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Evaluation of the Von Rosenberg's method for the convection-diffusion equation

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Abstract

An explicit numerical scheme developed by Von Rosenberg for the convection-diffusion equation in one spatial dimension is reviewed and analyzed. The convergence of this scheme is outlined and a comparative study was established with an explicit standard finite difference scheme. The results show that Von Rosenberg's scheme considerably improves the stability condition and accuracy in difficult problems associated with very small diffusion coefficients. This study extends and corrects the original work presented by Von Rosenberg, which represents an original contribution.

Keywords: Convection, diffusion, explicit, finite difference, stability.

1 Introduction

Numerical methods are very important because they solve complex problems in engineering and sciences [1]. In particular, numerical methods are widely used for solving partial differential equations. The purpose of this research is the study of an explicit numerical scheme developed by D.U. Von Rosenberg, [2], for the one dimensional convection diffusion equation. This scheme has

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impressive performance at solving convection diffusion problems with large speeds and low diffusion coefficients.

Von Rosenberg's method was presented for first time in [2], it was adapted to the variable coefficient case in [3], and it was extended to multidimensional problems in [4]. However, a complete analysis of convergence of the method, even in one dimension, is not available in the technical literature to the best of our knowledged, so this paper fills that gap.

The content of this paper has been distributed in six sections. First section is this short introduction. Second section gives a description of the convection diffusion equation in the context of this article. Third section states Von Rosenberg's method. Section four is devoted to converge analysis of the method. Finally, sections five and six contain a numerical comparative study and the conclusions.

2 Equations

In this paper the convection diffusion equation is represented by the following equation

$$D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}. \tag{1}$$

In this equation u is the fluid property, x is the spatial variable ($x \geq 0$), t is the time variable ($t \geq 0$), D is the diffusion coefficient and v is the fluid velocity. For convenience and without lost of generality, v and D are positive constants.

The convection and diffusion equation represent mathematically a physical process of heat transfer that simultaneously combines both forms of transmission. However, there are other physical processes such as fluid flow in porous media which is also governed by the same equation. [5, 6]. Consequently, it will be assumed that all variables in Eq. (1) are dimensionless.

The general upstream boundary condition for Eq. (1) for all t is

$$v(g(t) - u(0,t)) + D\frac{\partial u}{\partial x}(0,t) = 0. \tag{2}$$

As D approaches 0, this condition simplifies to

$$u(0,t) = g(t) \tag{3}$$

and g is a given function. Since D is small for the cases of interest, the numerical solutions were obtained with Eq. (3). Upstream condition is similar to Eq. (3) but evaluated at the upper side of the space interval of integration, namely $u(L,t) = h(t)$, h a known function, if x in Eq. (1) runs on $[0, L]$. In this work and [2] $h(t)$ is the null function if no explicit mention of it is made. In order to solve the convection diffusion equations an initial condition is required

$$u(x, 0) = f(x) \tag{4}$$

where f is a given function.

3 Von Rosenberg's Method

Next we present the Von Rosenberg's method (VR) using one dimensional convection-diffusion equation (1), it combines the Crank-Nicholson and Finite Difference (FD) methods.

Von Rosenberg's method is explicit and consequently the relationship must be satisfied

$$\frac{\triangle x}{v \triangle t} = 1$$

where we call $Q = \frac{\triangle x}{v \triangle t}$ for easy reference.

When the method is stated, we use the next parameter

$$R = \frac{2D}{v \triangle x},$$

which plays a fundamental role in this section. In these expressions $\triangle t$ is the time step and $\triangle x$ is the size of the mesh block in space.

