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Numerical solution of mixed Volterra-Fredholm integral equations using iterative method via two-variables Bernstein polynomials

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Abstract

This paper is concerned with the numerical solution of mixed Volterra-Fredholm integral equations, based on iterative method and two variable Bernstein polynomials. In the main result, this method has several benefits in proposing an efficient and simple scheme with good degree of accuracy. Our second main result is to prove the convergence of the method, and to derive an upper bound under assumptions. Numerical experiments are performed for the approximation of the solution of two examples to demonstrate the accuracy and integrity of the method.

Keywords: Two-variable Bernstein polynomials; Mixed Volterra-Fredholm integral equations; Iterative method.

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1 Introduction

The objective of this study is to propose an iterative approach for numerical solutions of linear mixed Volterra-Fredholm integral equations as:

$$u(x,y) = u_0(x,y) + \int_0^x \int_0^1 \xi(x,s,y,t)u(s,t)dsdt, \tag{1}$$

where, $u(x,y)$ is an unknown function, $u_0(x,y)$ and $\xi(x,s,y,t)$ are analytical functions on unit square $[0,1]\times[0,1]$ and $[0,1]\times[0,1]\times[0,1]\times[0,1]$, respectively. The existence and uniqueness of the solution for Eq. (1) are reviewed in [1].

In the recent scientific literature many authors attempted to extend numerical method to approximate the solution of Eq.(1) [2–4]. In this paper, we introduce an efficient and convenient numerical scheme based upon iterated method and two-dimensional Bernstein polynomials (2D-BPs). The advantage of the method over other existing methods is its simplicity of performance.

Bernstein polynomials are very useful in many field of mathematics especially numerical analysis. These kind of polynomials have been applied in finding the solution of types of differential equations, integral equations and approximation theory [5–7].

The sections of this paper are listed as follows: In Section 2, some general concepts concerning 2D-BPs and some elementary concept related to this method are reported. In Section 3, the method is applied to solve mixed Volterra-Fredholm integral equations. Section 4 contains error analysis results and assumptions of the scheme. In Section 5, proposed method is applied for two examples to illustrate the accuracy and implementation of the scheme. Finally, Section 6 presents a brief conclusion of this paper.

2 Preliminaries

The B-functions of mn-th degree are defined on the interval $[0,1]\times[0,1]$ as [3, 5–7]

$$B_{(i,m)(j,n)}(x,y) = \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} x^i(1-x)^{m-i}y^j(1-y)^{n-j},$$

for $i = 0,1,...,m$, $j = 0,1,...,n$, and m and n are positive integers.

The set of 2D-BPs can be defined as a $(m+1)(n+1)$-vector $\Upsilon(x,y)$:

$$\Upsilon(x,y) = \big[B_{(0,m)(0,n)}(x,y), ..., B_{(0,m)(n,n)}(x,y), ...,$$

$$B_{(m,m)(0,n)}(x,y), ..., B_{(m,m)(n,n)}(x,y)\big]^T,$$

where, $(x,y) \in [0,1] \times [0,1]$.

Suppose f is a function of two real variables on the unit square $x \in [0,1]$ and $y \in [0,1]$, then the Bernstein polynomial of two variables, of degree (n,m), corresponding to the function f, can be obtained as following form:

$$(B_{n,m}f)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\frac{i}{m}, \frac{j}{n}\right) B_{(i,m)(j,n)}(x,y) = F^T\Upsilon(x,y).$$

It is obvious that

$$B_{(i,m)(j,n)}(x,y) = B_{(i,m)}(x)B_{(j,n)}(y),$$

where $B_{(i,m)}(x)$ is one-dimensional Bernstein polynomials of m-th degree are defined by:

$$B_{(i,m)}(x) = \left(\begin{array}{c} m \\ i \end{array}\right) x^i(1-x)^{m-i}.$$

With considering following definition

$$C(m,n,i,j,k,t) := \left(\begin{array}{c} m \\ i \end{array}\right)\left(\begin{array}{c} n \\ j \end{array}\right)\left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} n \\ t \end{array}\right),$$

the following equality holds

$$\int_0^1\int_0^1 B_{(i,m)(j,n)}(x,y)B_{(k,m)(t,n)}(x,y)dxdy =$$

$$\frac{C(m,n,i,j,k,t)}{(2n+1)(2m+1)\left(\begin{array}{c} 2n \\ j+t \end{array}\right)\left(\begin{array}{c} 2m \\ i+k \end{array}\right)}.$$

