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Complex neutrosophic graphs

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Abstract

In this research article, we introduce the notion of complex neutrosophic graphs ($\mathfrak{cn}$-graphs, for short) and discuss some basic operations related to $\mathfrak{cn}$-graphs. We describe these operations with some examples. We also present energy of complex neutrosophic graphs.

Keywords: Complex neutrosophic sets, complex neutrosophic graphs, energy.

1 Introduction

Fuzzy set theory was introduced by Zadeh [1]. Applications of these sets have been broadly studied in other aspects such as control [2], reasoning [1], pattern recognition [2], engineering [3], etc. Atanassov [4] proposed the extended form of fuzzy set by adding a new component, called intuitionistic fuzzy sets (IFSs). The idea of IFSs is more meaningful as well as intensive due to the presence of degree of truth and falsity membership. Smarandache [5, 6] introduced the thought of neutrosophic sets by combining the non-standard analysis. Neutrosophic set theory is applied to image segmentation [7], topology [8], decision making [9], robotics [10], physics [11] and in many more real life problems. See also [12–16].

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Buckley [17] and Nguyen et al. [18] combined complex numbers with fuzzy sets. On the other hand, Ramot et al. [19, 20] extended the range of membership to "unit circle in the complex plane", unlike the others who limited the range to $[0, 1]$. Further this concept has been studied in IFSs [21] and Samarandache's neutrosophic sets [22].

A graph is a mathematical object containing points (vertices) and connections (edges), and is a convenient way of interpreting information involving the relationship between different objects. However, due to some reasons, in practical applications of graph theory, different types of uncertainties are frequently encountered. To handle these uncertainties, Kaufmann [23] introduced the theory of fuzzy graphs based on Zadeh's fuzzy relations. Later, Rosenfeld [24] put forward another elaborated definition of fuzzy graph with fuzzy vertex and fuzzy edges and developed the structure of fuzzy graphs. Mordeson and Peng [25] defined some operations on fuzzy graphs. All the concepts on crisp graph theory do not have similarities in fuzzy graphs. Dhavaseelan et al. [26] defined strong neutrosophic graphs. Akram and Shahzadi [27] first studied single-valued neutrosophic graphs. Further several new concepts on neutrosophic graphs with their applications were discussed in [28–31]. On the other hand, Akram and Shahzadi [29] have shown that there are some flaws in Broumi et al. [30]'s definitions. His definitions are not useful for the study of applied network models. Recently, Akram and Naz [32] determined the energy of Pythagorean fuzzy graphs. In this paper, we provide the new concept of complex neutrosophic graphs with some fundamental operations. We also describe energy of complex neutrosophic graphs.

2 Preliminaries and basic definitions

Definition 1. *[19] Let $U \neq \varnothing$. A complex fuzzy set (CFS) $\mathcal{A}$, is an object of the form*

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) : x \in U\} = \{(x, u_{\mathcal{A}}(x)e^{i\omega_{\mathcal{A}}(x)}) : x \in U\},$$

where $i = \sqrt{-1}$, $u_{\mathcal{A}}(x) \in [0, 1]$ and $0 < \omega_{\mathcal{A}}(x) < 2\pi$.

Definition 2. *[22] Let $U \neq \varnothing$. A complex neutrosophic set ($\mathfrak{cn}$-set) $\mathcal{A}$, is an object of the form*

$$\begin{aligned}\mathcal{A} &= \{(x, \mathfrak{T}_{\mathcal{A}}(x), \mathfrak{I}_{\mathcal{A}}(x), \mathfrak{F}_{\mathcal{A}}(x)) : x \in U\}\\ &= \{(x, s_{\mathcal{A}}(x)e^{i\alpha_{\mathcal{A}}(x)}, t_{\mathcal{A}}(x)e^{i\beta_{\mathcal{A}}(x)}, u_{\mathcal{A}}(x)e^{i\gamma_{\mathcal{A}}(x)}) : x \in U\},\end{aligned}$$

where $i = \sqrt{-1}$, $s_{\mathcal{A}}(x), t_{\mathcal{A}}(x), u_{\mathcal{A}}(x) \in [0,1]$, $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x) \in [0, 2\pi]$ *and* $0^- \leq s_{\mathcal{A}}(x) + t_{\mathcal{A}}(x) + u_{\mathcal{A}}(x) \leq 3^+$.

Definition 3. *[22] Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\mathfrak{cn}$*-sets in* $\mathcal{X}$, *where*

$$\begin{aligned} \mathcal{A} &= \{(x, \mathfrak{T}_{\mathcal{A}}(x), \mathfrak{I}_{\mathcal{A}}(x), \mathfrak{F}_{\mathcal{A}}(x)) : x \in \mathcal{X}\} \\ \text{and } \mathcal{B} &= \{(x, \mathfrak{T}_{\mathcal{B}}(x), \mathfrak{I}_{\mathcal{B}}(x), \mathfrak{F}_{\mathcal{B}}(x)) : x \in \mathcal{X}\}. \end{aligned}$$

Then for all $x \in \mathcal{X}$,

(1) $\mathcal{A} \subset \mathcal{B}$ *if and only if* $s_{\mathcal{A}}(x) < s_{\mathcal{B}}(x)$, $t_{\mathcal{A}}(x) > t_{\mathcal{B}}(x)$, $u_{\mathcal{A}}(x) > u_{\mathcal{B}}(x)$ *for amplitude terms and* $\alpha_{\mathcal{A}}(x) < \alpha_{\mathcal{B}}(x)$, $\beta_{\mathcal{A}}(x) > \beta_{\mathcal{B}}(x)$, $\gamma_{\mathcal{A}}(x) > \gamma_{\mathcal{B}}(x)$ *for phase terms.*

(2) $\mathcal{A} = \mathcal{B}$ *if and only if* $s_{\mathcal{A}}(x) = s_{\mathcal{B}}(x)$, $t_{\mathcal{A}}(x) = t_{\mathcal{B}}(x)$, $u_{\mathcal{A}}(x) = u_{\mathcal{B}}(x)$ *for amplitude terms and* $\alpha_{\mathcal{A}}(x) = \alpha_{\mathcal{B}}(x)$, $\beta_{\mathcal{A}}(x) = \beta_{\mathcal{B}}(x)$, $\gamma_{\mathcal{A}}(x) = \gamma_{\mathcal{B}}(x)$ *for phase terms.*

(3) $\mathcal{A} \cup \mathcal{B} = \{(x, \mathfrak{T}_{\mathcal{A}\cup\mathcal{B}}(x), \mathfrak{I}_{\mathcal{A}\cup\mathcal{B}}(x), \mathfrak{F}_{\mathcal{A}\cup\mathcal{B}}(x)) : x \in \mathcal{X}\}$
where

$$\begin{aligned} \mathfrak{T}_{\mathcal{A}\cup\mathcal{B}}(x) &= s_{\mathcal{A}\cup\mathcal{B}}(x) e^{i\alpha_{\mathcal{A}\cup\mathcal{B}}(x)} = \max\{s_{\mathcal{A}}(x), s_{\mathcal{B}}(x)\} e^{i \max\{\alpha_{\mathcal{A}}(x), \alpha_{\mathcal{B}}(x)\}}, \\ \mathfrak{I}_{\mathcal{A}\cup\mathcal{B}}(x) &= t_{\mathcal{A}\cup\mathcal{B}}(x) e^{i\beta_{\mathcal{A}\cup\mathcal{B}}(x)} = \min\{t_{\mathcal{A}}(x), t_{\mathcal{B}}(x)\} e^{i \min\{\beta_{\mathcal{A}}(x), \beta_{\mathcal{B}}(x)\}}, \\ \mathfrak{F}_{\mathcal{A}\cup\mathcal{B}}(x) &= u_{\mathcal{A}\cup\mathcal{B}}(x) e^{i\gamma_{\mathcal{A}\cup\mathcal{B}}(x)} = \min\{u_{\mathcal{A}}(x) u_{\mathcal{B}}(x)\} e^{i \min\{\gamma_{\mathcal{A}}(x), \gamma_{\mathcal{B}}(x)\}}. \end{aligned}$$

(4) $\mathcal{A} \cap \mathcal{B} = \{(x, \mathfrak{T}_{\mathcal{A}\cap\mathcal{B}}(x), \mathfrak{I}_{\mathcal{A}\cap\mathcal{B}}(x), \mathfrak{F}_{\mathcal{A}\cap\mathcal{B}}(x)) : x \in \mathcal{X}\}$
where

$$\begin{aligned} \mathfrak{T}_{\mathcal{A}\cap\mathcal{B}}(x) &= s_{\mathcal{A}\cap\mathcal{B}}(x) e^{i\alpha_{\mathcal{A}\cap\mathcal{B}}(x)} = \min\{s_{\mathcal{A}}(x), s_{\mathcal{B}}(x)\} e^{i \min\{\alpha_{\mathcal{A}}(x), \alpha_{\mathcal{B}}(x)\}}, \\ \mathfrak{I}_{\mathcal{A}\cap\mathcal{B}}(x) &= t_{\mathcal{A}\cap\mathcal{B}}(x) e^{i\beta_{\mathcal{A}\cap\mathcal{B}}(x)} = \max\{t_{\mathcal{A}}(x), t_{\mathcal{B}}(x)\} e^{i \max\{\beta_{\mathcal{A}}(x), \beta_{\mathcal{B}}(x)\}}, \\ \mathfrak{F}_{\mathcal{A}\cap\mathcal{B}}(x) &= u_{\mathcal{A}\cap\mathcal{B}}(x) e^{i\gamma_{\mathcal{A}\cap\mathcal{B}}(x)} = \max\{u_{\mathcal{A}}(x) u_{\mathcal{B}}(x)\} e^{i \max\{\gamma_{\mathcal{A}}(x), \gamma_{\mathcal{B}}(x)\}}. \end{aligned}$$

Definition 4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\mathfrak{cn}$*-sets in* $\mathcal{X}$, *where*
$\mathcal{A} = \{(x, \mathfrak{T}_{\mathcal{A}}(x), \mathfrak{I}_{\mathcal{A}}(x), \mathfrak{F}_{\mathcal{A}}(x)) : x \in \mathcal{X}\}$ *and*
$\mathcal{B} = \{(x, \mathfrak{T}_{\mathcal{B}}(x), \mathfrak{I}_{\mathcal{B}}(x), \mathfrak{F}_{\mathcal{B}}(x)) : x \in \mathcal{X}\}$.
Then $\mathfrak{T}_{\mathcal{A}}(x) \leq \mathfrak{T}_{\mathcal{B}}(x)$, $\mathfrak{I}_{\mathcal{A}}(x) \geq \mathfrak{I}_{\mathcal{B}}(x)$, $\mathfrak{F}_{\mathcal{A}}(x) \geq \mathfrak{F}_{\mathcal{B}}(x)$ *if and only if*
$|\mathfrak{T}_{\mathcal{A}}(x)| \leq |\mathfrak{T}_{\mathcal{B}}(x)|$, $|\mathfrak{I}_{\mathcal{A}}(x)| \geq |\mathfrak{I}_{\mathcal{B}}(x)|$, $|\mathfrak{F}_{\mathcal{A}}(x)| \geq |\mathfrak{F}_{\mathcal{B}}(x)|$ *and*
$\alpha_{\mathcal{A}}(x) \leq \alpha_{\mathcal{B}}(x)$, $\beta_{\mathcal{A}}(x) \geq \beta_{\mathcal{B}}(x)$, $\gamma_{\mathcal{A}}(x) \geq \gamma_{\mathcal{B}}(x)$.