To approximate the diffusion term we use a Crank-Nicholson type approximation obtaining

$$D\frac{\partial^2 u}{\partial x^2} \approx \frac{D}{2(\Delta x)^2}[u_{i+1}^n - 2u_i^n + u_{i-1}^n + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}].$$

Then taking into account $Q = 1$ the last expression is equivalent to

$$D\frac{\partial^2 u}{\partial x^2} \approx (\frac{1}{\Delta t})\ (\frac{R}{4})[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n]. \tag{5}$$

To approximate the convection term two different discretizations are averaged as functions of R. The first one is an explicit approximation type upwind

$$(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x})_{i-1/2,n+1/2} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} + v\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

which taking into account that $Q = 1$ transforms into

$$(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x})_{i-1/2,n+1/2} \approx \frac{u_i^{n+1} - u_{i-1}^n}{\Delta t}. \qquad (6)$$

The second of the approximations is a centered type Crank-Nicholson approach

$$(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x})_{i,n+1/2} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} + v\frac{1}{2}(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}).$$

Which considering the restriction $Q = 1$ can be written as

$$(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x})_{i,n+1/2} \approx \frac{1}{\Delta t}[(u_i^{n+1} - u_i^n) + \frac{1}{4}(u_{i+1}^{n+1} - u_{i-1}^{n+1} + u_{i+1}^n - u_{i-1}^n)]. \quad (7)$$

Combining Eq. (6) with Eq. (7) using as weights $(1 - R)$ and R respectively, it results

$$(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x})_{i-1/2(1-R),n+1/2}$$
$$\approx \frac{1}{\Delta t}[u_i^{n+1} - u_{i-1}^n + \frac{R}{4}(3u_{i-1}^n + u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - 4u_i^n)]. \qquad (8)$$

Substituting expressions Eq. (8) and Eq. (5) as approximations in Eq. (1) results

$$\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2}\right)$$
$$\approx \frac{1}{2(\Delta t)}\left[(2+R)u_i^{n+1} - (2-R)u_{i-1}^n - R\left(u_i^n + u_{i-1}^{n+1}\right)\right]. \qquad (9)$$

If initial and boundary conditions, as described in the previous section, are taken into account then the Von Rosenberg's method becomes an explicit scheme.

4 Convergence study

In this section a convergence analysis of the Von Rosenberg's method is presented under the boundary conditions described in section 3. For simplicity, it will be assumed that $g(t) = 1$ and $f(x) = 0$.

4.1 Consistency

Let $G \subset \mathbb{R}^2$ an open set and $u : G \to \mathbb{R}$. Using the following notation

$$u(x_i, t_n) = u_i^n$$

The truncation error is obtained by substituting the Taylor's formula u, around the point where the finite differences Von Rosenberg method are centered [2], ie, $x_{i-(1/2)(1-R)}, t_{n+1/2}$.

Substituting the Taylor's formula for u, on the right side of the following expression

$$\begin{aligned} &\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2}\right) \\ &\approx \frac{1}{2(\Delta t)}\left[(2+R)u_i^{n+1} - (2-R)u_{i-1}^n - Ru_{i-1}^{n+1} - Ru_i^n\right]. \end{aligned} \tag{10}$$

And recalling that $Q = \frac{\Delta x}{v\Delta t} = 1$ and $R = \frac{2D}{v\Delta x}$, so $\frac{(\Delta x)^2}{2(\Delta t)}R = D$, we obtain the expansion of the one dimensional convection-diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} &= \frac{[(2+R)u_i^{n+1} - (2-R)u_{i-1}^n - Ru_{i-1}^{n+1} - Ru_i^n]}{2(\Delta t)} \\ &+ [1+3R^2]v\frac{\partial^3 u}{\partial x^3}\frac{(\Delta x)^2}{24} + [1-R^2]\frac{\partial^3 u}{\partial x^2 t}\frac{(\Delta x)^2}{8} \\ &+ \frac{\partial^3 u}{\partial x t^2}\frac{(\Delta x)(\Delta t)}{8} + \frac{\partial^3 u}{\partial t^3}\frac{(\Delta t)^2}{24} + H.O.T. \end{aligned}$$

With truncation error

$$\begin{aligned} &[1+3R^2]v\frac{\partial^3 u}{\partial x^3}\frac{(\Delta x)^2}{24} + [1-R^2]\frac{\partial^3 u}{\partial x^2 t}\frac{(\Delta x)^2}{8} + \frac{\partial^3 u}{\partial x t^2}\frac{(\Delta x)(\Delta t)}{8} + \frac{\partial^3 u}{\partial t^3}\frac{(\Delta t)^2}{24} \\ &+ H.O.T. \end{aligned}$$

Recall that $H.O.T.$ is the sum of remainder terms in Taylor's formula. This truncation error is of quadratic order for both space and time, ie, $O((\Delta x)^2, (\Delta t)^2)$.