Also,

$$\int_0^x\int_0^1 B_{(i,m)(j,n)}(s,y)B_{(k,m)(t,n)}(s,y)dsdy$$

$$= C(m,n,i,j,k,t)\int_0^x\int_0^1 s^i(1-s)^{m-i}y^j(1-y)^{n-j}s^k(1-s)^{m-k}y^t(1-y)^{n-t}dsdy$$

$$= C(m,n,i,j,k,t)\int_0^x\int_0^1 s^{i+k}(1-s)^{2m-i-k}y^{j+t}(1-y)^{2n-j-t}dsdy$$

$$= \frac{C(m,n,i,j,k,t)}{(2m+1)\begin{pmatrix} 2m \\ i+k \end{pmatrix}}\int_0^x \sum_{\alpha=0}^{2n-j-t}(-1)^\alpha \begin{pmatrix} 2n-j-t \\ \alpha \end{pmatrix} y^{j+t+\alpha}dy$$

$$= \frac{C(m,n,i,j,k,t)\sum_{\alpha=0}^{2n-j-t}(-1)^\alpha \begin{pmatrix} 2n-j-t \\ \alpha \end{pmatrix}\frac{x^{j+t+\alpha+1}}{(j+t+\alpha+1)}}{(2m+1)\begin{pmatrix} 2m \\ i+k \end{pmatrix}}. \tag{2}$$

If we define $(m+1)(n+1)\times(m+1)(n+1)$-matrix $M(x)$, such that the components of $M(x)$ are elements of Eq. (2), then

$$\int_0^x\int_0^1 \Upsilon(s,y)\Upsilon^T(s,y)dsdy = M(x).$$

3 Solving mixed type Volterra-Fredholm intgral equations by 2D-BPs

In this section, iterative method and two-variable Bernstein polynomials are used to solve mixed Volterra-Fredholm integral equations.

For the integral equation (1) iterative process is

$$u_{p+1}(x,y) = u_0(x,y) + \int_0^x\int_0^1 \xi(x,s,y,t)u_p(s,t)dsdt, \quad p=0,1,... \tag{3}$$

In general, the integrals occurring in the above iterative method cannot found analytically and thus will have to be approximated by suitable method. Hence, the functions $u_0(x,y)$ and $\xi(x,s,y,t)$ can be approximated with respect to 2D-BPs as:

$$u_0(x,y) \simeq \Upsilon^T(x,y)U_0 := (Bu)_0(x,y), \tag{4}$$

$$\xi(x,s,y,t) \simeq \Upsilon^T(x,y)\Xi\Upsilon(s,t) := (B\xi)(x,s,y,t), \tag{5}$$

where, the $(m+1)(n+1)\times 1$-vector U_0 and matrix Ξ are 2D-BPs coefficients of $u_0(x,y)$, and $\xi(x,s,y,t)$, respectively.

Thus, it is introduced a iterative procedure based on Bernstein polynomials. So, following iterative method gives the approximate solution of Eq.(1)

$$(Bu)_{p+1}(x,y) := u_0(x,y) + \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_p(s,t)dsdt, \quad p = 0,1,2,... \tag{6}$$

replacing $p = 0$ and by substituting (4-5) in integral part of the equation (6), we have

$$(Bu)_1(x,y) = u_0(x,y) + \int_0^x \int_0^1 \Upsilon^T(x,y)\Xi\Upsilon(s,t)\Upsilon^T(s,t)U_0 dsdt,$$

using previous relations

$$\int_0^x \int_0^1 \Upsilon^T(x,y)\Xi\Upsilon(s,t)\Upsilon^T(s,t)U_0 dsdt =$$

$$\Upsilon^T(x,y)\Xi \int_0^x \int_0^1 \Upsilon(s,t)\Upsilon^T(s,t)U_0 dsdt$$
$$= \Upsilon^T(x,y)\Xi M(x)U_0.$$

So,

$$(Bu)_1(x,y) = u_0(x,y) + \Upsilon^T(x,y)\Xi M(x)U_0 \simeq \Upsilon^T(x,y)(I + \Xi M(x))U_0.$$

This procedure is continued for $p = 1$

$$(Bu)_2(x,y) := u_0(x,y) + \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_1(s,t)dsdt,$$
$$= u_0(x,y) + \Upsilon^T(x,y)\Xi M(x)U_0 +$$
$$\Upsilon^T(x,y)\Xi \int_0^x \int_0^1 \Upsilon(s,t)\Upsilon^T(s,t)\Xi M(s)U_0 dsdt,$$

where the last term leads to calculate integral in the following form

$$\int_0^x \int_0^1 s^{i+k+\beta}(1-s)^{2m-i-k}y^{j+t}(1-y)^{2n-j-t}dsdy$$

$$= \frac{\sum_{\alpha=0}^{2n-j-t}(-1)^{\alpha}\begin{pmatrix} 2n-j-t \\ \alpha \end{pmatrix}\frac{x^{j+t+\alpha+1}}{(j+t+\alpha+1)}}{(2m+\beta+1)\begin{pmatrix} 2m+\beta \\ i+k+\beta \end{pmatrix}},$$

and this process to be continued.

