l). \tag{46}$$

This is a Cauchy problem for a linear system of ordinary differential equations with constant coefficients with respect to $y_0(t)$, $y_N(t)$. It is always possible to find the exact solution to this problem. Let this solution be found. Then taking into account the expressions of the found solution $y_0(t)$ and $y_N(t)$, we can rewrite (40), (43) in the following matrix form

$$\frac{dy(t)}{dt} + P(t)y(t) + \sum_{k=1}^{m} b_k y(t_k) = f(t), \quad 0 < t \le T,$$

$$y(0) = \varphi_0,$$

where the unknown is $y(t) = \{y_1(t), \ldots, y_{N-1}(t)\}$. This is a nonlocal problem for a linear system of ordinary differential equation. It can be solved by the method described in [28] and we refer the reader to this paper.

3.2 Maximum principle and some theorems following from this principle

Consider first problem (40)-(43) and prove the following theorem for the solution of this problem.

By analogy to (5)-(7) we suppose that in (36)-(39) and in (40)-(43)

$$
\begin{aligned}
&b_k \geq 0, \;\; k=1,\ldots,m, \quad b+\sum_{k=1}^{m} b_k < 0, \\
&\beta_1, \alpha_2 \leq 0, \quad \alpha_1+\beta_1>0, \quad \alpha_2+\beta_2>0.
\end{aligned} \tag{47}
$$

Theorem 5. *(Maximum principle)*

Let functions $y_n(t), n=0,1,\ldots,N$, *satisfy problem* (40)-(43) *and conditions* (47) *are fulfilled. If in* (40)-(43) $f_n(t) \leq 0$ *(*$f_n(t) \geq 0$*),* $n=1,\ldots,N-1$, $0 \leq t \leq T$, $\mu_1(t) \leq 0$, $\mu_2(t) \leq 0$ *(*$\mu_1(t) \geq 0$, $\mu_2(t) \geq 0$*),* $0 \leq t \leq T$, *then the solution* $y_n(t), n=0,1,\ldots,N$, *can not attain the greatest positive (the least negative) value in the interval* $(0,T]$.

Proof. We prove only the first part of the theorem for the greatest positive value. The rest part is completely analogous. We use the reasonings by contradiction and is analogous to the proof of Theorem 1. Let there exists a point $t_0 \in (0,T]$ wherein the solution of (40)-(43) accepts the greatest positive value for $n=n_0$

$$
y_{n_0}(t_0) = \max_{0<t\leq T, 0\leq n\leq N} y_n(t) = M > 0.
$$

Let $0<n<N$. Consider equation (40) for $n=n_0$ at the point $t=t_0$. Since for all $0 \leq n \leq N$ we have $y_n(t_0) \leq y_{n_0}(t_0)$, we infer, taking into account (47),

$$
\begin{aligned}
f_{n_0}(t_0) = {} & \frac{dy_n(t)}{dt} \\
& - \frac{1}{h}\left[\frac{k(x_{n_0+1},t_0)+k(x_{n_0},t_0)}{2}\frac{y_{n_0+1}(t_0)-y_{n_0}(t_0)}{h}\right. \\
& \left. - \frac{k(x_{n_0},t_0)+k(x_{n_0-1},t_0)}{2}\frac{y_{n_0}(t_0)-y_{n_0-1}(t_0)}{h}\right] \\
& - b y_{n_0}(t_0) - \sum_{k=1}^{m} b_k y_{n_0}(t_k) \\
\geq {} & -b y_{n_0}(t_0) - \sum_{k=1}^{m} b_k y_{n_0}(t_k) \\
\geq {} & -\left(b+\sum_{k=1}^{m} b_k\right) y_{n_0}(t_0) > 0.
\end{aligned}
$$

This contradicts the condition $f_{n_0}(t_0) \le 0$.

Let now $n_0 = 0$. Then, under conditions (47), we have from (41)

$$\mu_1(t_0) = \frac{dy_0(t_0)}{dt} + \alpha_1 y_0(t_0) + \beta_1 y_N(t_0) \ge (\alpha_1 + \beta_1) y_0(t_0) > 0$$

and this contradicts the assumption $\mu_1(t) \le 0$.