Therefore, by the above study, the scheme Von Rosenberg is consistent, since the truncation error vanishes when $\Delta x \to 0$ y $\Delta t \to 0$.

Observation

In the article by Von Rosenberg [2] there is an error in the expression of the truncation error. The correct truncation error is the one presented in this subsection.

4.2 Stability

To study the stability of Von Rosenberg method will use the **Gershgorin circle theorem** [6, 7].

The difference equation for the Von Rosenberg equation for an homogeneous convection-diffusion equations is:

$$A\ U^{n+1} = B\ U^n + R\ F_1 + (2 - R)\ F_2$$

where

$$A = \begin{bmatrix} (2+R) & 0 & \dots & 0 \\ -R & (2+R) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -R & (2+R) \end{bmatrix}, \ B = \begin{bmatrix} R & 0 & \dots & 0 \\ (2-R) & R & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & (2-R) & R \end{bmatrix}$$

$$\begin{aligned} U^{n+1} &= (u_1^{n+1}, u_2^{n+1}, \ \dots \ , u_{N-1}^{n+1})^t \\ U^n &= (u_1^n, u_2^n, \ \dots \ , u_{N-1}^n)^t \\ F_1 &= (u_0^{n+1}, 0, \ \dots \ , 0)^t \\ F_2 &= (u_0^n, 0, \ \dots \ , 0)^t \end{aligned}$$

The stability study is perfomed by applying Gershgorin theorem to estimate the eigenvalues of these matrices. We begin by estimating the eigenvalues of matrix A, where we obtain

$$2 \leq \lambda \leq 2(R+1) \qquad (11)$$

For making the study of the eigenvalues of matrix A^{-1} the reciprocal of the inequality Eq. (11) is used.

$$\frac{1}{2} \geq \frac{1}{\lambda} \geq \frac{1}{2(R+1)} \tag{12}$$

We will do the same study for the matrix B, where we obtain

$$2(R-1) \leq \beta \leq 2 \tag{13}$$

Estimation of the eigenvalues for $A^{-1}B$, is obtained by multiplying inequalities Eq. (12) and Eq. (13).

$$1 \geq \beta . \frac{1}{\lambda} \geq \frac{R-1}{R+1} \tag{14}$$

As $\frac{R-1}{R+1} \geq -1$ because $R = \frac{2D}{v\ \Delta x} > 0$ then

$$\left| \frac{\beta}{\lambda} \right| \leq 1$$

and, the Von Rosenbetg method is stable when the condition $Q = 1$ is satisfied.

4.3 Convergence

The convergence of the Von Rosenberg method follows by an application of the well known **Lax equivalence theorem** [6, 7].

5 Numerical experiments

Two numerical examples will be presented and analyzed, each example will be composed of two graphics, the diffusion of the first graph will be 0.001 and the second 0.1. For each example the Von Rosenberg method shows a better performance than an explicit standard finite difference scheme. In order to appreciate the accuracy of the Von Rosenberg method a table with the errors between analytic and numerical solutions is presented with each figure. Those tables show that Von Rosenberg method produces better approximations. Table 1 shows some notation used in the analysis of the examples.

Tab. 1: Notation used for tables of examples.