3 Complex neutrosophic graphs

Definition 5. *A* $\mathfrak{cn}$*-relation on* $\mathcal{X} \neq \varnothing$*, is a* $\mathfrak{cn}$*-subset of* $\mathcal{X} \times \mathcal{X}$ *of the form* $\mathcal{B} = \{(xy, \mathfrak{T}_{\mathcal{B}}(xy), \mathfrak{I}_{\mathcal{B}}(xy), \mathfrak{F}_{\mathcal{B}}(xy)) : xy \in \mathcal{X} \times \mathcal{X}\}$ *such that*

$$\mathfrak{T}_{\mathcal{B}}(xy) = s_{\mathcal{B}}(xy)e^{i\alpha_{\mathcal{B}}(xy)},\ \mathfrak{I}_{\mathcal{B}}(xy) = t_{\mathcal{B}}(xy)e^{i\beta_{\mathcal{B}}(xy)},\ \mathfrak{F}_{\mathcal{B}}(xy) = u_{\mathcal{B}}(xy)e^{i\gamma_{\mathcal{B}}(xy)}$$

for all $x, y \in \mathcal{X}$.

Definition 6. *A* $\mathfrak{cn}$*-graph on* $\mathcal{X} \neq \varnothing$*, is a pair* $\mathbb{G} = (\mathcal{A}, \mathcal{B})$ *where,* $\mathcal{A}$ *is a* $\mathfrak{cn}$*-set on* $\mathcal{X}$ *and* $\mathcal{B}$ *is a* $\mathfrak{cn}$*-relation in* $\mathcal{X}$ *such that*

$$\begin{aligned}
s_{\mathcal{B}}(xy)e^{i\alpha_{\mathcal{B}}(xy)} &\leq \min\{s_{\mathcal{A}}(x), s_{\mathcal{A}}(y)\}e^{i\min\{\alpha_{\mathcal{A}}(x),\alpha_{\mathcal{A}}(y)\}},\\
t_{\mathcal{B}}(xy)e^{i\beta_{\mathcal{B}}(xy)} &\leq \min\{t_{\mathcal{A}}(x), t_{\mathcal{A}}(y)\}e^{i\min\{\beta_{\mathcal{A}}(x),\beta_{\mathcal{A}}(y)\}},\\
u_{\mathcal{B}}(xy)e^{i\gamma_{\mathcal{B}}(xy)} &\leq \max\{u_{\mathcal{A}}(x), u_{\mathcal{A}}(y)\}e^{i\max\{\gamma_{\mathcal{A}}(x),\gamma_{\mathcal{A}}(y)\}}
\end{aligned}$$

for all $x, y \in \mathcal{X}$*.* $\mathcal{A}$ *and* $\mathcal{B}$ *are called the complex neutrosophic vertex set and the complex neutrosophic edge set of* $\mathbb{G}$*, respectively. Here* $\mathcal{B}$ *is the complex neutrosophic relation on* $\mathcal{A}$.

Example 1. *Consider a graph* $\mathbb{G}^* = (\mathcal{V}, \mathrm{E})$ *such that* $\mathcal{X} = \{a, b, c\}$*,* $\mathrm{E} = \{ab, bc, ac\}$*. Let* $\mathcal{A}$ *be a* $\mathfrak{cn}$*-subset of* $\mathcal{X}$ *and let* $\mathcal{B}$ *be a* $\mathfrak{cn}$*-subset of* $\mathrm{E} \subseteq \mathcal{X} \times \mathcal{X}$*, as given:*

$$\mathcal{A} = \left(\begin{array}{c} \frac{(0.2e^{i\ \pi/2}, 0.4e^{i\ 2\pi/5}, 0.5e^{i\ \pi/4})}{a}, \frac{(0.5e^{i\ 3\pi/4}, 0.6e^{i\ \pi/2}, 0.1e^{i\ \pi/8})}{b}, \\ \frac{(0.4e^{i\ \pi/2}, 0.5e^{i\ \pi}, 0.2e^{i\ \pi/3})}{c} \end{array} \right).$$

$$\mathcal{B} = \left(\begin{array}{c} \frac{(0.1e^{i\ \pi/2}, 0.4e^{i\ \pi/3}, 0.3e^{i\ \pi/6})}{ab}, \frac{(0.2e^{i\ 2\pi/5}, 0.4e^{i\ 2\pi/5}, 0.1e^{i\ \pi/4})}{bc}, \\ \frac{(0.1e^{i\ \pi/4}, 0.3e^{i\ \pi/4}, 0.4e^{i\ 2\pi/7})}{ac} \end{array} \right).$$

By routine calculations, it can be observed that the graph shown in Fig. 1 is a $\mathfrak{cn}$*-graph.*

Definition 7. *The Cartesian product* $\mathbb{G}_1 \times \mathbb{G}_2$ *of two* $\mathfrak{cn}$*-graphs is defined as a pair* $\mathbb{G}_1 \times \mathbb{G}_2 = (\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B}_1 \times \mathcal{B}_2)$*, such that:*

1. $s_{\mathcal{A}_1\times\mathcal{A}_2}(x_1, x_2)e^{i\alpha_{\mathcal{A}_1\times\mathcal{A}_2}(x_1,x_2)} = \min\{s_{\mathcal{A}_1}(x_1), s_{\mathcal{A}_2}(x_2))\}e^{i\min\{\alpha_{\mathcal{A}_1}(x_1),\alpha_{\mathcal{A}_2}(x_2)\}}$,
   $t_{\mathcal{A}_1\times\mathcal{A}_2}(x_1, x_2)e^{i\beta_{\mathcal{A}_1\times\mathcal{A}_2}(x_1,x_2)} = \min\{t_{\mathcal{A}_1}(x_1), t_{\mathcal{A}_2}(x_2))\}e^{i\min\{\beta_{\mathcal{A}_1}(x_1),\beta_{\mathcal{A}_2}(x_2)\}}$,
   $u_{\mathcal{A}_1\times\mathcal{A}_2}(x_1, x_2)e^{i\gamma_{\mathcal{A}_1\times\mathcal{A}_2}(x_1,x_2)} = \max\{u_{\mathcal{A}_1}(x_1), u_{\mathcal{A}_2}(x_2))\}e^{i\max\{\gamma_{\mathcal{A}_1}(x_1),\gamma_{\mathcal{A}_2}(x_2)\}}$,
   for all $x_1,\ x_2 \in \mathcal{X}$,

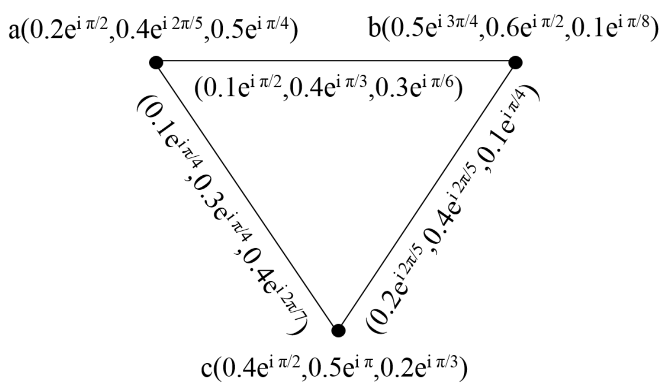

Fig. 1: $\mathfrak{cn}$-graph

2. $s_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\alpha_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))}$
$= \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{B}_2}(x_2y_2))\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{B}_2}(x_2y_2)\}}$,
$t_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\beta_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))}$
$= \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{B}_2}(x_2y_2))\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{B}_2}(x_2y_2)\}}$,
$u_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\gamma_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))}$
$= \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{B}_2}(x_2y_2))\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{B}_2}(x_2y_2)\}}$,
for all $x \in \mathcal{X}_1$, *and* $x_2y_2 \in \mathrm{E}_2$,

3. $s_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,z)(y_1,z))e^{i\alpha_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,z)(y_1,z))}$
$= \min\{s_{\mathcal{B}_1}(x_1y_1), s_{\mathcal{A}_2}(z))\}e^{i\min\{\alpha_{\mathcal{B}_1}(x_1y_1),\alpha_{\mathcal{A}_2}(z)\}}$,
$t_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,z)(y_1,z))e^{i\beta_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,z)(y_1,z))}$
$= \min\{t_{\mathcal{B}_1}(x_1y_1), t_{\mathcal{A}_2}(z))\}e^{i\min\{\beta_{\mathcal{B}_1}(x_1y_1),\beta_{\mathcal{A}_2}(z)\}}$,
$u_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,z)(y_1,z))e^{i\gamma_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,z)(y_1,z))}$
$= \max\{u_{\mathcal{B}_1}(x_1y_1), u_{\mathcal{A}_2}(z))\}e^{i\max\{\gamma_{\mathcal{B}_1}(x_1y_1),\gamma_{\mathcal{A}_2}(z)\}}$,
for all $z \in \mathcal{X}_2$, *and* $x_1y_1 \in \mathrm{E}_1$.

Example 2. *Consider the two* $\mathfrak{cn}$*-graphs, as shown in Fig. 2. Then, their corresponding Cartesian product* $\mathbb{G}_1 \times \mathbb{G}_2$ *is shown in Fig. 3.*

Proposition 1. The Cartesian product of two $\mathfrak{cn}$-graphs is a $\mathfrak{cn}$-graph.

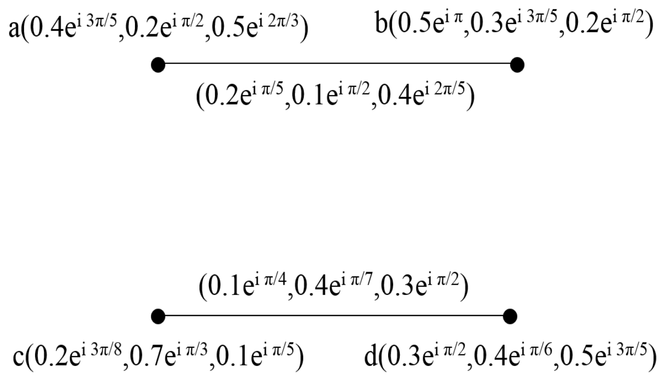

Fig. 2: $\mathfrak{cn}$-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$

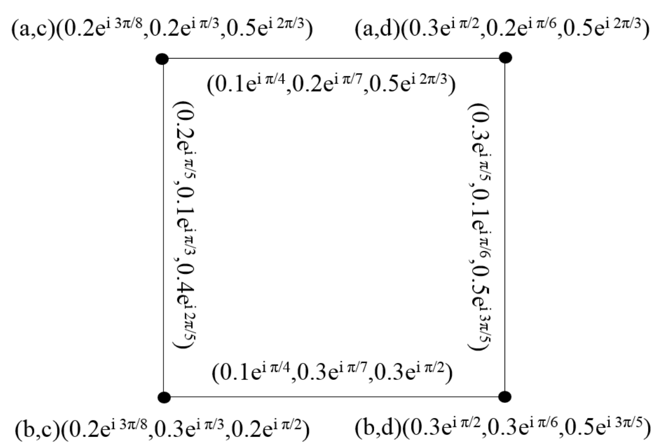

Fig. 3: $\mathfrak{cn}$-graph $\mathbb{G}_1 \times \mathbb{G}_2$

Proof: The conditions for $\mathcal{A}_1 \times \mathcal{A}_2$ are obvious, therefore, we verify only conditions for $\mathcal{B}_1 \times \mathcal{B}_2$.
Let $x \in \mathcal{X}_1$, and $x_2y_2 \in E_2$. Then

