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On the Modified Methods for Irreducible Linear Systems with L-Matrices

Seyyed Ahmad Edalatpanah [1]

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Abstract

Milaszewicz, [Milaszewic J.P, Linear Algebra. Appl. 93,1987, 161−170] presented new preconditioner for linear system in order to improve the convergence rates of Jacobi and Gauss-Seidel iterative methods. Li et al.,[Li Y., C., Li, S. Wu, Appl. Math.Comput. 186, 2007, 379−388] applied this preconditioner and provided convergence theorem for modified AOR method. Yun and Kim [Yun J.H., S.W. Kim, Appl. Math. Comput. 201, 2008, 56-64] pointed out some errors in Li et al.$'s$ theorem and provided some correct results for convergence of the preconditioned AOR method. In this paper, we analyze their convergence properly and propose a new theorem for irreducible modified AOR method. In particular, based on directed graph, we prove that the convergence theorem of Li et al. is true, without any additional assumptions.

Keywords: Preconditioning, accelerated overrelaxation (AOR), convergence analysis, L-matrix, directed graph.

1 Introduction

Let us consider the following linear systems

$$Ax = b, \tag{1}$$

[1]Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran (saedalatpanah@gmail.com).

where $A \in R^{n\times n}$ and $b, x \in R^n$. Many of the problems that arise in technological, industrial and science situations are linear systems and there are some reliable methods for solving this class of problems; see [1–12] and the references therein. The basic iterative method for solving Eq.(1) can be represented as:

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b,\ i = 0, 1, ... \tag{2}$$

where $x^{(0)}$ is an initial vector and $M^{-1}N$ is the iteration matrix. For simplicity, but without loss of generality, we assume that $diag(A) = I$ and,

$$A = I - L - U, \tag{3}$$

where I, is the identity matrix, L and U are strictly lower and strictly upper triangular matrices of A, respectively. If $A = M-N$, where M is nonsingular, then the basic iterative method for solving Eq.(1) is Eq.(2). This iterative process converges to the unique solution $x = A^{-1}b$ for any initial vector value, if and only if the spectral radius of the iteration matrix is smaller than one; i.e, $\rho(M^{-1}N) < 1$.There are some special iterative methods for solving a linear system Eq.(1) based on Eq.(2). For example, the *accelerated overrelaxation iterative method* (AOR) is as follows ;

$$x^{(i+1)} = L_{r,w}x^{(i)} + (I - rL)^{-1}wb,\ i = 0, 1, ...$$

where (w, r) are real parameters with $w \neq 0$, and:

$$M_{r,w} = \frac{1}{w}(I - rL),\ N_{r,w} = \frac{1}{w}[(1 - w)I + (w - r)L + wU],$$

$$T_{r,w} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU]. \tag{4}$$

A preconditioner is defined as an auxiliary approximate solver which will be combined with an iterative method. Furthermore, according to critical importance of spectral radius, in preconditioning, we find a more desired spectral radius. Therefore, the basic idea of preconditioned iterative methods is to transform Eq.(1) into the following preconditioned form

$$PAx = Pb,\ P \in R^{n\times n}. \tag{5}$$

where P is a linear operator and called the preconditioner. Let

$$\bar{A} = PA = \bar{D} - \bar{L} - \bar{U}. \tag{6}$$

If we use the AOR method for the modified linear system (5), then we get the preconditioned AOR iterative method whose iteration matrix is

$$\bar{T}_{r,w} = (\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}], \tag{7}$$

In literature, various authors have suggested different models of (I+S)-type preconditioner for the above mentioned problem [1–6, 8–12] This is a paper motivated by studying of some previous works and possible gaps or missing details in there in. Milaszewicz [1], presented the preconditioner

$$P = I + S. \tag{8}$$

where the elements of the first column below the diagonal of A eliminate and,

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ -a_{31} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & 0 & \cdots & 0 \end{pmatrix}. \tag{9}$$

Then $\bar{A} = (I+S)A = I - L - U + S - SL - SU = \bar{D} - \bar{L} - \bar{U}$, where,

$$\bar{D} = I - \bar{D}_1,\ \bar{L} = L - S + SL + \bar{L}_1,\ \bar{U} = U + \bar{U}_1, \tag{10}$$

where, $\bar{D}_1, \bar{L}_1, \bar{U}_1$ are diagonal, strictly lower and strictly upper triangular parts of $SU = \bar{D}_1 + \bar{L}_1 + \bar{U}_1$, respectively. Furthermore, for $(i = 2, \cdots, n)$ we have:

$$\bar{A} = (\bar{a}_{ij}) = (a_{ij}) - (a_{i1})(a_{1j}). \tag{11}$$

Hence, when A is Z-matrix (see the following definition), for any $j \neq i, 1$; we have:

$$a_{ij} \neq 0 \Rightarrow \bar{a}_{ij} \neq 0. \tag{12}$$

Li et.al [2], applied the Milaszewicz's preconditioner and provided convergence theorem for the preconditioned AOR iterative method. However, since this theorem was proved under the incorrect lemma, Yun and Kim in [3] stated that this theorem is not generally true. These authors, under additional assumptions provided some correct results for convergence of the preconditioned AOR method. In this paper, we propose a new theorem for irreducible L-matrix. In particular, based on directed graph, we prove that the mentioned convergence theorem of [2] is true, without any additional assumptions.