Notation	Description
$ua(x,t)$	Analytical solution values
$u(x,t)$	Values approach to numerical solution by the Von Rosenberg method
$udf(x,t)$	Values approach to numerical solution by the Finite Difference method
$\lvert ua(x,t) - u(x,t)\rvert$	Absolute error for the solutions obtained by the Von Rosenberg method
$\lvert ua(x,t) - udf(x,t)\rvert$	Absolute error for the solutions obtained by the Finite Difference method

Example 1

For this example, we will use the boundary condition Eq. (3) where $g(t) = 1$ and initial condition Eq. (4) where $f(x) = 0$. The spatial and temporal domain of this example is $[0, 8]$. For this first example, by the Von Rosenberg method, forty steps were used in space ($N = 40$) and forty steps in time ($M = 40$) and the finite difference method used $N = 40$ and $M = 1000$. Fig. 1 shows the first graph of Example 1 (Example 1a).

Table 2 shows some numerical results. Fig. 2 shows the second graph in Example 1 (Example 1b).

Table 3 shows some corresponding numerical results associated to Fig. 2.

In the graphs for Example 1 it can be seen that the Von Rosenberg method is the best approximation to the analytical solution. This can also be seen in the tables, where errors by the method of Von Rosenberg are smaller than the Finite Difference method. Consequently, Von Rosenberg's method is the most favorable option for the numerical approximations for the one-dimensional convection-diffusion equation, with the boundary condition and initial condition considered in this example.

Example 2

For this example the spatial interval $[0, 8]$ is used and the time interval $[0, 4]$, the fluid velocity is $v = 1$ and the time chart is $t = 4$. The boundary conditions are

$$u(0,t) = 0, \quad u(8,t) = 0, \quad t > 0.$$

The initial condition for this example,

$$\begin{cases} u(x,0) = 1, & \text{for} \quad 0.1 < x < 1.1 \\ u(x,0) = 0, & \text{in another case} \end{cases}$$

For the Von Rosenberg method, steps in space and time are, $N = 40$ and $M = 40$. For the finite difference method, steps in space are, $N = 40$ and the $\Delta t = \frac{(\Delta x)^2}{4(\Delta x\ v+2D)}$.
For the one dimensional convection-diffusion equation with boundary conditions and initial condition mentioned above, we don't have analytical solution, so we take as analytic solution the finite difference approximation with $N = 4000$.
Fig. 3 shows the first graph of Example 2 (Example 2a).
Table 4 shows some numerical results. Fig. 4 shows the second graph of Example 2 (Example 2b).
Some numerical results are shown in the table 5.
Results from Example 2 show that numerical approximation produced by Von Rosenberg method are the best.

6 Conclusions

The results obtained indicate that the Von Rosenberg method produces the best approximations for the solutions of the convection-diffusion equation. As we have seen in the examples, always the method of Von Rosenberg brings the best results. To demonstrate the accuracy of the Von Rosenberg method, very small broadcast values were used in the examples (this corresponds to $0 < R \leq 1$), it is in that case where the other numerical methods with few steps of space and time do not converge so quickly. For very high broadcasts, $D > 1$, not showed in the examples, numerical tests were performed and the solution obtained by the method of Von Rosenberg converges to the analytical solution.

Main advantages of Von Rosenberg method are its easy implementation and formulation, it is the combination of Centered Finite Difference method and the implicit Crank-Nicholson method. Another advantage is that the truncation error is of quadratic order for both space and time. The analytical study of convergence of the method of Von Rosenberg corrects the consistency analysis presented in [2] and complement the stability analysis, which is omitted in [2]. Consequently the convergence study is an original contribution of this work.