Theorem 1. *Suppose that* $\xi : [0,1] \times [0,1] \times [0,1] \times [0,1] \to R$ *and* $u_0(x,y) : [0,1] \times [0,1] \to R$ *are continuous functions such that*

$$S := \sup_{0 \le x,s,y,t \le 1} |\xi(x,s,y,t)| < 1 \tag{7}$$

then there is a unique continuous function $u(x,y) : [0,1] \times [0,1] \to R$ *that satisfied Eq.(1) and it can be achieved by the following successive approximations method:*

$$u_p(x,y) := u_0(x,y) + \int_0^x \int_0^1 \xi(x,s,y,t) u_{p-1}(s,t) dsdt, \quad p = 1,2,...$$

Proof: The integral equation may be written as a fixed point equation $\Theta u = u$ where the map Θ and Γ are defined as follow:

$$\Theta u := u_0(x,y) + \int_0^x \int_0^1 \xi(x,s,y,t) u(s,t) dsdt,$$

$$\Gamma u := \int_0^x \int_0^1 \xi(x,s,y,t) u(s,t) dsdt.$$

It suffices to show that Θ is a contraction map on the normed space $X = C([0,1] \times [0,1])$ with the norm $\| \,.\, \|_\infty$.
The operator Θ, defined on the space $C([0,1] \times [0,1])$ of continuous functions on $[0,1] \times [0,1]$ endowed with max norm and with values in the space $C([0,1] \times [0,1])$ is bounded. The operator is in fact compact.
In addition, Θ is a contraction since, for any $u_1, u_2 \in C([0,1] \times [0,1])$, we have

$$\| \Theta u_1 - \Theta u_2 \|_\infty = \sup_{0 \le x,y \le 1} \left| \int_0^x \int_0^1 \xi(x,s,y,t)(u_1(x,y) - u_2(x,y)) dsdt \right|$$

$$\le \sup_{0 \le x,y \le 1} \int_0^x \int_0^1 |\xi(x,s,y,t)| |(u_1(x,y) - u_2(x,y))| dsdt$$

$$\le \|u_1(x,y) - u_2(x,y)\|_\infty \sup_{0 \le x,y,s,t \le 1} \int_0^x \int_0^1 |\xi(x,s,y,t)| dsdt$$

$$\le S \|u_1(x,y) - u_2(x,y)\|_\infty.$$

The result can be achieved by the contraction mapping theorem.

Furthermore, the sequence of successive approximations, $u_p(x,y)$ converges to the solution $u(x,y)$. Suppose

$$e_p(x,y) := u_p(x,y) - u(x,y),$$

where, $u(x,y)$ is the exact solution of Eq. (1). Moreover,

$$u = u_0 + \Gamma u.$$

By subtracting (3) from (1)

$$e_{p+1} = \Gamma e_p,$$

then

$$\|e_{p+1}\|_\infty = \|\Gamma e_p\|_\infty \leq S\|e_p\|_\infty, \tag{8}$$

to obtain the error bound

$$u_{p+1} - u_p = u_{p+1} - u + u - u_p = e_{p+1} - e_p,$$

so

$$e_p = e_{p+1} - (u_{p+1} - u_p),$$

as result

$$\|e_p\|_\infty \leq \|e_{p+1}\|_\infty + \|u_{p+1} - u_p\|_\infty,$$

from (8)

$$\|e_p\|_\infty \leq S\|e_p\|_\infty + \|u_{p+1} - u_p\|_\infty,$$

as a result

$$\|e_p\|_\infty \leq \frac{\|u_{p+1} - u_p\|_\infty}{1-S}. \tag{9}$$

Inequality (9) shows a computable bound on the error, and the inequality holds when $S < 1$. This last condition guarantee that the sequence u_p converges to u and if $S > 1$, then inequality (9) is invalid and the sequence u_p may not converge.
Also, the following error bounds hold and show the speed of convergence:

$$\| e_p \|_\infty \leq \frac{S^p}{1-S}\|u_1 - u_0\|_\infty,$$

$$\| e_p \|_\infty \leq \frac{S^{p+1}}{1-S}S_0, \tag{10}$$

where, $S_0 = \sup_{0\leq x,y\leq 1} \| u_0(x,y) \|$.

4 Error analysis

Let us denote by $[0,1]^r$ the unit cube in $\mathbb{R}^r$, and $m_1, m_2, ..., m_r$ natural numbers.

Theorem 2. *([8, 9]) Let $\|X\| = \left(\sum_{i=1}^{r}(x_i)^2\right)^{1/2}$, where $X \in [0,1]^r$ and if $u(X)$ is continuous in $[0,1]^r$, and*

$$w(u;\delta) = \sup_{\|X_1 - X_2\| \leq \delta} \|u(X_1) - u(X_2)\| \quad X_1, X_2 \in [0,1]^r$$

then

$$\|B_{m_1,m_2,...,m_r}u(X) - u(X)\| \leq \frac{5}{4} w\left(u; m^{-1/2}\right) \tag{11}$$

where $m = \sum_{i=1}^{r} m_i$ [10].

Theorem 3. *([11]) If $u : [0,1]^r \to R$ is a continuous function satisfying the Lipschitz condition*

$$\|u(X) - u(Y)\| \leq L\|X - Y\|,$$

on $[0,1]^r$, then the inequality

$$\|(B_{m_1,m_2,...,m_r}u)(X) - u(X)\| < \frac{L}{2}\left(\sum_{i=1}^{r} \frac{1}{m_i}\right)^2, \tag{12}$$

hold.

For more details on error bound see [3].