$$\begin{aligned}
&s_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\alpha_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))}\\
&= \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{B}_2}(x_2y_2))\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{B}_2}(x_2y_2)\}}\\
&\leq \min\{s_{\mathcal{A}_1}(x), \min\{s_{\mathcal{A}_2}(x_2), s_{\mathcal{A}_2}(y_2)\}\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\min\{\alpha_{\mathcal{A}_2}(x_2),\alpha_{\mathcal{A}_2}(y_2)\}\}}\\
&= \min\left\{\begin{array}{c}\min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(x_2)\},\\ \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(y_2)\}\end{array}\right\}e^{i\min\left\{\begin{array}{c}\min\{\alpha_{\mathcal{A}_1}(x), \alpha_{\mathcal{A}_2}(x_2)\},\\ \min\{\alpha_{\mathcal{A}_1}(x), \alpha_{\mathcal{A}_2}(y_2)\}\end{array}\right\}}\\
&= \min\{s_{\mathcal{A}_1\times\mathcal{A}_2}(x,x_2), s_{\mathcal{A}_1\times\mathcal{A}_2}(x,y_2)\}e^{i\min\{\alpha_{\mathcal{A}_1\times\mathcal{A}_2}(x,x_2),\alpha_{\mathcal{A}_1\times\mathcal{A}_2}(x,y_2)\}},
\end{aligned}$$

$$\begin{aligned}
&t_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\beta_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))}\\
&= \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{B}_2}(x_2y_2))\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{B}_2}(x_2y_2)\}}\\
&\leq \min\{t_{\mathcal{A}_1}(x), \min\{t_{\mathcal{A}_2}(x_2), t_{\mathcal{A}_2}(y_2)\}\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\min\{\beta_{\mathcal{A}_2}(x_2),\beta_{\mathcal{A}_2}(y_2)\}\}}\\
&= \min\left\{\begin{array}{c}\min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(x_2)\},\\ \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(y_2)\}\end{array}\right\}e^{i\min\left\{\begin{array}{c}\min\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{A}_2}(x_2)\},\\ \min\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{A}_2}(y_2)\}\end{array}\right\}}\\
&= \min\{t_{\mathcal{A}_1\times\mathcal{A}_2}(x,x_2), t_{\mathcal{A}_1\times\mathcal{A}_2}(x,y_2)\}e^{i\min\{\beta_{\mathcal{A}_1\times\mathcal{A}_2}(x,x_2),\beta_{\mathcal{A}_1\times\mathcal{A}_2}(x,y_2)\}},
\end{aligned}$$

$$\begin{aligned}
&u_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\gamma_{\mathcal{B}_1\times\mathcal{B}_2}((x,x_2)(x,y_2))}\\
&= \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{B}_2}(x_2y_2))\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{B}_2}(x_2y_2)\}}\\
&\leq \max\{u_{\mathcal{A}_1}(x), \max\{u_{\mathcal{A}_2}(x_2), u_{\mathcal{A}_2}(y_2)\}\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\max\{\gamma_{\mathcal{A}_2}(x_2),\gamma_{\mathcal{A}_2}(y_2)\}\}}\\
&= \max\left\{\begin{array}{c}\max\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(x_2)\},\\ \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(y_2)\}\end{array}\right\}e^{i\max\left\{\begin{array}{c}\max\{\gamma_{\mathcal{A}_1}(x), \gamma_{\mathcal{A}_2}(x_2)\},\\ \max\{\gamma_{\mathcal{A}_1}(x), \gamma_{\mathcal{A}_2}(y_2)\}\end{array}\right\}}\\
&= \max\{u_{\mathcal{A}_1\times\mathcal{A}_2}(x,x_2), u_{\mathcal{A}_1\times\mathcal{A}_2}(x,y_2)\}e^{i\max\{\gamma_{\mathcal{A}_1\times\mathcal{A}_2}(x,x_2),\gamma_{\mathcal{A}_1\times\mathcal{A}_2}(x,y_2)\}},
\end{aligned}$$

Similarly, we can prove it for $z \in \mathcal{X}_2$, and $x_1y_1 \in E_1$. $\square$

Definition 8. *The composition $\mathbb{G}_1 \circ \mathbb{G}_2$ of two* $\mathfrak{cn}$ *-graphs is defined as a pair $\mathbb{G}_1 \circ \mathbb{G}_2 = (\mathcal{A}_1 \circ \mathcal{A}_2,\ \mathcal{B}_1 \circ \mathcal{B}_2)$, such that:*

1. $s_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2)e^{i\alpha_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2)} = \min\{s_{\mathcal{A}_1}(x_1), s_{\mathcal{A}_2}(x_2))\}e^{i\min\{\alpha_{\mathcal{A}_1}(x_1),\alpha_{\mathcal{A}_2}(x_2)\}}$,

$t_{\mathcal{A}_1 \circ \mathcal{A}_2}(x_1, x_2) e^{i\beta_{\mathcal{A}_1 \circ \mathcal{A}_2}(x_1,x_2)} = \min\{t_{\mathcal{A}_1}(x_1), t_{\mathcal{A}_2}(x_2))\} e^{i \min\{\beta_{\mathcal{A}_1}(x_1), \beta_{\mathcal{A}_2}(x_2)\}}$,
$u_{\mathcal{A}_1 \circ \mathcal{A}_2}(x_1, x_2) e^{i\gamma_{\mathcal{A}_1 \circ \mathcal{A}_2}(x_1,x_2)} = \max\{u_{\mathcal{A}_1}(x_1), u_{\mathcal{A}_2}(x_2))\} e^{i \max\{\gamma_{\mathcal{A}_1}(x_1), \gamma_{\mathcal{A}_2}(x_2)\}}$,
for all $x_1,\ x_2 \in \mathcal{X}$,

2. $s_{\mathcal{B}_1 \circ \mathcal{B}_2}((x, x_2)(x, y_2)) e^{i\alpha_{\mathcal{B}_1 \circ \mathcal{B}_2}((x,x_2)(x,y_2))}$
$= \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{B}_2}(x_2 y_2))\} e^{i \min\{\alpha_{\mathcal{A}_1}(x), \alpha_{\mathcal{B}_2}(x_2 y_2)\}}$,
$t_{\mathcal{B}_1 \circ \mathcal{B}_2}((x, x_2)(x, y_2)) e^{i\beta_{\mathcal{B}_1 \circ \mathcal{B}_2}((x,x_2)(x,y_2))}$
$= \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{B}_2}(x_2 y_2))\} e^{i \min\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{B}_2}(x_2 y_2)\}}$,
$u_{\mathcal{B}_1 \circ \mathcal{B}_2}((x, x_2)(x, y_2)) e^{i\gamma_{\mathcal{B}_1 \circ \mathcal{B}_2}((x,x_2)(x,y_2))}$
$= \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{B}_2}(x_2 y_2))\} e^{i \max\{\gamma_{\mathcal{A}_1}(x), \gamma_{\mathcal{B}_2}(x_2 y_2)\}}$,
for all $x \in \mathcal{X}_1$, *and* $x_2 y_2 \in \mathrm{E}_2$,

3. $s_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z)) e^{i\alpha_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1,z)(y_1,z))}$
$= \min\{s_{\mathcal{B}_1}(x_1 y_1), s_{\mathcal{A}_2}(z))\} e^{i \min\{\alpha_{\mathcal{B}_1}(x_1 y_1), \alpha_{\mathcal{A}_2}(z)\}}$,
$t_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z)) e^{i\beta_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1,z)(y_1,z))}$
$= \min\{t_{\mathcal{B}_1}(x_1 y_1), t_{\mathcal{A}_2}(z))\} e^{i \min\{\beta_{\mathcal{B}_1}(x_1 y_1), \beta_{\mathcal{A}_2}(z)\}}$,
$u_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z)) e^{i\gamma_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1,z)(y_1,z))}$
$= \max\{u_{\mathcal{B}_1}(x_1 y_1), u_{\mathcal{A}_2}(z))\} e^{i \max\{\gamma_{\mathcal{B}_1}(x_1 y_1), \gamma_{\mathcal{A}_2}(z)\}}$,
for all $z \in \mathcal{X}_2$, *and* $x_1 y_1 \in \mathrm{E}_1$.

4. $s_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, x_2)(y_1, y_2)) e^{i\alpha_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1,x_2)(y_1,y_2))}$
$= \min\{s_{\mathcal{A}_2}(x_2), s_{\mathcal{A}_2}(y_2), s_{\mathcal{B}_1}(x_1 y_1)\} e^{i \min\{\alpha_{\mathcal{A}_2}(x_2), \alpha_{\mathcal{A}_2}(y_2), \alpha_{\mathcal{B}_1}(x_1 y_1)\}}$,
$t_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, x_2)(y_1, y_2)) e^{i\beta_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1,x_2)(y_1,y_2))}$
$= \min\{t_{\mathcal{A}_2}(x_2), t_{\mathcal{A}_2}(y_2), t_{\mathcal{B}_1}(x_1 y_1)\} e^{i \min\{\beta_{\mathcal{A}_2}(x_2), \beta_{\mathcal{A}_2}(y_2), \beta_{\mathcal{B}_1}(x_1 y_1)\}}$,
$u_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, x_2)(y_1, y_2)) e^{i\gamma_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1,x_2)(y_1,y_2))}$
$= \max\{u_{\mathcal{A}_2}(x_2), u_{\mathcal{A}_2}(y_2), u_{\mathcal{B}_1}(x_1 y_1)\} e^{i \max\{\gamma_{\mathcal{A}_2}(x_2), \gamma_{\mathcal{A}_2}(y_2), \gamma_{\mathcal{B}_1}(x_1 y_1)\}}$,
for all $x_2, y_2 \in \mathcal{X}_2$, $x_2 \neq y_2$ *and* $x_1 y_1 \in \mathrm{E}_1$.