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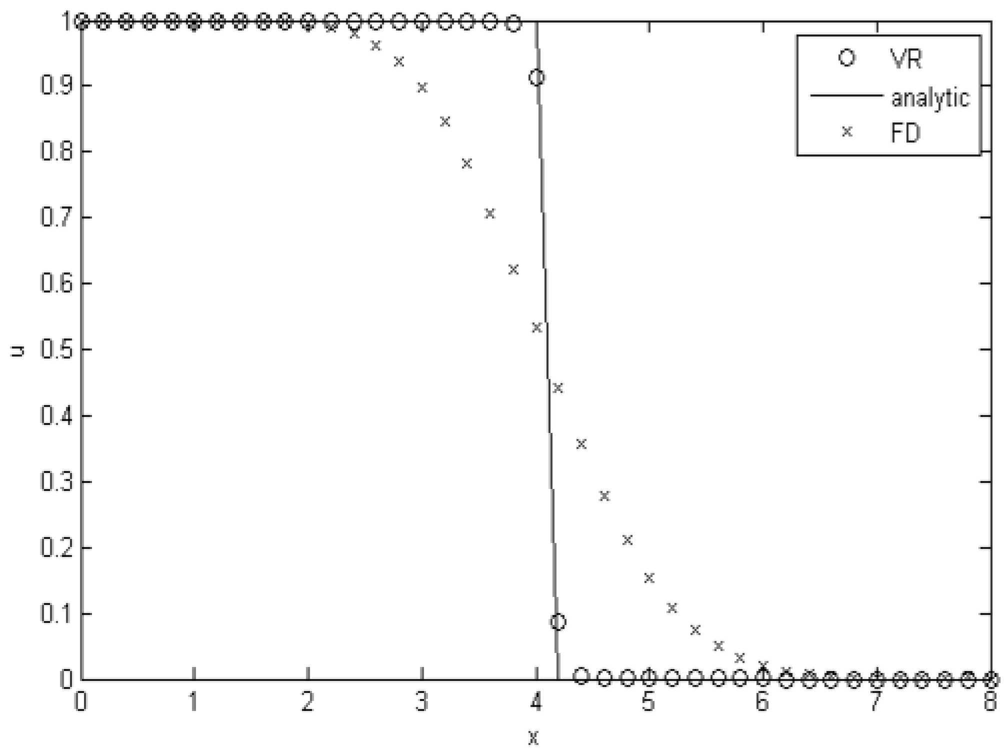

Fig. 1: Comparison of the solutions obtained with (VR) and (FD) methods vs. analytical solution ($D = 0.001$, $t = 4$).

Tab. 2: Results of Example 1a ($D = 0.001$, $t = 4$).

x	$ua(x,t)$	$u(x,t)$	$udf(x,t)$	$\|ua(x,t) - u(x,t)\|$	$\|ua(x,t) - udf(x,t)\|$
0	1.00000	1.00000	1.00000	0.00000	0.00000
2.4	1.00000	1.00000	0.97992	0.00000	0.02007
4.4	0.00000	0.00443	0.35569	0.00443	0.35569
6.4	0.00000	0.00000	0.00713	0.00000	0.00713
8	0.00000	0.00000	0.00000	0.00000	0.00000