Theorem 4. *Let $u(x,y)$ be a solution of equation (1) and $(Bu)_p(x,y)$ achieve from equation (6), and*
1. $\|u_i(x,y)\| < S_i$, for $0 \leq i \leq p-1$, where $S_i = \sup_{0 \leq x,y \leq 1} \|u_i(x,y)\|$.
2. $\|(B\xi)(x,s,y,t)\| \leq \tilde{S} < 1$, for $x,y,s,t \in [0,1]$.
3. $m_1 = m_2 = ... = m_r = m$.
4. $G := Max\{S_0, S_1, ..., S_{p-1}\}$.
Then

$$sup_{0 \leq x,y \leq 1}(\|u(x,y) - (Bu)_p(x,y)\|) \leq$$

$$G\left(\frac{S^{p+1}}{1-S}\right) + \frac{5}{4}G\left(\frac{1-\tilde{S}^p}{1-\tilde{S}}\right) w\left(\xi; (4m)^{-1/2}\right) + \frac{5}{4}\tilde{S}^p w\left(u_0; (2m)^{-1/2}\right).$$

Proof: From Eq.(3) and Eq.(6) we have

$$\|u_i(x,y)-(Bu)_i(x,y)\| \leq$$

$$\left\|\int_0^x\int_0^1 \xi(x,s,y,t)u_{i-1}(s,t)dsdt-\int_0^x\int_0^1(B\xi)(x,s,y,t)(Bu)_{i-1}(s,t)dsdt\right\|,$$

for $1 \leq i \leq p$.

At first, for $i=1$

$$\|u_1(x,y)-(Bu)_1(x,y)\|$$

$$\leq \left\|\int_0^x\int_0^1 \xi(x,s,y,t)u_0(s,t)dsdt-\int_0^x\int_0^1(B\xi)(x,s,y,t)(Bu)_0(s,t)dsdt\right\|$$

$$\leq \left\|\int_0^x\int_0^1 \xi(x,s,y,t)u_0(s,t)-(B\xi)(x,s,y,t)u_0(s,t)dsdt\right\|+$$

$$\left\|\int_0^x\int_0^1(B\xi)(x,s,y,t)u_0(s,t)-(B\xi)(x,s,y,t)(Bu)_0(s,t)dsdt\right\|$$

$$\leq \int_0^x\int_0^1 \|\xi(x,s,y,t)u_0(s,t)-(B\xi)(x,s,y,t)u_0(s,t)\|\,dsdt+$$

$$\int_0^x\int_0^1 \|(B\xi)(x,s,y,t)u_0(s,t)-(B\xi)(x,s,y,t)(Bu)_0(s,t)\|\,dsdt$$

$$\leq \int_0^x\int_0^1 \|\xi(x,s,y,t)-(B\xi)(x,s,y,t)\|\,\|u_0(s,t)\|\,dsdt+$$

$$\int_0^x\int_0^1 \|(B\xi)(x,s,y,t)\|\|u_0(s,t)-(Bu)_0(s,t)\|dsdt$$

$$\leq \|\xi(x,s,y,t)-(B\xi)(x,s,y,t)\|\|u_0(x,y)\|+$$

$$\|(B\xi)(x,s,y,t)\|\|u_0(x,y)-(Bu)_0(x,y)\|.$$

For $i=2$

$$\|u_2(x,y)-(Bu)_2(x,y)\|$$

$$\leq \left\|\int_0^x\int_0^1 \xi(x,s,y,t)u_1(s,t)dsdt-\int_0^x\int_0^1(B\xi)(x,s,y,t)(Bu)_1(s,t)dsdt\right\|$$

$$\leq \int_0^x\int_0^1 \|\xi(x,s,y,t)u_1(s,t)-(B\xi)(x,s,y,t)u_1(s,t)\|\,dsdt+$$

$$\int_0^x \int_0^1 \|(B\xi)(x,s,y,t)u_1(s,t) - (B\xi)(x,s,y,t)(Bu)_1(s,t)\| \, dsdt$$

$$\leq \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_1(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\| \|u_1(x,y) - (Bu)_1(x,y)\|$$

$$\leq \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_1(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\| [\|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_0(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\| \|u_0(x,y) - (Bu)_0(x,y)\|].$$

At last, for $i = p$

$$\|u_p(x,y) - (Bu)_p(x,y)\| \leq$$

$$\left\| \int_0^x \int_0^1 \xi(x,s,y,t)u_{p-1}(s,t)dsdt - \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_{p-1}(s,t)dsdt \right\|$$

$$\leq \int_0^x \int_0^1 \|\xi(x,s,y,t)u_{p-1}(s,t) - (B\xi)(x,s,y,t)u_{p-1}(s,t)\| \, dsdt +$$

$$\int_0^x \int_0^1 \|(B\xi)(x,s,y,t)u_{p-1}(s,t) - (B\xi)(x,s,y,t)(Bu)_{p-1}(s,t)\| \, dsdt$$

$$\leq \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_{p-1}(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\| \|u_{p-1}(x,y) - (Bu)_{p-1}(x,y)\|$$