Example 3. *Consider the two* $\mathfrak{cn}$*-graphs, as shown in Fig. 4. Then, their composition* $\mathbb{G}_1 \circ \mathbb{G}_2$ *is shown in Fig. 5.*

Proposition 2. *The composition of two* $\mathfrak{cn}$*-graphs is a* $\mathfrak{cn}$*-graph.*

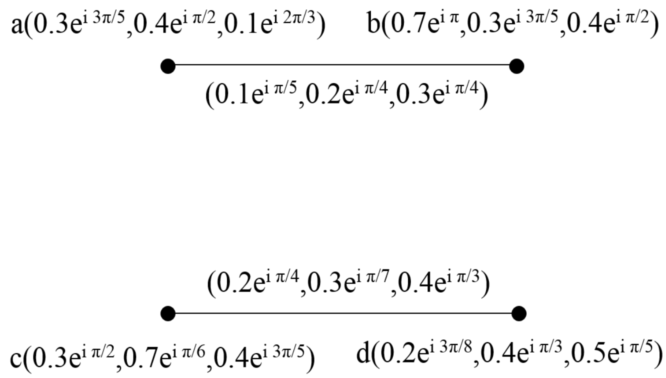

Fig. 4: $\mathfrak{cn}$-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$

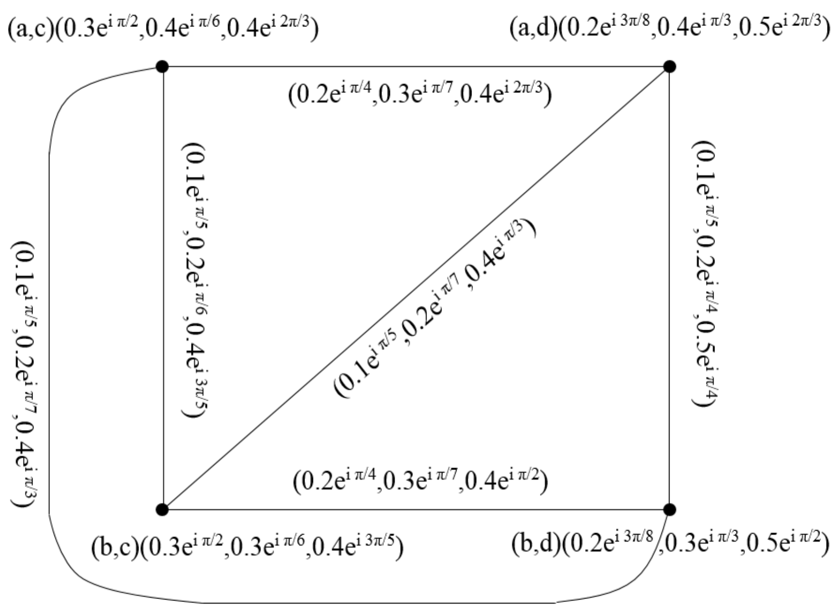

Fig. 5: $\mathfrak{cn}$-graph $\mathbb{G}_1 \circ \mathbb{G}_2$

Proof: As in the previous proof, we verify the conditions for $\mathcal{B}_1 \circ \mathcal{B}_2$ only. We prove the 2nd case:

$$\begin{aligned}
&s_{\mathcal{B}_1\circ\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\alpha_{\mathcal{B}_1\circ\mathcal{B}_2}((x,x_2)(x,y_2))}\\
&=\min\{s_{\mathcal{A}_1}(x), s_{\mathcal{B}_2}(x_2y_2))\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{B}_2}(x_2y_2)\}}\\
&\leq \min\{s_{\mathcal{A}_1}(x), \min\{s_{\mathcal{A}_2}(x_2), s_{\mathcal{A}_2}(y_2)\}\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\min\{\alpha_{\mathcal{A}_2}(x_2),\alpha_{\mathcal{A}_2}(y_2)\}\}}\\
&=\min\left\{\begin{array}{c}\min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(x_2)\},\\ \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(y_2)\}\end{array}\right\}e^{i\min\left\{\begin{array}{c}\min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{A}_2}(x_2)\},\\ \min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{A}_2}(y_2)\}\end{array}\right\}}\\
&=\min\{s_{\mathcal{A}_1\circ\mathcal{A}_2}(x,x_2), s_{\mathcal{A}_1\circ\mathcal{A}_2}(x,y_2)\}e^{i\min\{\alpha_{\mathcal{A}_1\circ\mathcal{A}_2}(x,x_2),\alpha_{\mathcal{A}_1\circ\mathcal{A}_2}(x,y_2)\}},
\end{aligned}$$

$$\begin{aligned}
&t_{\mathcal{B}_1\circ\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\beta_{\mathcal{B}_1\circ\mathcal{B}_2}((x,x_2)(x,y_2))}\\
&=\min\{t_{\mathcal{A}_1}(x), t_{\mathcal{B}_2}(x_2y_2))\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{B}_2}(x_2y_2)\}}\\
&\leq \min\{t_{\mathcal{A}_1}(x), \min\{t_{\mathcal{A}_2}(x_2), t_{\mathcal{A}_2}(y_2)\}\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\min\{\beta_{\mathcal{A}_2}(x_2),\beta_{\mathcal{A}_2}(y_2)\}\}}\\
&=\min\left\{\begin{array}{c}\min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(x_2)\},\\ \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(y_2)\}\end{array}\right\}e^{i\min\left\{\begin{array}{c}\min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{A}_2}(x_2)\},\\ \min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{A}_2}(y_2)\}\end{array}\right\}}\\
&=\min\{t_{\mathcal{A}_1\circ\mathcal{A}_2}(x,x_2), t_{\mathcal{A}_1\circ\mathcal{A}_2}(x,y_2)\}e^{i\min\{\beta_{\mathcal{A}_1\circ\mathcal{A}_2}(x,x_2),\beta_{\mathcal{A}_1\circ\mathcal{A}_2}(x,y_2)\}},
\end{aligned}$$

$$\begin{aligned}
&u_{\mathcal{B}_1\circ\mathcal{B}_2}((x,x_2)(x,y_2))e^{i\gamma_{\mathcal{B}_1\circ\mathcal{B}_2}((x,x_2)(x,y_2))}\\
&=\max\{u_{\mathcal{A}_1}(x), u_{\mathcal{B}_2}(x_2y_2))\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{B}_2}(x_2y_2)\}}\\
&\leq \max\{u_{\mathcal{A}_1}(x), \max\{u_{\mathcal{A}_2}(x_2), u_{\mathcal{A}_2}(y_2)\}\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\max\{\gamma_{\mathcal{A}_2}(x_2),\gamma_{\mathcal{A}_2}(y_2)\}\}}\\
&=\max\left\{\begin{array}{c}\max\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(x_2)\},\\ \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(y_2)\}\end{array}\right\}e^{i\max\left\{\begin{array}{c}\max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{A}_2}(x_2)\},\\ \max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{A}_2}(y_2)\}\end{array}\right\}}\\
&=\max\{u_{\mathcal{A}_1\circ\mathcal{A}_2}(x,x_2), u_{\mathcal{A}_1\circ\mathcal{A}_2}(x,y_2)\}e^{i\max\{\gamma_{\mathcal{A}_1\circ\mathcal{A}_2}(x,x_2),\gamma_{\mathcal{A}_1\circ\mathcal{A}_2}(x,y_2)\}},
\end{aligned}$$

Similarly, we can prove it for $z \in \mathcal{X}_2$, and $x_1y_1 \in E_1$. In the case x_2, $y_2 \in \mathcal{X}_2$,

$x_2 \neq y_2$ and $x_1y_1 \in E_1$, we have,

$$
\begin{aligned}
& s_{\mathcal{B}_1\circ\mathcal{B}_2}((x_1,x_2)(y_1,y_2))e^{i\alpha_{\mathcal{B}_1\circ\mathcal{B}_2}((x_1,x_2)(y_1,y_2))} \\
&= \min\{s_{\mathcal{A}_2}(x_2), s_{\mathcal{A}_2}(y_2), s_{\mathcal{B}_1}(x_1y_1)\}e^{i\min\{\alpha_{\mathcal{A}_2}(x_2),\alpha_{\mathcal{A}_2}(y_2),\alpha_{\mathcal{B}_1}(x_1y_1)\}} \\
&\leq \min\left\{\begin{array}{c} s_{\mathcal{A}_2}(x_2), s_{\mathcal{A}_2}(y_2), \\ \min\{s_{\mathcal{A}_1}(x_1), s_{\mathcal{A}_1}(y_1)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \alpha_{\mathcal{A}_2}(x_2), \alpha_{\mathcal{A}_2}(y_2), \\ \min\{\alpha_{\mathcal{A}_1}(x_1), \alpha_{\mathcal{A}_1}(y_1)\} \end{array}\right\}} \\
&= \min\left\{\begin{array}{c} \min\{s_{\mathcal{A}_1}(x_1), s_{\mathcal{A}_2}(x_2)\}, \\ \min\{s_{\mathcal{A}_1}(y_1), s_{\mathcal{A}_2}(y_2)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \min\{\alpha_{\mathcal{A}_1}(x_1), \alpha_{\mathcal{A}_2}(x_2)\}, \\ \min\{\alpha_{\mathcal{A}_1}(y_1), \alpha_{\mathcal{A}_2}(y_2)\} \end{array}\right\}} \\
&= \min\{s_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2), s_{\mathcal{A}_1\circ\mathcal{A}_2}(y_1,y_2)\}e^{i\min\{\alpha_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2),\alpha_{\mathcal{A}_1\circ\mathcal{A}_2}(y_1,y_2)\}}
\end{aligned}
$$

$$
\begin{aligned}
& t_{\mathcal{B}_1\circ\mathcal{B}_2}((x_1,x_2)(y_1,y_2))e^{i\beta_{\mathcal{B}_1\circ\mathcal{B}_2}((x_1,x_2)(y_1,y_2))} \\
&= \min\{t_{\mathcal{A}_2}(x_2), t_{\mathcal{A}_2}(y_2), t_{\mathcal{B}_1}(x_1y_1)\}e^{i\min\{\beta_{\mathcal{A}_2}(x_2),\beta_{\mathcal{A}_2}(y_2),\beta_{\mathcal{B}_1}(x_1y_1)\}} \\
&\leq \min\left\{\begin{array}{c} t_{\mathcal{A}_2}(x_2), t_{\mathcal{A}_2}(y_2), \\ \min\{t_{\mathcal{A}_1}(x_1), t_{\mathcal{A}_1}(y_1)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \beta_{\mathcal{A}_2}(x_2), \beta_{\mathcal{A}_2}(y_2), \\ \min\{\beta_{\mathcal{A}_1}(x_1), \beta_{\mathcal{A}_1}(y_1)\} \end{array}\right\}} \\
&= \min\left\{\begin{array}{c} \min\{t_{\mathcal{A}_1}(x_1), t_{\mathcal{A}_2}(x_2)\}, \\ \min\{t_{\mathcal{A}_1}(y_1), t_{\mathcal{A}_2}(y_2)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \min\{\beta_{\mathcal{A}_1}(x_1), \beta_{\mathcal{A}_2}(x_2)\}, \\ \min\{\beta_{\mathcal{A}_1}(y_1), \beta_{\mathcal{A}_2}(y_2)\} \end{array}\right\}} \\
&= \min\{t_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2), t_{\mathcal{A}_1\circ\mathcal{A}_2}(y_1,y_2)\}e^{i\min\{\beta_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2),\beta_{\mathcal{A}_1\circ\mathcal{A}_2}(y_1,y_2)\}}
\end{aligned}
$$

$$
\begin{aligned}
& u_{\mathcal{B}_1\circ\mathcal{B}_2}((x_1,x_2)(y_1,y_2))e^{i\gamma_{\mathcal{B}_1\circ\mathcal{B}_2}((x_1,x_2)(y_1,y_2))} \\
&= \max\{u_{\mathcal{A}_2}(x_2), u_{\mathcal{A}_2}(y_2), u_{\mathcal{B}_1}(x_1y_1)\}e^{i\max\{\gamma_{\mathcal{A}_2}(x_2),\gamma_{\mathcal{A}_2}(y_2),\gamma_{\mathcal{B}_1}(x_1y_1)\}} \\
&\leq \max\left\{\begin{array}{c} u_{\mathcal{A}_2}(x_2), u_{\mathcal{A}_2}(y_2), \\ \max\{u_{\mathcal{A}_1}(x_1), u_{\mathcal{A}_1}(y_1)\} \end{array}\right\} e^{i\max\left\{\begin{array}{c} \gamma_{\mathcal{A}_2}(x_2), \gamma_{\mathcal{A}_2}(y_2), \\ \max\{\gamma_{\mathcal{A}_1}(x_1), \gamma_{\mathcal{A}_1}(y_1)\} \end{array}\right\}} \\
&= \max\left\{\begin{array}{c} \max\{u_{\mathcal{A}_1}(x_1), u_{\mathcal{A}_2}(x_2)\}, \\ \max\{u_{\mathcal{A}_1}(y_1), u_{\mathcal{A}_2}(y_2)\} \end{array}\right\} e^{i\max\left\{\begin{array}{c} \max\{\gamma_{\mathcal{A}_1}(x_1), \gamma_{\mathcal{A}_2}(x_2)\}, \\ \max\{\gamma_{\mathcal{A}_1}(y_1), \gamma_{\mathcal{A}_2}(y_2)\} \end{array}\right\}} \\
&= \max\{u_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2), u_{\mathcal{A}_1\circ\mathcal{A}_2}(y_1,y_2)\}e^{i\max\{\gamma_{\mathcal{A}_1\circ\mathcal{A}_2}(x_1,x_2),\gamma_{\mathcal{A}_1\circ\mathcal{A}_2}(y_1,y_2)\}}.
\end{aligned}
$$

This completes the proof. □

Definition 9. *The union* $\mathbb{G}_1 \cup \mathbb{G}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ *of two* $\mathfrak{cn}$*-graphs is defined as follows:*

1. $s_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = s_{\mathcal{A}_1}(x)e^{i\alpha_{\mathcal{A}_1}(x)}$,
   $t_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = t_{\mathcal{A}_1}(x)e^{i\beta_{\mathcal{A}_1}(x)}$,
   $u_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = u_{\mathcal{A}_1}(x)e^{i\gamma_{\mathcal{A}_1}(x)}$,
   for $x \in \mathcal{X}_1$ *and* $x \notin \mathcal{X}_2$.