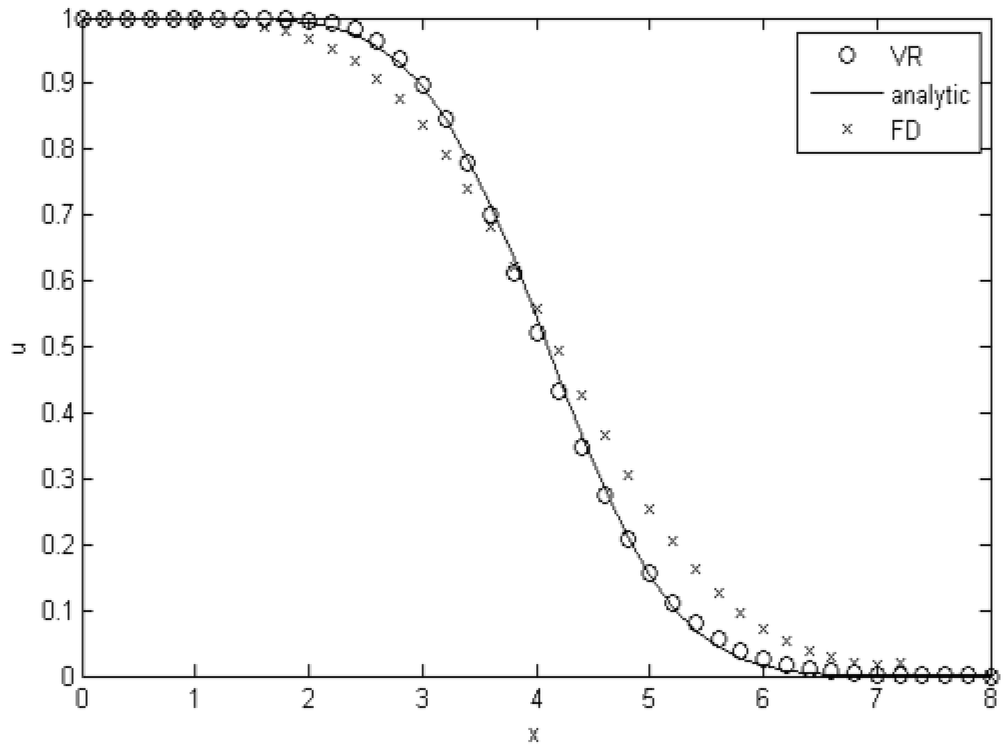

Fig. 2: Comparison of the solutions obtained with (VR) and (FD) methods vs. analytical solution ($D = 0.1$, $t = 4$).

Tab. 3: Results of Example 1b ($D = 0.1$, $t = 4$).

x	$ua(x,t)$	$u(x,t)$	$udf(x,t)$	$\|ua(x,t) - u(x,t)\|$	$\|ua(x,t) - udf(x,t)\|$
0	1.00000	1.00000	1.00000	0.00000	0.00000
2.4	0.97422	0.98182	0.93346	0.00759	0.04076
4.4	0.36537	0.34938	0.42798	0.01599	0.06260
6.4	0.00457	0.00999	0.03825	0.00541	0.03367
8	0.00000	0.00000	0.00000	0.00000	0.00000

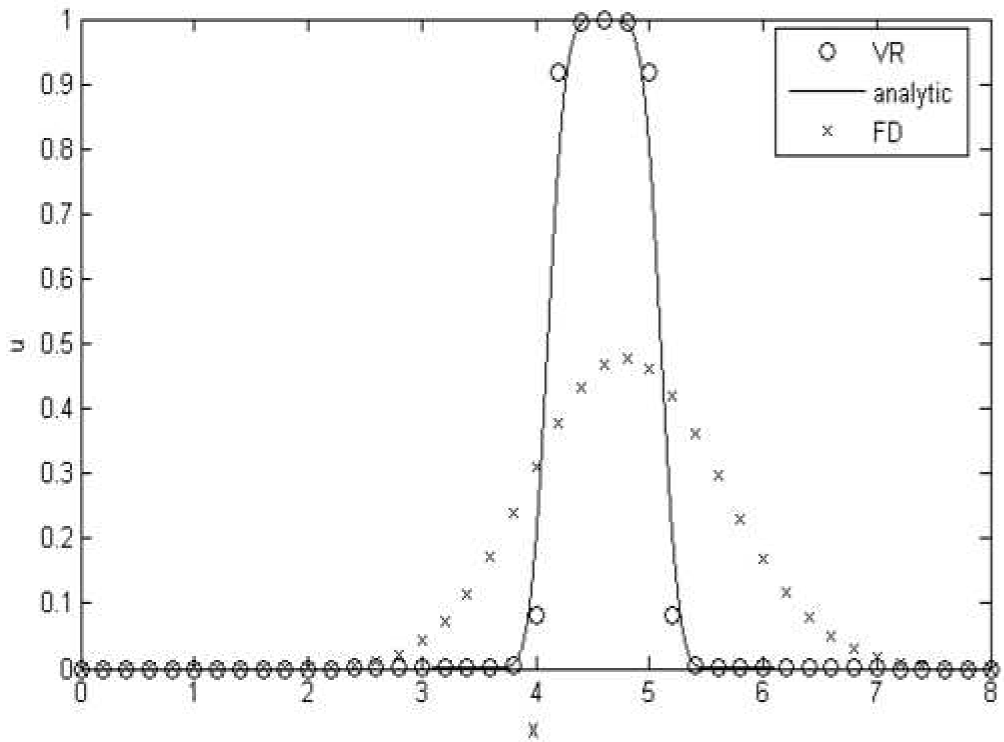

Fig. 3: Comparison of the solutions obtained with (VR) and (FD) methods vs. analytical solution ($D = 0.001$, $t = 4$).