$$\leq \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \, \|u_{p-1}(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\| \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_{p-2}(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\|^2 \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_{p-3}(x,y)\| + \ldots$$

$$+ \|(B\xi)(x,s,y,t)\|^{p-1} \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \|u_0(x,y)\| +$$

$$\|(B\xi)(x,s,y,t)\|^p \|u_0(x,y) - (Bu)_0(x,y)\|$$

$$\leq \|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\| \Big[S_{p-1} + \tilde{S}S_{p-2} + \tilde{S}^2 S_{p-3} + \ldots$$

$$+ \tilde{S}^{p-1} S_0 \Big] + \tilde{S}^p \|u_0(x,y) - (Bu)_0(x,y)\| ,$$

from Eq.(11), it's achieved that

$$\|u_p(x,y) - (Bu)_p(x,y)\| \leq$$

$$\frac{5}{4}w\left(\xi;(4m)^{-1/2}\right)\left[S_{p-1}+\tilde{S}S_{p-2}+\tilde{S}^2S_{p-3}+...+\tilde{S}^{p-1}S_0\right]+$$
$$\frac{5}{4}\tilde{S}^p w\left(u_0;(2m)^{-1/2}\right).$$

By choosing $G := Max\{S_0, S_1, ..., S_{p-1}\}$, we get

$$\begin{aligned}\|u_p(x,y)-(Bu)_p(x,y)\| \leq & \frac{5}{4}G\left(\frac{1-\tilde{S}^p}{1-\tilde{S}}\right)w\left(\xi;(4m)^{-1/2}\right) \\ & +\frac{5}{4}\tilde{S}^p w\left(u_0;(2m)^{-1/2}\right),\end{aligned} \tag{13}$$

Also, the following inequality holds

$$\|u(x,y)-(Bu)_p(x,y)\| = \|u(x,y)-u_p(x,y)+u_p(x,y)-(Bu)_p(x,y)\|$$
$$\leq \|u(x,y)-u_p(x,y)\|+\|u_p(x,y)-(Bu)_p(x,y)\|,$$

finally from (10) and (13) the desired result is reached

$$\|u(x,y)-(Bu)_p(x,y)\|$$
$$\leq \frac{S^{p+1}}{1-S}S_0+\frac{5}{4}G\left(\frac{1-\tilde{S}^p}{1-\tilde{S}}\right)w\left(\xi;(4m)^{-1/2}\right)+\frac{5}{4}\tilde{S}^p w\left(u_0;(2m)^{-1/2}\right)$$
$$\leq G\left(\frac{S^{p+1}}{1-S}\right)+\frac{5}{4}G\left(\frac{1-\tilde{S}^p}{1-\tilde{S}}\right)w\left(\xi;(4m)^{-1/2}\right)+\frac{5}{4}\tilde{S}^p w\left(u_0;(2m)^{-1/2}\right).$$

Theorem 5. *Let $u(x,y)$ be solution of equation (1) and $(Bu)_p(x,y)$ achieve from equation (6), and*
1. $\|u_i(x,y)\| < S_i$, for $0 \leq i \leq p-1$.
2. $\|(B\xi)(x,s,y,t)\| \leq \tilde{S} < \sqrt{\frac{1}{2}}$, for $x,y,s,t \in [0,1]$.
3. $u_0(x,y)$, $\xi(x,s,y,t)$ are continuous functions satisfying the Lipschitz condition, with Lipschitz constant $L \geq 0$.
4. $m_1 = m_2 = ... = m_r = m$.
5. $G := Max\{S_0, S_1, ..., S_p\}$.
Then

$$sup_{0\leq x,y\leq 1}(\|u(x,y)-(Bu)_p(x,y)\|_2^2)$$
$$\leq 2\frac{2S^{2(p+1)}}{1-2S^2}S_0^2+2\left[\left(\frac{1-(2\tilde{S}^2)^p}{1-2\tilde{S}^2}\right)\frac{128L^2G^2}{m^4}+2^p\tilde{S}^{2p}\left(\frac{4L^2}{m^4}\right)\right].$$

Proof: From Eq.(3) and Eq.(6) for $0 < i \leq p$ we have

$$\|u_i(x,y) - (Bu)_i(x,y)\|_2^2$$

$$\leq \left\| \int_0^x \int_0^1 \xi(x,s,y,t) u_{i-1}(s,t) dsdt \right.$$

$$\left. - \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_{i-1}(s,t) dsdt \right\|_2^2 .$$