The situation with $n_0 = N$ is completely analogous. This proves the theorem. $\square$

From this theorem we obtain in the standard way the following assertion.

Theorem 6. *Let conditions* (47) *are fulfilled. Let the right sides of equations* (40) *and boundary conditions* (41), (42) *satisfy the conditions*

$$f_n(t) \le 0 \quad (f_n(t) \ge 0), \quad n = 1, \dots, N-1, \quad 0 \le t \le T,$$

$$\mu_1(t) \le 0,\ \mu_2(t) \le 0 \quad (\mu_1(t) \ge 0,\ \mu_2(t) \ge 0), \quad 0 \le t \le T.$$

If $y_n(0) \ge 0$ *(*$y_n(0) \le 0$*),* $n = 0, 1, \dots, N$*, then* $y_n(t) \ge 0$ *(*$y_n(t) \le 0$*),* $n = 0, 1, \dots, N, \quad 0 \le t \le T$.

Corollary 1. *Let conditions* (47) *be fulfilled. Then the homogeneous problem corresponding to problem* (40)-(43) *have only the trivial solution* $y_n(t) = 0$, $n = 0, 1, \dots, N$.

Theorem 7. *Let* $y_n(t)$*,* $n = 0, 1, \dots, N$*, be a solution to problem* (40)-(43) *and let* $\widetilde{y}_n(t)$*,* $n = 0, 1, \dots, N$*, be a solution to the same problem but with another corresponding data* $\widetilde{f}_n(t)$*,* $n = 1, \dots, N-1$*,* $\widetilde{\mu}_1(t)$*,* $\widetilde{\mu}_2(t)$*,* $\widetilde{\varphi}(x_n)$*,* $n = 0, 1, \dots, N$*, respectively. If*

$$|f_n(t)| \le \widetilde{f}_n(t), \quad |\mu_1(t)| \le \widetilde{\mu}_1(t), \quad |\mu_2(t)| \le \widetilde{\mu}_2(t), \quad |\varphi(x_n)| \le \widetilde{\varphi}(x_n),$$

then $|y_n(t)| \le \widetilde{y}_n(t)$*,* $n = 0, 1, \dots, N$*,* $0 \le t \le T$*.*

For the proof of this theorem it is sufficient to consider functions $\widetilde{y}_n(t) + y_n(t)$ and $\widetilde{y}_n(t) - y_n(t)$ apply Theorem 5.

3.3 Convergence of straight lines method

We now use the maximum principle and in particular the comparison theorem to prove the convergence of the solution of problem (40)-(43) to the solution of problem (36)-(39). We assume that the exact solution $u(x,t)$ has in $D = \{0 < x < l, 0 < t \le T\}$ bounded derivatives in x up to the fourth order and the coefficient $k(x,t)$ has in D bounded derivatives in x up to the third order and we denote

$$\begin{aligned} K &= \sup_D \max\{|k(x,t)|, |k'_x(x,t)|, |k''_x(x,t)|, |k'''_x(x,t)|\}, \\ M &= \sup_D \max\{|u'(x,t)|, |u''_x(x,t)|, |u'''_x(x,t)|, |u^{IV}_x(x,t)|\}, \end{aligned} \tag{48}$$

Let $u(x_n,t)$ be the value of the exact solution of problem (36)-(39) on the straight line $x = x_n$ and let $y_n(t)$, $n = 0,1,\ldots,N$, be the solution of problem (40)-(43). Introduce the auxiliary function

$$z_n(t) = y_n(t) - u(x_n,t), \quad n = 0,1,\ldots,N, 0 \le t \le T.$$

For this function we get

$$\begin{aligned} \frac{dz_n(t)}{dt} =& \frac{1}{h}\Big[\frac{k(x_{n+1},t)+k(x_n,t)}{2}\frac{z_{n+1}(t)-z_n(t)}{h} \\ &- \frac{k(x_n,t)+k(x_{n-1},t)}{2}\frac{z_n(t)-z_{n-1}(t)}{h}\Big] \\ &+ bz_n(t) + \sum_{k=1}^{m} b_k z_n(t_k) + h^2 R_n(t), \\ &n = 1,\ldots,N-1, \quad 0 < t \le T, \end{aligned} \tag{49}$$