2. $s_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = s_{\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_2}(x)}$,
   $t_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = t_{\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_2}(x)}$,
   $u_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = u_{\mathcal{A}_2}(x)e^{i\gamma_{\mathcal{A}_2}(x)}$,
   for $x \in \mathcal{X}_2$ *and* $x \notin \mathcal{X}_1$.

3. $s_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = \max\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(x)\}e^{i\max\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{A}_2}(x)\}}$,
   $t_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = \max\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(x)\}e^{i\max\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{A}_2}(x)\}}$,
   $u_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = \min\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(x)\}e^{i\min\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{A}_2}(x)\}}$,
   for $x \in \mathcal{X}_1 \cap \mathcal{X}_2$.

4. $s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = s_{\mathcal{B}_1}(xy)e^{i\alpha_{\mathcal{B}_1}(xy)}$,
   $t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = t_{\mathcal{B}_1}(xy)e^{i\beta_{\mathcal{B}_1}(xy)}$,
   $u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = u_{\mathcal{B}_1}(xy)e^{i\gamma_{\mathcal{B}_1}(xy)}$,
   for $xy \in \mathrm{E}_1$ *and* $xy \notin \mathrm{E}_2$.

5. $s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = s_{\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_2}(xy)}$,
   $t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = t_{\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_2}(xy)}$,
   $u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = u_{\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_2}(xy)}$,
   for $xy \in \mathcal{X}_2$ *and* $xy \notin \mathcal{X}_1$.

6. $s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = \max\{s_{\mathcal{B}_1}(xy), s_{\mathcal{B}_2}(xy)\}e^{i\max\{\alpha_{\mathcal{B}_1}(xy),\alpha_{\mathcal{B}_2}(xy)\}}$,
   $t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = \max\{t_{\mathcal{B}_1}(xy), t_{\mathcal{B}_2}(xy)\}e^{i\max\{\beta_{\mathcal{B}_1}(xy),\beta_{\mathcal{B}_2}(xy)\}}$,
   $u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = \min\{u_{\mathcal{B}_1}(xy), u_{\mathcal{B}_2}(xy)\}e^{i\min\{\gamma_{\mathcal{B}_1}(xy),\gamma_{\mathcal{B}_2}(xy)\}}$,
   for $xy \in \mathcal{X}_1 \cap \mathcal{X}_2$.

Example 4. *Consider the two* $\mathfrak{cn}$*-graphs, as shown in Fig. 6 and Fig. 7. Then, their corresponding union* $\mathbb{G}_1 \cup \mathbb{G}_2$ *is shown in Fig. 8.*

Proposition 3. *The union of two* $\mathfrak{cn}$*-graphs is a* $\mathfrak{cn}$*-graph.*

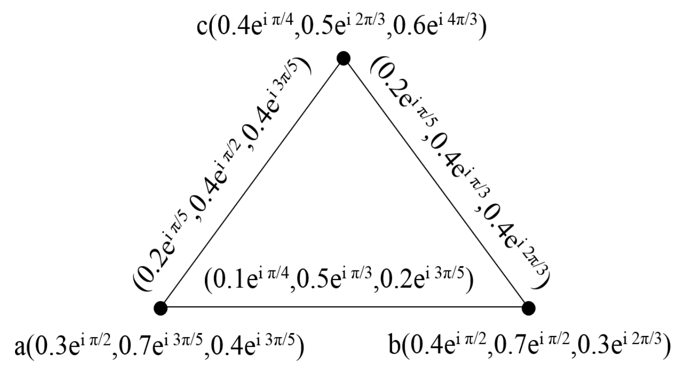

Fig. 6: $\mathfrak{cn}$-graph $\mathbb{G}_1$

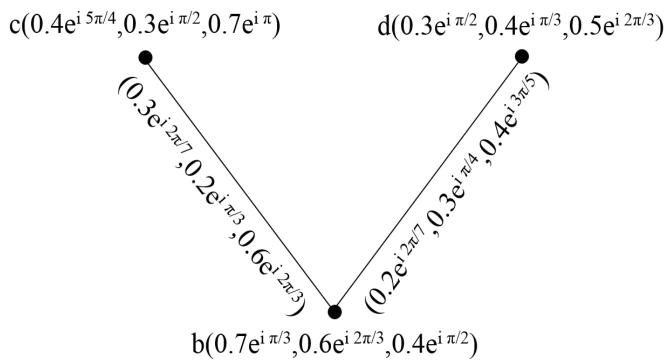

Fig. 7: $\mathfrak{cn}$-graph $\mathbb{G}_2$

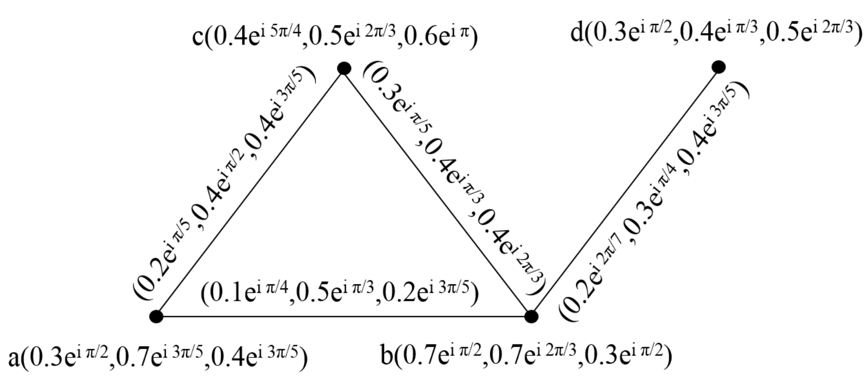

Fig. 8: $\mathfrak{cn}$-graph $\mathbb{G}_1 \cup \mathbb{G}_2$

Proof: Since all the conditions for $\mathcal{A}_1 \cup \mathcal{A}_2$ are automatically satisfied therefore, we verify only conditions for $\mathcal{B}_1 \cup \mathcal{B}_2$. In the case, when $xy \in E_1 \cap E_2$. Then

$$
\begin{aligned}
& s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} \\
& = \max\{s_{\mathcal{B}_1}(xy), s_{\mathcal{B}_2}(xy)\}e^{i\max\{\alpha_{\mathcal{B}_1}(xy),\alpha_{\mathcal{B}_2}(xy)\}} \\
& \leq \max\left\{\begin{array}{c} \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_1}(y)\}, \\ \min\{s_{\mathcal{A}_2}(x), s_{\mathcal{A}_2}(y)\} \end{array}\right\} e^{i\max\left\{\begin{array}{c} \min\{\alpha_{\mathcal{A}_1}(x), \alpha_{\mathcal{A}_1}(y)\}, \\ \min\{\alpha_{\mathcal{A}_2}(x), \alpha_{\mathcal{A}_2}(y)\} \end{array}\right\}} \\
& = \min\left\{\begin{array}{c} \max\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(x)\}, \\ \max\{s_{\mathcal{A}_1}(y), s_{\mathcal{A}_2}(y)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \max\{\alpha_{\mathcal{A}_1}(x), \alpha_{\mathcal{A}_2}(x)\}, \\ \max\{\alpha_{\mathcal{A}_1}(y), \alpha_{\mathcal{A}_2}(y)\} \end{array}\right\}} \\
& = \min\{s_{\mathcal{A}_1\cup\mathcal{A}_2}(x), s_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}
\end{aligned}
$$

$$
\begin{aligned}
& t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} \\
& = \max\{t_{\mathcal{B}_1}(xy), t_{\mathcal{B}_2}(xy)\}e^{i\max\{\beta_{\mathcal{B}_1}(xy),\beta_{\mathcal{B}_2}(xy)\}} \\
& \leq \max\left\{\begin{array}{c} \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_1}(y)\}, \\ \min\{t_{\mathcal{A}_2}(x), t_{\mathcal{A}_2}(y)\} \end{array}\right\} e^{i\max\left\{\begin{array}{c} \min\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{A}_1}(y)\}, \\ \min\{\beta_{\mathcal{A}_2}(x), \beta_{\mathcal{A}_2}(y)\} \end{array}\right\}} \\
& = \min\left\{\begin{array}{c} \max\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(x)\}, \\ \max\{t_{\mathcal{A}_1}(y), t_{\mathcal{A}_2}(y)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \max\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{A}_2}(x)\}, \\ \max\{\beta_{\mathcal{A}_1}(y), \beta_{\mathcal{A}_2}(y)\} \end{array}\right\}} \\
& = \min\{t_{\mathcal{A}_1\cup\mathcal{A}_2}(x), t_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}
\end{aligned}
$$

$$
\begin{aligned}
& u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} \\
& = \min\{u_{\mathcal{B}_1}(xy), u_{\mathcal{B}_2}(xy)\}e^{i\min\{\gamma_{\mathcal{B}_1}(xy),\gamma_{\mathcal{B}_2}(xy)\}} \\
& \leq \min\left\{\begin{array}{c} \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_1}(y)\}, \\ \max\{u_{\mathcal{A}_2}(x), u_{\mathcal{A}_2}(y)\} \end{array}\right\} e^{i\min\left\{\begin{array}{c} \max\{\gamma_{\mathcal{A}_1}(x), \gamma_{\mathcal{A}_1}(y)\}, \\ \max\{\gamma_{\mathcal{A}_2}(x), \gamma_{\mathcal{A}_2}(y)\} \end{array}\right\}} \\
& = \max\left\{\begin{array}{c} \min\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(x)\}, \\ \min\{u_{\mathcal{A}_1}(y), u_{\mathcal{A}_2}(y)\} \end{array}\right\} e^{i\max\left\{\begin{array}{c} \min\{\gamma_{\mathcal{A}_1}(x), \gamma_{\mathcal{A}_2}(x)\}, \\ \min\{\gamma_{\mathcal{A}_1}(y), \gamma_{\mathcal{A}_2}(y)\} \end{array}\right\}} \\
& = \max\{u_{\mathcal{A}_1\cup\mathcal{A}_2}(x), u_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}
\end{aligned}
$$