At first, for $i = 1$

$$\|u_1(x,y) - (Bu)_1(x,y)\|_2^2$$

$$\leq \left\| \int_0^x \int_0^1 \xi(x,s,y,t) u_0(s,t) dsdt - \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_0(s,t) dsdt \right\|_2^2$$

$$\leq 2[\left\| \int_0^x \int_0^1 \xi(x,s,y,t) u_0(s,t) - (B\xi)(x,s,y,t) u_0(s,t) dsdt \right\|_2^2 +$$

$$\|(B\xi)(x,s,y,t)u_0(s,t) - (B\xi)(x,s,y,t)(Bu)_0(s,t)dsdt\|_2^2]$$

$$\leq 2[\int_0^x \int_0^1 \|\xi(x,s,y,t)u_0(s,t) - (B\xi)(x,s,y,t)u_0(s,t)\|_2^2 \, dsdt +$$

$$\int_0^x \int_0^1 \|(B\xi)(x,s,y,t)u_0(s,t) - (B\xi)(x,s,y,t)(Bu)_0(s,t)\|_2^2 dsdt]$$

$$\leq 2sup_{0\leq x,s,y,t\leq 1} \|(\xi(x,s,y,t) - (B\xi)(x,s,y,t))u_0(x,y)\|_2^2 +$$

$$2sup_{0\leq x,s,y,t\leq 1} \|(B\xi)(x,s,y,t)(u_0(x,y) - (Bu)_0(x,y))\|_2^2$$

$$\leq 2(\frac{L^2}{4}(\frac{4}{m})^4 S_0^2 + \tilde{S}^2 \frac{L^2}{4}(\frac{2}{m})^4) = \frac{128L^2}{m^4} S_0^2 + 2\frac{4L^2\tilde{S}^2}{m^4}.$$

The last term is satisfied by Theorem 3.
Now, for $i = 2$

$$\|u_2(x,y) - (Bu)_2(x,y)\|_2^2$$

$$\leq \left\| \int_0^x \int_0^1 \xi(x,s,y,t) u_1(s,t) dsdt - \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_1(s,t) dsdt \right\|_2^2$$

$$\leq 2[\int_0^x \int_0^1 \|\xi(x,s,y,t)u_1(s,t) - (B\xi)(x,s,y,t)u_1(s,t)\|_2^2 dsdt +$$

$$\int_0^x \int_0^1 \|(B\xi)(x,s,y,t)u_1(s,t) - (B\xi)(x,s,y,t)(Bu)_1(s,t)\|_2^2 dsdt]$$

$$\leq 2sup_{0\leq x,s,y,t\leq 1}\|(\xi(x,s,y,t) - (B\xi)(x,s,y,t))u_1(x,y)\|_2^2 +$$

$$2sup_{0\leq x,s,y,t\leq 1}\|(B\xi)(x,s,y,t)(u_1(x,y) - (Bu)_1(x,y))\|_2^2$$

$$\leq \frac{128L^2S_1^2}{m^4} + 2\tilde{S}^2(\frac{128L^2}{m^4}S_0^2 + 2\frac{4L^2\tilde{S}^2}{m^4}).$$

$$\leq \frac{128L^2}{m^4}(S_1^2 + 2\tilde{S}^2S_0^2) + 2^2\frac{4L^2\tilde{S}^4}{m^4}.$$

For $i = p$

$$\|u_p(x,y) - (Bu)_p(x,y)\|_2^2$$

$$\leq \left\|\int_0^x \int_0^1 \xi(x,s,y,t)u_{p-1}(s,t)dsdt - \int_0^x \int_0^1 (B\xi)(x,s,y,t)(Bu)_{p-1}(s,t)dsdt\right\|_2^2$$

$$\leq 2[\int_0^x \int_0^1 \|\xi(x,s,y,t)u_{p-1}(s,t) - (B\xi)(x,s,y,t)u_{p-1}(s,t)\|_2^2 dsdt +$$

$$\int_0^x \int_0^1 \|(B\xi)(x,s,y,t)u_{p-1}(s,t) - (B\xi)(x,s,y,t)(Bu)_{p-1}(s,t)\|_2^2 dsdt]$$

$$\leq 2sup_{0\leq x,y\leq 1}|u_{p-1}(x,y)|^2 sup_{0\leq x,s,y,t\leq 1}\|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\|_2^2 +$$

$$2sup_{0\leq x,s,y,t\leq 1}|(B\xi)(x,s,y,t)|^2 sup_{0\leq x,y\leq 1}\|u_{p-1}(x,y) - (Bu)_{p-1}(x,y)\|_2^2$$

$$\leq 2sup_{0\leq x,y\leq 1}\|\xi(x,s,y,t) - (B\xi)(x,s,y,t)\|_2^2$$

$$\left[S_{p-1}^2 + 2\tilde{S}^2S_{p-2}^2 + 2^2\tilde{S}^4S_{p-3}^2 + ... + 2^{p-1}\tilde{S}^{2(p-1)}S_0^2\right] +$$

$$2^p\tilde{S}^{2p}sup_{0\leq x,y\leq 1}\|u_0(x,y) - (Bu)_0(x,y)\|_2^2,$$

by choosing $G := Max\{S_0, S_1, \cdots, S_{p-1}\}$, and corresponding to $\tilde{S}^2 < 1/2$ we get

$$\|u_p(x,y) - (Bu)_p(x,y)\|_2^2 \leq (\frac{(2\tilde{S}^2)^p - 1}{2\tilde{S}^2 - 1})\frac{128L^2G^2}{m^4} + 2^p\tilde{S}^{2p}(\frac{4L^2}{m^4}).$$