Tab. 4: Results of Example 2a ($D = 0.001$, $t = 4$).

x	$ua(x,t)$	$u(x,t)$	$udf(x,t)$	$\|ua(x,t) - u(x,t)\|$	$\|ua(x,t) - udf(x,t)\|$
0	0.00000	0.00000	0.00000	0.00000	0.00000
2.4	0.00000	0.00000	0.00490	0.00000	0.00490
4.4	0.99230	0.99600	0.43360	0.00370	0.55870
6.4	0.00000	0.00000	0.07910	0.00000	0.07910
8	0.00000	0.00000	0.00000	0.00000	0.00000

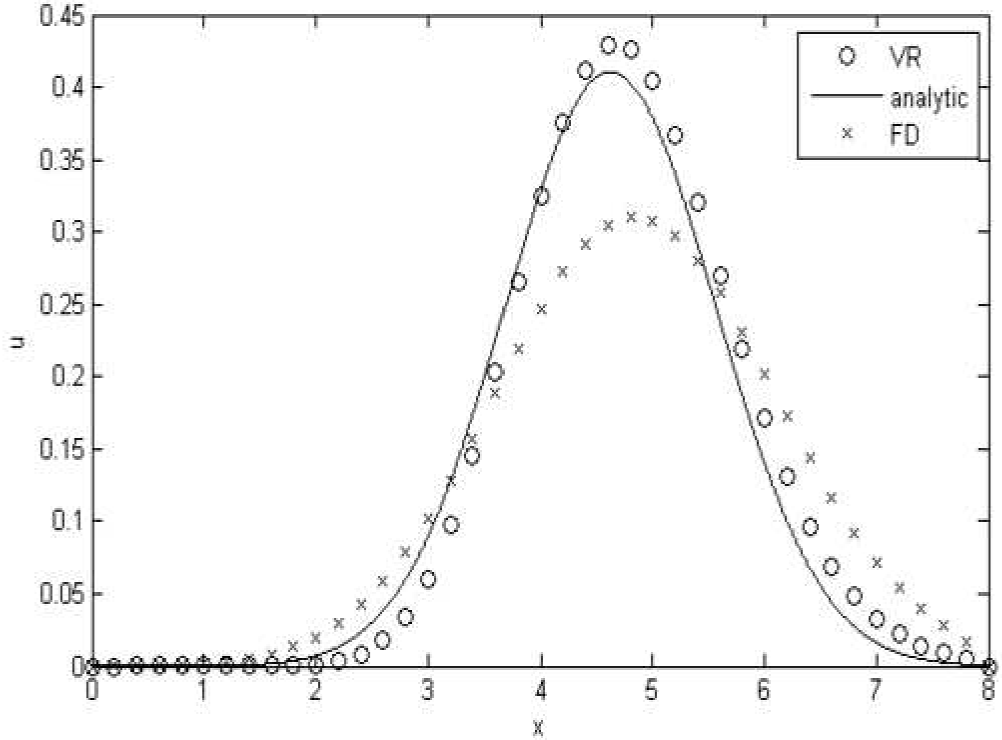

Fig. 4: Comparison of the solutions obtained with (VR) and (FD) methods vs. analytical solution ($D = 0.1$, $t = 4$).

Tab. 5: Results of Example 2b ($D = 0.1$, $t = 4$).

x	$ua(x,t)$	$u(x,t)$	$udf(x,t)$	$\lvert ua(x,t) - u(x,t)\rvert$	$\lvert ua(x,t) - udf(x,t)\rvert$
0	0.00000	0.00000	0.00000	0.00000	0.00000
2.4	0.02290	0.00850	0.04240	0.01440	0.01950
4.4	0.39890	0.41220	0.29180	0.01330	0.10710
6.4	0.06870	0.09670	0.14330	0.02800	0.07460
8	0.00000	0.00000	0.00000	0.00000	0.00000