If $xy \in E_1$ and $xy \notin E_2$, then

$$\begin{aligned}
s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} &\leq \min\{s_{\mathcal{A}_1\cup\mathcal{A}_2}(x), s_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}\\
t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} &\leq \min\{t_{\mathcal{A}_1\cup\mathcal{A}_2}(x), t_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}\\
u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} &\leq \max\{u_{\mathcal{A}_1\cup\mathcal{A}_2}(x), u_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}
\end{aligned}$$

If $xy \in E_2$ and $xy \notin E_1$, then

$$\begin{aligned}
s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} &\leq \min\{s_{\mathcal{A}_1\cup\mathcal{A}_2}(x), s_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}\\
t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} &\leq \min\{t_{\mathcal{A}_1\cup\mathcal{A}_2}(x), t_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}\\
u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} &\leq \max\{u_{\mathcal{A}_1\cup\mathcal{A}_2}(x), u_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}}
\end{aligned}$$

This completes the proof. □

Definition 10. *The join $\mathbb{G}_1+\mathbb{G}_2 = (\mathcal{A}_1+\mathcal{A}_2, \mathcal{B}_1+\mathcal{B}_2)$ of two* $\mathfrak{cn}$*-graphs, where, $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$, is defined as follows:*

1. $$\begin{cases} s_{\mathcal{A}_1+\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1+\mathcal{A}_2}(x)} = s_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} \\ t_{\mathcal{A}_1+\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1+\mathcal{A}_2}(x)} = t_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} \\ u_{\mathcal{A}_1+\mathcal{A}_2}(x)e^{i\gamma_{\mathcal{A}_1+\mathcal{A}_2}(x)} = u_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} \end{cases} \quad \text{if } x \in \mathcal{X}_1 \cup \mathcal{X}_2,$$

2. $$\begin{cases} s_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1+\mathcal{B}_2}(x)} = s_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(x)} \\ t_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1+\mathcal{B}_2}(x)} = t_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(x)} \\ u_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1+\mathcal{B}_2}(x)} = u_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1\cup\mathcal{B}_2}(x)} \end{cases} \quad \text{if } xy \in \mathrm{E}_1 \cap \mathrm{E}_2,$$

3. $$\begin{cases} s_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1+\mathcal{B}_2}(xy)} = \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{A}_2}(y)\}} \\ t_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1+\mathcal{B}_2}(xy)} = \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{A}_2}(y)\}} \\ u_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1+\mathcal{B}_2}(xy)} = \max u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{A}_2}(y)\}} \end{cases} \quad \text{if } xy \in \acute{E},$$

   where $\acute{E}$ is the set of all edges joining the vertices of $\mathcal{X}_1$ and $\mathcal{X}_2$.

Example 5. *Consider the two* $\mathfrak{cn}$*-graphs, as shown in Fig. 9. Then, their corresponding join $\mathbb{G}_1+\mathbb{G}_2$ is shown in Fig. 10.*

Proposition 4. *The join of two* $\mathfrak{cn}$*-graphs is a* $\mathfrak{cn}$*-graph.*

Proof: Let $\mathbb{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathbb{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be $\mathfrak{cn}$-graphs of the graphs $\mathbb{G}_1^* = (\mathcal{X}_1, E_1)$ and $\mathbb{G}_2^* = (\mathcal{X}_2, E_2)$, respectively. We prove that $\mathbb{G}_1+\mathbb{G}_2 =$

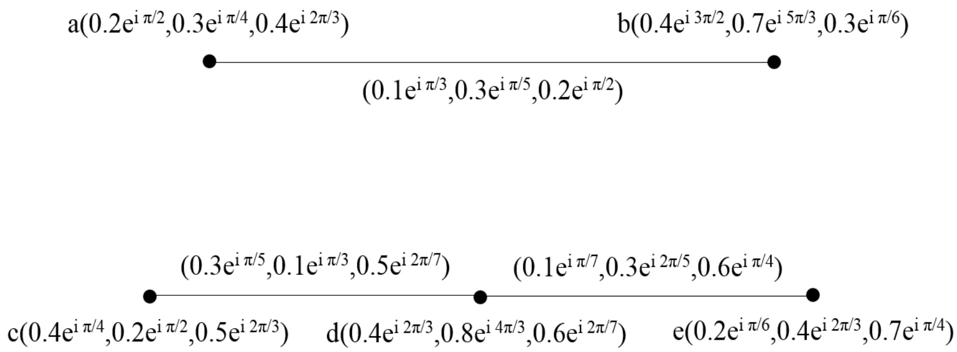

Fig. 9: $\mathfrak{cn}$-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$

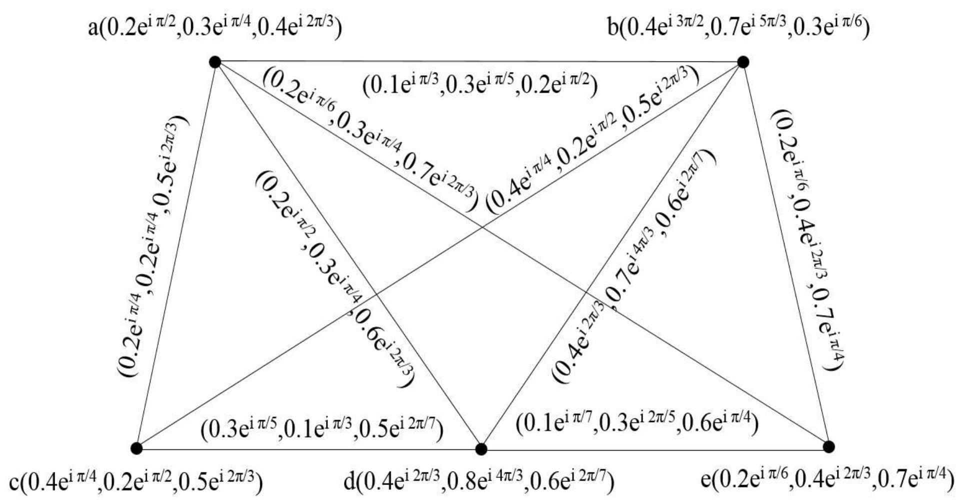

Fig. 10: $\mathfrak{cn}$-graph $\mathbb{G}_1+\mathbb{G}_2$

$(\mathcal{A}_1{+}\mathcal{A}_2,\, \mathcal{B}_1{+}\mathcal{B}_2)$ is a $\mathfrak{cn}$-graph of the graph $\mathbb{G}_1^*{+}\mathbb{G}_2^*$. In view of Proposition 3 it is sufficient to verify the case when $xy \in \acute{E}$. In this case we have

$$\begin{aligned} s_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1+\mathcal{B}_2}(xy)} &= \min\{s_{\mathcal{A}_1}(x), s_{\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{A}_2}(y)\}} \\ &\le \min\{s_{\mathcal{A}_1\cup\mathcal{A}_2}(x), s_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}} \\ &= \min\{s_{\mathcal{A}_1+\mathcal{A}_2}(x), s_{\mathcal{A}_1+\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1+\mathcal{A}_2}(x),\alpha_{\mathcal{A}_1+\mathcal{A}_2}(y)\}} \end{aligned}$$

$$\begin{aligned} t_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1+\mathcal{B}_2}(xy)} &= \min\{t_{\mathcal{A}_1}(x), t_{\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1}(x),\beta_{\mathcal{A}_2}(y)\}} \\ &\le \min\{t_{\mathcal{A}_1\cup\mathcal{A}_2}(x), t_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}} \\ &= \min\{t_{\mathcal{A}_1+\mathcal{A}_2}(x), t_{\mathcal{A}_1+\mathcal{A}_2}(y)\}e^{i\min\{\beta_{\mathcal{A}_1+\mathcal{A}_2}(x),\beta_{\mathcal{A}_1+\mathcal{A}_2}(y)\}} \end{aligned}$$

$$\begin{aligned} u_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\gamma_{\mathcal{B}_1+\mathcal{B}_2}(xy)} &= \max\{u_{\mathcal{A}_1}(x), u_{\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1}(x),\gamma_{\mathcal{A}_2}(y)\}} \\ &\le \max\{u_{\mathcal{A}_1\cup\mathcal{A}_2}(x), u_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(x),\gamma_{\mathcal{A}_1\cup\mathcal{A}_2}(y)\}} \\ &= \max\{u_{\mathcal{A}_1+\mathcal{A}_2}(x), u_{\mathcal{A}_1+\mathcal{A}_2}(y)\}e^{i\max\{\gamma_{\mathcal{A}_1+\mathcal{A}_2}(x),\gamma_{\mathcal{A}_1+\mathcal{A}_2}(y)\}} \end{aligned}$$

This completes the proof. □

Definition 11. *Let $\mathbb{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathbb{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be two $\mathfrak{cn}$-graphs. A homomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ is a mapping $g : \mathcal{X}_1 \to \mathcal{X}_2$ such that:*

1. $\begin{cases} s_{\mathcal{A}_1}(x_1)e^{i\alpha_{\mathcal{A}_1}(x_1)} \le s_{\mathcal{A}_2}(g(x_1))e^{i\alpha_{\mathcal{A}_2}(g(x_1))} \\ t_{\mathcal{A}_1}(x_1)e^{i\beta_{\mathcal{A}_1}(x_1)} \le t_{\mathcal{A}_2}(g(x_1))e^{i\beta_{\mathcal{A}_2}(g(x_1))} \\ u_{\mathcal{A}_1}(x_1)e^{i\gamma_{\mathcal{A}_1}(x_1)} \le u_{\mathcal{A}_2}(g(x_1))e^{i\gamma_{\mathcal{A}_2}(g(x_1))} \end{cases}$ *for all* $x_1 \in \mathcal{X}_1$,

2. $\begin{cases} s_{\mathcal{B}_1}(x_1y_1)e^{i\alpha_{\mathcal{B}_1}(x_1y_1)} \le s_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\alpha_{\mathcal{B}_2}(g(x_1)g(y_1))} \\ t_{\mathcal{B}_1}(x_1y_1)e^{i\beta_{\mathcal{B}_1}(x_1y_1)} \le t_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\beta_{\mathcal{B}_2}(g(x_1)g(y_1))} \\ u_{\mathcal{B}_1}(x_1y_1)e^{i\gamma_{\mathcal{B}_1}(x_1y_1)} \le u_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\gamma_{\mathcal{B}_2}(g(x_1)g(y_1))} \end{cases}$ *for all* $x_1y_1 \in \mathrm{E}_1$.