Also, the following inequality holds

$$\|u(x,y) - (Bu)_p(x,y)\|_2^2 = \|u(x,y) - u_p(x,y) + u_p(x,y) - (Bu)_p(x,y)\|_2^2$$

$$\leq 2\|u(x,y) - u_p(x,y)\|_2^2 + 2\|u_p(x,y) - (Bu)_p(x,y)\|_2^2.$$

Moreover,

$$\|u(x,y)-u_p(x,y)\|_2^2 = \left\|\int_0^x\int_0^1 \xi(x,s,y,t)(u(s,t)-u_{p-1}(s,t))dsdt\right\|_2^2$$

$$\leq \int_0^x\int_0^1 \|\xi(x,s,y,t)\,(u(s,t)-u_{p-1}(s,t))\|_2^2\, dsdt \leq$$

$$\int_0^x\int_0^1 \|\xi(x,s,y,t)\|_2^2\, \|(u(s,t)-u_{p-1}(s,t))\|_2^2\, dsdt$$

$$\leq S^2\,\|u(x,y)-u_{p-1}(x,y)\|_2^2$$

$$\leq S^4\,\|u(x,y)-u_{p-2}(x,y)\|_2^2 \leq \cdots \leq S^{2p}\,\|u(x,y)-u_0(x,y)\|_2^2. \qquad (14)$$

Also,

$$\|u(x,y)-u_0(x,y)\|_2^2 \leq 2S^2\,\|u(x,y)-u_0(x,y)\|_2^2 + 2\,\|u_1(x,y)-u_0(x,y)\|_2^2,$$

so,

$$\|u(x,y)-u_0(x,y)\|_2^2 \leq \frac{2}{1-2S^2}\,\|u_1(x,y)-u_0(x,y)\|_2^2. \qquad (15)$$

It is achieved from Eq.(14) and Eq.(15)

$$\|u(x,y)-u_p(x,y)\|_2^2 \leq \frac{2S^{2(p+1)}}{1-2S^2}S_0^2.$$

Finally

$$sup_{0\leq x,y\leq 1}\left(\|u(x,y)-(Bu)_p(x,y)\|_2^2\right)$$

$$\leq 2\frac{2S^{2(p+1)}}{1-2S^2}S_0^2 + 2\left[\left(\frac{1-(2\tilde{S}^2)^p}{1-2\tilde{S}^2}\right)\frac{128L^2G^2}{m^4} + 2^p\tilde{S}^{2p}\left(\frac{4L^2}{m^4}\right)\right],$$

which concludes the argument.

5 Numerical examples

Here, It is presented two numerical examples to demonstrate performance and applicability of the 2D-BPs methods in this paper. The results are compared with the exact solutions by calculating the following error function:

$$E_p(x,y) = \left|(Bu)_p(x,y) - u(x,y)\right|, \qquad p = 0,1,2,...$$

where, $u(x,y)$ denote the exact solution of the given examples, and $(Bu)_p(x,y)$ be the approximate solution by the presented method.

Example 1. *([3]) Consider the following linear mixed Volterra-Fredholm integral equation,*

$$u(x,y) = u_0(x,y) + \int_0^x \int_0^1 y^2 e^{-t} u(s,t)dtds, \qquad x,y \in [0,1),$$

where, $u_0(x,y) = x^2e^y - \frac{1}{3}x^3y^2$ *with the exact solution* $u(x,y) = x^2e^y$*, for* $0 \leq x,y < 1$.

At first, it is checked that the conditions of Theorem 1 are satisfied. The numerical results are shown in Tables 1, 2 and 3 the second column contains the absolute error $E_p(x,y)$ and relative error is in third column.

Tab. 1: Results of Example 1, with $m = n = 4$ and $p = 2$.

(x,y)	Absolute error	Relative error
$x=y=0.0$	0	0
$x=y=0.1$	7.859×10^{-6}	3.555×10^{-4}
$x=y=0.2$	1.136×10^{-4}	1.163×10^{-3}
$x=y=0.3$	5.144×10^{-4}	2.117×10^{-3}
$x=y=0.4$	1.432×10^{-3}	3.000×10^{-3}
$x=y=0.5$	3.023×10^{-3}	3.667×10^{-3}
$x=y=0.6$	5.286×10^{-3}	4.029×10^{-3}
$x=y=0.7$	7.968×10^{-3}	4.038×10^{-3}
$x=y=0.8$	1.047×10^{-2}	3.677×10^{-3}
$x=y=0.9$	1.177×10^{-2}	2.955×10^{-3}
$x=y=1.0$	1.030×10^{-2}	1.896×10^{-3}

For comparing the current method with other methods, numerical results form [3] and [4] are presented in Table 4. As seen in table, our method has

Tab. 2: Results of Example 1, with $m = n = 4$ and $p = 5$.