2 Main Results

We begin with some basic notations and preliminary results which we refer to later.

Definition 1. *([6, 7]). A matrix $A = (a_{ij})$ is called a Z-matrix if for any $i \neq j; a_{i,j} \leq 0$ and a Z-matrix is an L-matrix, if $a_{ii} > 0$. $A(2 : n, 2 : n)$ denotes the submatrix of A whose rows are indexed by $(2, 3, ..., n)$ and columns by $(2, 3, ..., n)$. For an $n \times n$ matrix A, the directed graph $\Gamma(A)$ of A is defined to be the pair (V, E), where $V = \{1, \cdots, n\}$ is a set of vertices and $E = \{(i, j) : a_{ij} \neq 0, i, j = 1, \cdots, n\}$ is a set of arcs. A path from i to j of length k in $\Gamma(A)$ is a sequence of vertices $\sigma = (i_0, i_1, \cdots, i_k)$ where $i_0 = i$ and $i_k = j$ such that $(i_0, i_1), (i_1, i_2), \cdots, (i_{k-1}, i_k)$ are arcs of $\Gamma(A)$. A directed graph $\Gamma(A)$ is strongly connected if for any two vertices i , j, there is a path from i to j in $\Gamma(A)$. A matrix $A_{n\times n}$ is said to be irreducible if $\Gamma(A)$ is strongly connected.*

Lemma 1. *([7]). If A be a nonnegative and irreducible matrix, then;*
(i) A has a positive real eigenvalue equal to its spectral radius,
(ii) For $\rho(A) > 0$, there corresponds an eigenvector $x \geq 0$,
(iii) $\rho(A)$ does not decrease when any entry of A is increased.

Theorem 1. *([2], Theorem 1). Let $T_{r,w}$ and $\bar{T}_{r,w}$ be the iteration matrices of Eqs.(4)and(7) of the AOR method and $0 \leq r \leq w \leq 1$. If A is an irreducible L-matrix and there exists a nonempty set $\alpha \subset N_1 = \{2, 3, \cdots, n\}$ such that*

$$\begin{cases} 0 < a_{1i}a_{i1} < 1, & \text{if } \ i \in \alpha, \\ \quad a_{1i}a_{i1} = 0, & \text{if } i \in N_1 - \alpha. \end{cases}$$

Then we have;
(i)If $\rho(T_{r,w}) < 1 \Rightarrow \rho(\bar{T}_{r,w}) < \rho(T_{r,w})$.
(ii)If $\rho(T_{r,w}) = 1 \Rightarrow \rho(\bar{T}_{r,w}) = \rho(T_{r,w})$.
(iii)If $\rho(T_{r,w}) > 1 \Rightarrow \rho(\bar{T}_{r,w}) > \rho(T_{r,w})$.

Since, above theorem was proved under the assumption that $\bar{T}_{r,w}$ is irreducible, Yun and Kim in [3] by a counterexample, show that it is not generally true; (see [3], Example 3.1) and under additional assumptions provided the following result corresponding to Theorem 1:

Theorem 2. *([3], Theorem 3.3). Let $T_{r,w}$ and $\bar{T}_{r,w}$ be defined by (4) and (7), $0 \leq r \leq w \leq 1$, and $A(2:n, 2:n)$ be an irreducible submatrix of A. If A is an L-matrix and there exists a nonempty set $\alpha \subset N_1 = \{2, 3, \cdots, n\}$ such that* $\begin{cases} 0 < a_{1i}a_{i1} < 1, & \text{if } \ i \in \alpha, \\ a_{1i}a_{i1} = 0, & \text{if } i \in N_1 - \alpha. \end{cases}$
Then we have;
(i)If $\rho(T_{r,w}) < 1 \Rightarrow \rho(\bar{T}_{r,w}) < \rho(T_{r,w})$.
(ii)If $\rho(T_{r,w}) = 1 \Rightarrow \rho(\bar{T}_{r,w}) = \rho(T_{r,w})$.
(iii)If $\rho(T_{r,w}) > 1 \Rightarrow \rho(\bar{T}_{r,w}) > \rho(T_{r,w})$.