   A bijective homomorphism with the property

3. $\begin{cases} s_{\mathcal{A}_1}(x_1)e^{i\alpha_{\mathcal{A}_1}(x_1)} = s_{\mathcal{A}_2}(g(x_1))e^{i\alpha_{\mathcal{A}_2}(g(x_1))} \\ t_{\mathcal{A}_1}(x_1)e^{i\beta_{\mathcal{A}_1}(x_1)} = t_{\mathcal{A}_2}(g(x_1))e^{i\beta_{\mathcal{A}_2}(g(x_1))} \\ u_{\mathcal{A}_1}(x_1)e^{i\gamma_{\mathcal{A}_1}(x_1)} = u_{\mathcal{A}_2}(g(x_1))e^{i\gamma_{\mathcal{A}_2}(g(x_1))} \end{cases}$ *for all* $x_1 \in \mathcal{X}_1$,

   is called a weak isomorphism. A bijective homomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ such that:

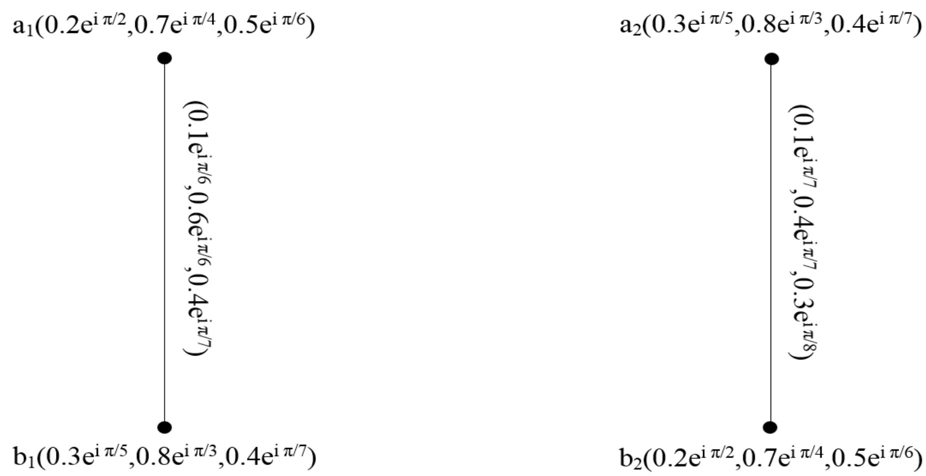

Fig. 11: $\mathfrak{cn}$-graphs $\mathbb{G}_1$ and $\mathbb{G}_2$

4. $\begin{cases} s_{\mathcal{B}_1}(x_1y_1)e^{i\alpha_{\mathcal{B}_1}(x_1y_1)} = s_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\alpha_{\mathcal{B}_2}(g(x_1)g(y_1))} \\ t_{\mathcal{B}_1}(x_1y_1)e^{i\beta_{\mathcal{B}_1}(x_1y_1)} = t_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\beta_{\mathcal{B}_2}(g(x_1)g(y_1))} \\ u_{\mathcal{B}_1}(x_1y_1)e^{i\gamma_{\mathcal{B}_1}(x_1y_1)} = u_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\gamma_{\mathcal{B}_2}(g(x_1)g(y_1))} \end{cases}$ *for all* $x_1y_1 \in \mathrm{E}_1$,

is called a weak co-isomorphism. A bijective mapping $g : \mathbb{G}_1 \to \mathbb{G}_2$ *satisfying* 3. *and* 4. *is called an isomorphism.*

Example 6. *Consider two* $\mathfrak{cn}$*-graphs, as shown in Fig. 11. Then, it is easy to see that the mapping* $g : \mathcal{X}_1 \to \mathcal{X}_2$ *defined by* $g(a_1) = b_2$ *and* $g(b_1) = a_2$ *is a weak isomorphism.*

4 Energy of complex neutrosophic graph

Gutman [33] in 1978 defined the concept of "graph energy". Anjali and Mathew [34] extended this idea to fuzzy graphs. Later, Praba et al. [35] extended this concept for intuitionistic fuzzy graphs and Thirunavukarasu et al. [36] extended this concept for complex fuzzy graphs. Here, we extended this concept to $\mathfrak{cn}$-graphs.

Definition 12. *Adjacency matrix of a* $\mathfrak{cn}$*-graph is written in the form of two adjacent matrices, one containing the entries as values of amplitude functions* *and the other containing the entries as values of phase functions i.e.,*

$$M(CNG) = (M(s_{ij}, t_{ij}, u_{ij}), M(\alpha_{ij}, \beta_{ij}, \gamma_{ij})).$$

Definition 13. *The eigenvalue of an adjacency matrix of $\mathfrak{cn}$-graph $M(CNG)$ is defined in the form (X, Y), where*
(i) the set of eigenvalues of $M(s_{ij}, t_{ij}, u_{ij}) = X$,
(ii) the set of eigenvalues of $M(\alpha_{ij}, \beta_{ij}, \gamma_{ij}) = Y$.

Definition 14. *The energy of a $\mathfrak{cn}$-graph is defined by*

$$\left(\sum_{(l_i,m_i,n_i)\in X} (|l_i|, |m_i|, |n_i|), \sum_{(e_i,f_i,g_i)\in X} (|e_i|, |f_i|, |g_i|) \right)$$

where $\sum_{(l_i,m_i,n_i)\in X} (|l_i|, |m_i|, |n_i|)$ is the energy of the Amplitude matrix denoted by

$$\mathrm{E}(s_{ij}(CNG), t_{ij}(CNG), u_{ij}(CNG))$$

and $\sum_{(e_i,f_i,g_i)\in X} (|e_i|, |f_i|, |g_i|)$ is the energy of the Phase matrix denoted by

$$\mathrm{E}(\alpha_{ij}(CNG), \beta_{ij}(CNG), \gamma_{ij}(CNG)).$$

Definition 15. *Let CNG be a $\mathfrak{cn}$-graph (without loops) with $|\mathcal{V}| = n$ and $|\mathrm{E}| = m$ and $M(CNG) = (M(s_{ij}, t_{ij}, u_{ij}), M(\alpha_{ij}, \beta_{ij}, \gamma_{ij}))$ be an adjacency matrix of $\mathfrak{cn}$-graph of CNG, then upper bound and lower bound of the energy of the $\mathfrak{cn}$-graph is*

(i) $\sqrt{2 \sum_{1\le i<j\le n} s_{ij}s_{ji} + n(n-1)|M(s_{ij})|^{\frac{2}{n}}} \le \mathrm{E}(s_{ij}(CNG))$
$\le \sqrt{2n \sum_{1\le i<j\le n} s_{ij}s_{ji}}$
where $|M(s_{ij})|$ is the determinant of $M(s_{ij})$.

(ii) $\sqrt{2 \sum_{1\le i<j\le n} t_{ij}t_{ji} + n(n-1)|M(t_{ij})|^{\frac{2}{n}}} \le \mathrm{E}(t_{ij}(CNG))$
$\le \sqrt{2n \sum_{1\le i<j\le n} t_{ij}t_{ji}}$
where $|M(t_{ij})|$ is the determinant of $M(t_{ij})$.

(iii) $\sqrt{2 \sum_{1\le i<j\le n} u_{ij}u_{ji} + n(n-1)|M(u_{ij})|^{\frac{2}{n}}} \le \mathrm{E}(u_{ij}(CNG))$
$\le \sqrt{2n \sum_{1\le i<j\le n} u_{ij}u_{ji}}$
where $|M(u_{ij})|$ is the determinant of $M(u_{ij})$.

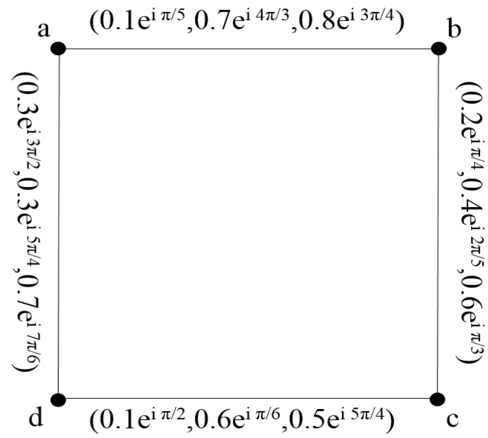

Fig. 12: $\mathfrak{cn}$-graph $\mathbb{G}$

(iv) $\sqrt{2\sum_{1\leq i<j\leq n}\alpha_{ij}\alpha_{ji}+n(n-1)|M(\alpha_{ij})|^{\frac{2}{n}}}\leq \mathrm{E}(\alpha_{ij}(CNG))$
$\leq \sqrt{2n\sum_{1\leq i<j\leq n}\alpha_{ij}\alpha_{ji}}$
where $|M(\alpha_{ij})|$ *is the determinant of* $M(\alpha_{ij})$.
(v) $\sqrt{2\sum_{1\leq i<j\leq n}\beta_{ij}\beta_{ji}+n(n-1)|M(\beta_{ij})|^{\frac{2}{n}}}\leq \mathrm{E}(\beta_{ij}(CNG))$
$\leq \sqrt{2n\sum_{1\leq i<j\leq n}\beta_{ij}\beta_{ji}}$
where $|M(\beta_{ij})|$ *is the determinant of* $M(\beta_{ij})$.
(vi) $\sqrt{2\sum_{1\leq i<j\leq n}\gamma_{ij}\gamma_{ji}+n(n-1)|M(\gamma_{ij})|^{\frac{2}{n}}}\leq \mathrm{E}(\gamma_{ij}(CNG))$
$\leq \sqrt{2n\sum_{1\leq i<j\leq n}\gamma_{ij}\gamma_{ji}}$
where $|M(\gamma_{ij})|$ *is the determinant of* $M(\gamma_{ij})$.

A numerical example to find out the energy of a $\mathfrak{cn}$-graph is given below.

Example 7. *Consider a graph* $\mathbb{G}^* = (\mathcal{V},\mathrm{E})$ *such that* $\mathcal{X}=\{a,b,c,d\}$, $\mathrm{E}=\{ab,bc,cd,da\}$. *Let* $\mathcal{B}$ *be a* $\mathfrak{cn}$*-subset of* $\mathrm{E}\subseteq\mathcal{X}\times\mathcal{X}$*, as given:*

$$\mathcal{B}=\begin{pmatrix} \frac{0.1e^{i\ \pi/5},0.7e^{i\ 4\pi/3},0.8e^{i\ 3\pi/4}}{ab}, \frac{0.2e^{i\ \pi/4},0.4e^{i\ 2\pi/5},0.6e^{i\ \pi/3}}{bc}, \\ \frac{0.1e^{i\ \pi/2},0.6e^{i\ \pi/6},0.5e^{i\ 5\pi/4}}{cd}, \frac{0.3e^{i\ 3\pi/2},0.3e^{i\ 5\pi/4},0.7e^{i\ 7\pi/6}}{da} \end{pmatrix}.$$