$$\frac{dz_0(t)}{dt} + \alpha_1 z_0(t) + \beta_1 z_N(t) = 0, \quad 0 < t \le T, \tag{50}$$

$$\frac{dz_N(t)}{dt} + \alpha_2 z_0(t) + \beta_2 z_N(t) = 0, \quad 0 < t \le T, \tag{51}$$

$$z_n(0) = 0, \quad n = 0,\ldots,N. \tag{52}$$

It can be directly verified on the base of the Taylor formula that

$$|R_n(t)| \le \frac{2}{3}KM,$$

where K and M are from (48).

Denote for all $n = 0, \ldots, N$ the functions

$$\widetilde{z}_n(t) = h^2 L \frac{1 - e^{bt}}{b_1 e^{bt_1} + b_2 e^{bt_2} + \cdots + b_m e^{bt_m} - (b + b_1 + \cdots + b_m)}, \quad 0 \le t \le T,$$

where L is a positive constant and will be chosen later. Under conditions (47) these functions are nonnegative functions. For them, we get after elementary calculations

$$\begin{aligned} \frac{d\widetilde{z}_n(t)}{dt} - \frac{1}{h}\Big[& \frac{k(x_{n+1},t)+k(x_n,t)}{2} \frac{\widetilde{z}_{n+1}(t)-\widetilde{z}_n(t)}{h} \\ & - \frac{k(x_n,t)+k(x_{n-1},t)}{2} \frac{\widetilde{z}_n(t)-\widetilde{z}_{n-1}(t)}{h}\Big] \\ & - b\widetilde{z}_n(t) - \sum_{k=1}^{m} b_k \widetilde{z}_n(t_k) \\ = {} & h^2 L, \quad n = 1, \ldots, N-1, \quad 0 < t \le T. \end{aligned} \tag{53}$$

On the other hand under conditions (47)

$$\begin{aligned} & \frac{d\widetilde{z}_0(t)}{dt} + \alpha_1 \widetilde{z}_0(t) + \beta_1 \widetilde{z}_N(t) \ge 0, \quad 0 < t \le T, \\ & \frac{d\widetilde{z}_N(t)}{dt} + \alpha_2 \widetilde{z}_0(t) + \beta_2 \widetilde{z}_N(t) \ge 0, \quad 0 < t \le T, \end{aligned} \tag{54}$$

$$\widetilde{z}_n(0) = 0, \quad n = 0, \ldots, N. \tag{55}$$

Let $L = \frac{2}{3} KM$. Then, comparing problem (49)-(52) with problem (53)-(55), we have from the comparison theorem

$$|z_n(t)| \le \widetilde{z}_n(t), \quad n = 0, \ldots, N, 0 < t \le T$$

that is for all $n = 0, \ldots, N$

$$\begin{aligned} \max_n |y_n(t) - u(x_n,t)| \ \le & \\ & \le \frac{h^2 L(1 - e^{bt})}{b_1 e^{bt_1} + b_2 e^{bt_2} + \cdots + b_m e^{bt_m} - (b + b_1 + \cdots + b_m)} \\ & \le Ch^2, \quad 0 < t \le T. \end{aligned} \tag{56}$$

Thus the following theorem holds.

Theorem 8. *Let the coefficients of problem* (36)-(39) *satisfy conditions* (47). *Then the solution of problem* (40)-(43) *converges to the solution of problem* (36)-(39) *and estimate* (56) *holds.*

As a conclusion we only mention again that problems of the (1)-(3) type arise in different mathematical models in various physical, financial, biological, social and engineering applications. Typically, different models on the base of loaded equations arise in the situations when some data of the models are unavailable for measurements. Such data are usually functions of the unknown solution itself. We deal with such situations, for example, in the case of different inverse problems and in the case of free boundary problems - see [1–26].

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