Now, we establish alternative results in the following theorems;

Theorem 3. *Let A be an irreducible L-matrix and there exists a nonempty set $\alpha \subset N_1 = \{2, 3, \cdots, n\}$ such that* $\begin{cases} 0 < a_{1i}a_{i1} < 1, & \text{if } \ i \in \alpha, \\ a_{1i}a_{i1} = 0, & \text{if } i \in N_1 - \alpha. \end{cases}$
Then $\bar{A}(2:n, 2:n)$ is an irreducible sub-matrix of $\bar{A}$.

Proof. Since A is an irreducible, then for any $i, j \in N_1 = \{2, 3, ..., n\}$ there is a path $\sigma = (i_0, i_1, \cdots, i_{k+1})$ from $i(= i_0)$ to $j(= i_{k+1})$ in a directed graph of A.
If $\forall s \in \{1, 2, \cdots, k\}; i_s > 1$, then from Eq.(12) we have $\sigma \in \Gamma(\bar{A})$.
If $\exists s \in \{1, 2, \cdots, k\}; i_s = 1$, then we have:

$$\sigma_1 = (i_0, i_1, \cdots, i_{s-1}), \sigma_2 = (i_{s+2}, i_{s+2}, \cdots, i_{k+1}), i_{s-1} \in N_1.$$

If $a_{i_{s-1},i_{s+1}} \neq 0$, then from Eq.(12), $\bar{\sigma} = (\sigma_1, \sigma_2) \in \Gamma(\bar{A})$;
If $a_{i_{s-1},i_{s+1}} = 0$, since A is an irreducible, there is a path $\xi = (i_{s-1}, p_1, p_2, \cdots, p_l, i_{s+1}) \in \Gamma(A)$ from i_{s-1} to i_{s+1}.
Now, if $\forall i; p_i > 1$, then from Eq.(12) we have $\bar{\sigma} = (\sigma_1, \xi, \sigma_2) \in \Gamma(\bar{A})$, and if $p_1 = 1, p_i > 1$, then $a_{p_1,p_2} = a_{1,p_2} \neq 0$, then from Eqs.(11) and (12) we get:

$$\bar{a}_{i_{s-1},p_2} = a_{i_{s-1},p_2} - (a_{i_{s-1},1})(a_{i_1,p_2}).$$

Hence, $\bar{A}(2:n, 2:n)$ is an irreducible sub-matrix of $\bar{A}$ and the proof is completed.□

Theorem 4. *Let A be an irreducible L-matrix and the conditions of Theorem 1 are satisfied; Then;*
(i)If $\rho(T_{r,w}) < 1 \Rightarrow \rho(\bar{T}_{r,w}) < \rho(T_{r,w})$.
(ii)If $\rho(T_{r,w}) = 1 \Rightarrow \rho(\bar{T}_{r,w}) = \rho(T_{r,w})$.
(iii)If $\rho(T_{r,w}) > 1 \Rightarrow \rho(\bar{T}_{r,w}) > \rho(T_{r,w})$.

Proof. Since A is an irreducible L-matrix, by Lemma 1:

$$\exists\ x > 0\ s.t\ T_{r,w}x = \lambda x, \tag{13}$$

where, $\lambda = \rho(T_{r,w})$ and,

$$T_{r,w} = (1-w)I + w(1-r)L + wU + H,$$

where, H is a nonnegative matrix. Moreover, with similar to the proof of Theorem 1, we get:

$$\bar{T}_{r,w}x - \lambda x = (\lambda - 1)\underbrace{((\bar{D} - r\bar{L})^{-1}(\bar{D} + r\bar{L} + (1-r)s)}_{z}x,$$

$$\bar{T}_{r,w} = (1-w)I + w(1-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U} + \bar{H} = \begin{pmatrix} 1-w \geq 0 & (\bar{T}_{12})_{1\times n-1} \geq 0 \\ (0)_{n-1\times 1} & (\bar{T}_{22})_{n-1\times n-1 \geq 0} \end{pmatrix}, \tag{14}$$

where, $\bar{H}$ is nonnegative matrix. By Theorem 3, $\bar{A}(2:n, 2:n)$ is an irreducible sub-matrix of $\bar{A}$ .Therefore, $\bar{T}_{22}$ is irreducible. Then by Lemma 1 and choosing:

$$x = \begin{pmatrix} (x_1)_{1\times 1} \\ (x_2)_{n-1\times 1} \end{pmatrix} \geq 0, z = \begin{pmatrix} (0)_{1\times 1} \\ (z_2)_{n-1\times 1} \end{pmatrix} \geq 0,$$

we have:

$$\begin{pmatrix} 1-w & (\bar{T}_{12})_{1\times n-1} \\ (0)_{n-1\times 1} & (\bar{T}_{22})_{n-1\times n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\lambda - 1) \begin{pmatrix} 0 \\ z_2 \end{pmatrix}.$$

Thus with continue in the proving process of Theorem 2 ([3], Theorem3.3) the proof is completed.□

3 Numerical experiments

In this section, we give some examples to illustrate the results obtained in previous sections.