The adjacency matrix of the graph is

$$M(CNG) = (M(s_{ij}, t_{ij}, u_{ij}), M(\alpha_{ij}, \beta_{ij}, \gamma_{ij})),$$

where

$$M(s_{ij}, t_{ij}, u_{ij}) = \begin{pmatrix} 0 & (0.1, 0.7, 0.8) & 0 & (0.3, 0.3, 0.7) \\ (0.1, 0.7, 0.8) & 0 & (0.2, 0.4, 0.6) & 0 \\ 0 & (0.2, 0.4, 0.6) & 0 & (0.1, 0.6, 0.5) \\ (0.3, 0.3, 0.7) & 0 & (0.1, 0.6, 0.5) & 0 \end{pmatrix}$$

and

$$M(\alpha_{ij}, \beta_{ij}, \gamma_{ij}) = \begin{pmatrix} 0 & (\pi/5, 4\pi/3, 3\pi/4) & 0 & (3\pi/2, 5\pi/4, 7\pi/6) \\ (\pi/5, 4\pi/3, 3\pi/4) & 0 & (\pi/4, 2\pi/5, \pi/3) & 0 \\ 0 & (\pi/4, 2\pi/5, \pi/3) & 0 & (\pi/2, \pi/6, 5\pi/4) \\ (3\pi/2, 5\pi/4, 7\pi/6) & 0 & (\pi/2, \pi/6, 5\pi/4) & 0 \end{pmatrix}$$

$$\textit{Eigenvalues of } M(s_{ij}) = \begin{pmatrix} 0 & 0.1 & 0 & 0.3 \\ 0.1 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.1 \\ 0.3 & 0 & 0.1 & 0 \end{pmatrix}$$

are $\{-0.3618, -0.1382, 0.1382, 0.3618\}$

$$\textit{Eigenvalues of } M(t_{ij}) = \begin{pmatrix} 0 & 0.7 & 0 & 0.3 \\ 0.7 & 0 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.3 & 0 & 0.6 & 0 \end{pmatrix}$$

are $\{-1.0055, -0.2984, 0.2984, 1.0055\}$

$$\textit{Eigenvalues of } M(u_{ij}) = \begin{pmatrix} 0 & 0.8 & 0 & 0.7 \\ 0.8 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0.5 \\ 0.7 & 0 & 0.5 & 0 \end{pmatrix}$$

are $\{-1.3190, -0.0155, 0.0155, 1.3190\}$

$$\textit{Eigenvalues of } M(\alpha_{ij}) = \begin{pmatrix} 0 & \pi/5 & 0 & 3\pi/2 \\ \pi/5 & 0 & \pi/4 & 0 \\ 0 & \pi/4 & 0 & \pi/2 \\ 3\pi/2 & 0 & \pi/2 & 0 \end{pmatrix}$$

are $\{-1.6035\pi, -0.1726\pi, 0.1726\pi, 1.6035\pi\}$

$$\textit{Eigenvalues of } M(\beta_{ij}) = \begin{pmatrix} 0 & 4\pi/3 & 0 & 5\pi/4 \\ 4\pi/3 & 0 & 2\pi/5 & 0 \\ 0 & 2\pi/5 & 0 & \pi/6 \\ 5\pi/4 & 0 & \pi/6 & 0 \end{pmatrix}$$

are $\{-2.4137\pi, -0.7243\pi, 0.7243\pi, 2.4137\pi\}$

$$\textit{Eigenvalues of } M(\gamma_{ij}) = \begin{pmatrix} 0 & 3\pi/4 & 0 & 7\pi/6 \\ 3\pi/4 & 0 & \pi/3 & 0 \\ 0 & \pi/3 & 0 & 5\pi/4 \\ 7\pi/6 & 0 & 5\pi/4 & 0 \end{pmatrix}$$

are $\{-1.8742\pi, -0.26\pi, 0.26\pi, 1.8742\pi\}$

Energy of Amplitude matrix $= (1, 2.6078, 2.669)$.

Energy of Phase matrix $= (3.5522\pi, 6.2761\pi, 4.2684\pi)$.

5 Conclusion

A graph is a mathematical object containing points (vertices) and connections (edges), and is a convenient way of interpreting information involving the relationship between different objects. We have defined $\mathfrak{cn}$-graphs and some operations, including union of $\mathfrak{cn}$-graphs, Cartesian product of $\mathfrak{cn}$-graphs, join of $\mathfrak{cn}$-graphs and composition of $\mathfrak{cn}$-graphs. We have aim to extend our work to: (1) Complex fuzzy soft graphs, (2) Complex intuitionistic fuzzy graphs.

References

[1] L.A. Zadeh. Fuzzy sets. *Inform. Control*, 8(3):338–353, 1965. doi: 10.1016/S0019-9958(65)90241-X.

[2] H.J. Zimmermann. *Fuzzy set theory and its applications,*. Originally published by Kluwer Academic Publishers in 1991, Second, Revised Edition. Springer Science+Business Media, LLC, 1991. ISBN 978-94-015-7951-3, doi: 10.1007/978-94-015-7949-0.

[3] M.J. Mendel. Fuzzy logic systems for engineering: a tutorial. *Proc. IEEE*, 83:345–377, 1995. doi: 10.1109/5.364485.

[4] K Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.*, 20(1):87–96, 1986. doi: 10.1016/S0165-0114(86)80034-3.

[5] F. Smarandache. *A unifying field in logics.* neutrosophy: neutrosophic probability, set and logic. Rehoboth:, American Research Press, 1999. ISBN 1879585766.

[6] F. Smarandache. Neutrosophic set - a generalization of the intuitionistic fuzzy set. *IEEE Int. Conf. Granular Comput.*, 1:38–42, 2006. doi: 10.1109/GRC.2006.1635754.

[7] Y. Guo and H.D. Cheng. A new neutrosophic approach to image segmentation. *Pattern Recognit.*, 42(5):587–595, 2009. doi: 10.1016/j.patcog.2008.10.002.

[8] F.G. Lupianez. On neutrosophic topology. *Kybernetes*, 37(6):797–800, 2018. doi: 10.1108/03684920810876990.

[9] P. Majumdar. Neutrosophic sets and its applications to decision making. *In: Acharjya D., Dehuri S., Sanyal S. (Eds.) Computational Intelligence for Big Data Analysis*, 19:97–115, 2015. doi: 10.1007/978-3-319-16598-1_4.

[10] F. Smarandache and L. Vladareanu. Applications of neutrosophic logic to robotics - an introduction. *IEEE Int. Conf. Granular Comput.*, 1:607–612, 2011. doi: 10.1109/GRC.2011.6122666.

[11] F. Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Prog. Phys.*, 4:143–146, 2013. http://www.ptep-online.com/2013/PP-35-17.PDF.

[12] M. Gulistan, S. Nawaz, and H. Hassan. Neutrosophic triplet non-associative semihypergroups with application. *Symmetry*, 10(11):ID:613, 2018. doi: 10.3390/sym10110613.

[13] M. Gulistan, N. Yaqoob, Z. Rashid, F. Smarandache, and H.A. Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):ID:203, 2018. doi: 10.3390/sym10060203.

[14] R.M. Hashim, M. Gulistan, and F. Smarandache. Applications of neutrosophic bipolar fuzzy sets in hope foundation for planning to build a children hospital with different types of similarity measures. *Symmetry*, 10:ID:331, 2018. doi: 10.3390/sym10080331.

[15] M. Khan, M. Gulistan, N. Yaqoob, and M. Shabir. Neutrosophic cubic (α, β)-ideals in semigroups with application. *J. Intell. Fuzzy Syst.*, 35(2):2469–2483, 2018. doi: 10.3233/JIFS-18112.

[16] J. Zhan, M. Khan, M. Gulistan, and A. Ali. Applications of neutrosophic cubic sets in multi-criteria decision-making. *Int. J. Uncertain. Quan.*, 7(5):377–394, 2017. doi: 10.1615/Int.J.UncertaintyQuantification.2017020446.

[17] J.J. Buckley. Fuzzy complex numbers. *Fuzzy Sets Syst.*, 33(3):333–345, 1989. doi: 10.1016/0165-0114(89)90122-X.

[18] H.T. Nguyen, A. Kandel, and V. Kreinovich. Complex fuzzy sets: towards new foundations. *IEEE Trans Fuzzy Syst.*, 2:1045–1048, 2000. doi: 10.1109/FUZZY.2000.839195.

[19] D. Ramot, R. Milo, M. Friedman, and A. Kandel. Complex fuzzy sets. *IEEE Trans Fuzzy Syst.*, 10(2):171–186, 2002. doi: 10.1109/91.995119.

[20] D. Ramot, M. Friedman, G. Langholz, and A. Kandel. Complex fuzzy logic. *IEEE Trans Fuzzy Syst.*, 11(4):450–461, 2003. doi: 10.1109/TFUZZ.2003.814832.

[21] A.M. Alkouri and A.R. Salleh. Complex Atanassov's intuitionistic fuzzy relation. *Abstr. Appl. Anal.*, Article ID 287382:1–18, 2013. doi: 10.1155/2013/287382.

[22] M. Ali and F. Smarandache. Complex neutrosophic set. *Neural Comput. Appl.*, 28(7):1817–1834, 2017. doi: 10.1007/s00521-015-2154-y.

[23] A. Kaufmann. *Introduction a la Thiorie des Sous-Ensemble Flous.* Neutrosophic graphs: A new dimension to graph theory 1. Masson et Cie, Paris, 1973. ISBN 2225366305.

[24] A. Rosenfeld. *Fuzzy graphs.* In: Zadeh L.A., Fu K.S., Shimura M. (Eds.) Fuzzy Sets and Their Applications. Academic Press, New York, 1975. http://www.oalib.com/references/13875421.

[25] J.N. Mordeson and C.S. Peng. Operations on fuzzy graphs. *Inf. Sci.*, 79(3-4):159–170, 1994. doi: 10.1016/0020-0255(94)90116-3.

[26] R. Dhavaseelan, R. Vikramaprasad, and V. Krishnaraj. Certain types of neutrosophic graphs. *Int. J. Math. Sci. Appl.*, 5(2):333–339, 2015. http://ijmsa.yolasite.com/resources/14.

[27] M. Akram and S. Shahzadi. Neutrosophic soft graphs with application. *J. Intell. Fuzzy Syst.*, 32(1):841–858, 2017. doi: 10.3233/JIFS-16090, https://www.mdpi.com/2075-1680/7/1/14/pdf.

[28] M. Akram. Single-valued neutrosophic planar graphs. *Int. J. Algebra Stat.*, 5(2):157–167, 2016. doi: 10.20454/ijas.2016.1207.

[29] M. Akram and G. Shahzadi. Operations on single-valued neutrosophic graphs. *J. Uncertain Syst.*, 11(2):1–26, 2017. http://fs.unm.edu/neut/OperationsOnSingleValued.pdf.

[30] S. Broumi, A. Bakali, M. Talea, and F. Smarandache. Isolated single valued neutrosophic graphs. *Neutrosophic Sets Syst.*, 11:74–78, 2016. http://fs.unm.edu/NSS/IsolatedSingleValuedNeutrosophicGraphs.pdf.

[31] S. Ashraf, S. Naz, H. Rashmanlou, and H.A. Malik. Regularity of graphs in single-valued neutrosophic environment. *J. Intell. Fuzzy Syst.*, 33(1):529–542, 2017. doi: 10.3233/JIFS-161960.

[32] M. Akram and S. Naz. Energy of pythagorean fuzzy graphs with applications. *Mathematics*, 6(8):136–162, 2018. doi: 10.3390/math6080136.

[33] I. Gutman. The energy of a graph. *Ber. Math. Statist. Sekt. Forschungszentram Graz.*, 103:1–22, 1978. https://www.scirp.org/(S(351jmbntvnsjt1aadkposzje))/reference/ReferencesPapers.aspx?ReferenceID=1513425.

[34] N. Anjali and S. Mathew. Energy of a fuzzy graph. *Ann. Fuzzy Math. Inform.*, 6(3):455–465, 2013. http://www.afmi.or.kr/articles_in_

[35] B. Praba, V.M. Chandrasekaran, and G. Deepa. Energy of an intuitionistic fuzzy graph. *Ital. J. Pure Appl. Math.*, 32:431–444, 2014. http://ijpam.uniud.it/online_issue/201432/40-PrabaChandrasekaranDeepa.pdf.

[36] P. Thirunavukarasu, R. Suresh, and K.K. Viswanathan. Energy of a complex fuzzy graph. *Journal of Mathematical Sciences and Engineering Applications*, 10(1):243–248, 2016. http://www.ascent-journals.com/IJMSEA/Vol10No1/20-suresh.pdf.