Example 1. Consider a 4×4 matrix A of the following form:

$$A = \begin{pmatrix} 1 & -0.1 & 0 & -0.1 \\ -0.1 & 1 & -0.1 & 0 \\ 0 & 0 & 1 & -0.1 \\ -0.1 & 0 & 0 & 1 \end{pmatrix}.$$

It is clear that A is an irreducible L-matrix and $\alpha = \{2, 4\}$ (i.e, $0 < a_{12}a_{21} < 1,\ 0 < a_{14}a_{41} < 1$, and $a_{13}a_{31} = 0$). The preconditioned matrix is:

$$\bar{A} = (I + S)A = \begin{pmatrix} 1 & -0.1 & 0 & -0.1 \\ 0 & 0.99 & -0.1 & -0.01 \\ 0 & 0 & 1 & -0.1 \\ 0 & -0.01 & 0 & 0.99 \end{pmatrix}.$$

And then,

$$\bar{A}(2:n, 2;n) = \begin{pmatrix} 0.99 & -0.1 & -0.01 \\ 0 & 1 & -0.1 \\ -0.01 & 0 & 0.99 \end{pmatrix},$$

is an irreducible sub-matrix of $\bar{A}$. Assume that $0 \leq r \leq w \leq 1$ and Eq.(7) we obtain:

$$\bar{T}_{r,w} = \begin{pmatrix} 1 - w & (\bar{T}_{12})_{1\times 3} \\ (0)_{3\times 1} & (\bar{T}_{22})_{3\times 3} \end{pmatrix},$$

$$\bar{T}_{12} = \begin{pmatrix} \frac{w}{10} & 0 & \frac{w}{10} \end{pmatrix},$$

$$\bar{T}_{22} = \begin{pmatrix} 1 - w & \frac{10w}{99} & \frac{w}{99} \\ 0 & 1 - w & \frac{w}{10} \\ \frac{w(1-r)}{99} & \frac{10rw}{9801} & 1 - w + \frac{rw}{9801} \end{pmatrix}.$$

where, $\bar{T}_{r,w}$ is nonnegative matrix. Furthermore, since $\bar{A}(2:n, 2;n)$ is an irreducible matrix, then $\bar{T}_{22}$ is irreducible. Therefore, we can see the results of Theorem 4.

For example, for $w = 1$ and $r = 0.8$, the spectral radius of AOR method is 0.0931, wheras the spectral radius of *preconditioned* AOR method with

Milaszewicz's preconditioner is 0.0286.

Example 2. (Application to the model convection-diffusion equation)

Consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + 2u_x + u_y + u_z = f(x, y, z),$$

on the unit cube domain $\Omega = [0, 1] \times [0, 1]$, with Dirichlet boundary conditions. When the seven-point finite difference discretization, for example, the centered differences to the diffusive terms, and the centered differences or the first-order upwind approximations to the convective terms are applied to the above model convection-diffusion equation, we get the system of linear equations Eq.(1) with the coefficient matrix

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z,$$

where, the equidistant step-size $h = 1/n + 1$ is used in the discretization on all of the three directions and the natural lexicographic ordering is employed to the unknowns. In addition,$\otimes$ denotes the Kronecker product, and T_x , T_y, and T_z are tridiagonal matrices given by:

$$T_x = tridiagonal[-\frac{2+2h}{12}, 1, -\frac{2-2h}{12}],$$
$$T_y = T_z = tridiagonal[-\frac{2+h}{12}, 0, -\frac{2-h}{12}].$$

For details, we refer to [13, 14]. Then, we solved the $n^3 \times n^3$ matrix yielded by the AOR scheme and *preconditioned* AOR method.
In Table 1, we report the CPU time for the corresponding schemes with different parameters. The initial approximation of $x^{(0)}$ is zero vector and we choose the right- hand side vector, such that $X = (1, 1, ..., 1)^T$ be the solution of $Ax = b$. The stopping criterion when the current iteration satisfies is $tol < eps = 10^{-10}$. Form the point of view of the CPU time, we can see that the *preconditioned* AOR method performs much better than AOR method.

4 Conclusion

In this paper, we have studied a preconditioned method for linear systems. Convergence properties of this method for irreducible L-matrix were dis-

Tab. 1: Numerical results for Example 2

n	$N = n^3$	w	r	AOR	$Preconditioned\ AOR$
5	125	0.8	0.5	0.0183	0.0049
7	343	0.8	0.5	0.0674	0.0092
9	729	0.95	0.9	0.1879	0.1509
10	1000	0.95	0.9	0.5343	0.3224

cussed. In particular, based on directed graph, we proved that the mentioned convergence theorem of [2] is true, without any additional assumptions.

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