(x, y)	Absolute error	Relative error
$x = y = 0.0$	0	0
$x = y = 0.1$	7.732×10^{-6}	3.498×10^{-4}
$x = y = 0.2$	1.113×10^{-4}	1.139×10^{-3}
$x = y = 0.3$	5.010×10^{-4}	2.061×10^{-3}
$x = y = 0.4$	1.386×10^{-3}	2.903×10^{-3}
$x = y = 0.5$	2.903×10^{-3}	3.522×10^{-3}
$x = y = 0.6$	5.027×10^{-3}	3.832×10^{-3}
$x = y = 0.7$	7.477×10^{-3}	3.789×10^{-3}
$x = y = 0.8$	9.632×10^{-3}	3.381×10^{-3}
$x = y = 0.9$	1.043×10^{-2}	2.619×10^{-3}
$x = y = 1.0$	8.323×10^{-3}	1.531×10^{-3}

Tab. 3: Results of Example 1, with $m = n = 6$ and $p = 5$.

(x, y)	Absolute error	Relative error
$x = y = 0.0$	0	0
$x = y = 0.1$	5.118×10^{-6}	2.315×10^{-4}
$x = y = 0.2$	7.369×10^{-5}	7.541×10^{-4}
$x = y = 0.3$	3.316×10^{-4}	1.364×10^{-3}
$x = y = 0.4$	9.176×10^{-4}	1.921×10^{-3}
$x = y = 0.5$	1.922×10^{-3}	2.331×10^{-3}
$x = y = 0.6$	3.329×10^{-3}	2.536×10^{-3}
$x = y = 0.7$	4.953×10^{-3}	2.507×10^{-3}
$x = y = 0.8$	6.386×10^{-3}	2.238×10^{-3}
$x = y = 0.9$	6.931×10^{-3}	1.734×10^{-3}
$x = y = 1.0$	5.551×10^{-3}	1.015×10^{-3}

less error in compare to methods based on operational matrices via Bernstein polynomials and triangular functions. This method can be run with increasing m, n and p until the results settle down to an appropriate accuracy.

Tab. 4: The absolute error function $E_p(x, y)$ of numerical results for Example 1 with three method for $m = n = 4$.

(x, y)	Iterative method via Bernstein polynomials $m = n = 4$ and$p = 8$	Operational matrices via Bernstein polynomials $m = n = 4$	Operational matrices via triangular functions $m = n = 4$
$x = y = 0.0$	0	2.265×10^{-8}	0
$x = y = 0.1$	2.976×10^{-4}	4.276×10^{-5}	1.6190×10^{-2}
$x = y = 0.2$	7.432×10^{-4}	2.001×10^{-4}	1.2381×10^{-2}
$x = y = 0.3$	9.534×10^{-4}	4.718×10^{-4}	1.3741×10^{-2}
$x = y = 0.4$	1.921×10^{-4}	7.694×10^{-4}	2.0272×10^{-2}
$x = y = 0.5$	1.021×10^{-3}	8.787×10^{-4}	6.8027×10^{-3}
$x = y = 0.6$	3.654×10^{-3}	4.215×10^{-4}	2.3771×10^{-2}
$x = y = 0.7$	2.004×10^{-3}	1.182×10^{-3}	2.0739×10^{-2}
$x = y = 0.8$	3.708×10^{-3}	4.755×10^{-3}	2.2988×10^{-2}
$x = y = 0.9$	1.154×10^{-3}	1.139×10^{-3}	3.0518×10^{-2}
$x = y = 1.0$	1.232×10^{-3}	2.253×10^{-3}	–

Example 2. *Consider the following linear mixed Volterra-Fredholm integral equation,*

$$u(x, y) = u_0(x, y) - \int_0^x \int_0^1 \frac{sint}{4} y e^{-3t} u(s, t) dt ds, \qquad x, y \in [0, 1),$$

where, $u_0(x, y) = cos(x)e^{3y} - \frac{1}{4} ysinx + \frac{1}{4} ysinxcos1$ *with the exact solution* $u(x, y) = e^{3y} cosx$, *for* $0 \leq x, y < 1$.

The numerical results are shown in Tables 5, 6, 7, and 8.

6 Conclusion

In this paper, we presented a accurate iterative approach to obtaining numerical solution for mixed Volterra-Fredholm integral equations. In general, the integrals occurring in the iterative method cannot found analytically and thus will have to be approximated by suitable method. It was introduced a iterative procedure based on Bernstein polynomials. The method yields

Tab. 5: Results of Example 2, with $m = n = 4$ and $p = 2$.

(x, y)	Absolute error	Relative error
$x = y = 0.0$	0	0
$x = y = 0.1$	1.060×10^{-4}	8.476×10^{-4}
$x = y = 0.2$	4.241×10^{-4}	2.510×10^{-3}
$x = y = 0.3$	9.508×10^{-4}	4.200×10^{-3}
$x = y = 0.4$	1.678×10^{-3}	5.577×10^{-3}
$x = y = 0.5$	2.594×10^{-3}	6.539×10^{-3}
$x = y = 0.6$	3.683×10^{-3}	7.102×10^{-3}
$x = y = 0.7$	4.924×10^{-3}	7.336×10^{-3}
$x = y = 0.8$	6.294×10^{-3}	7.325×10^{-3}
$x = y = 0.9$	7.765×10^{-3}	7.155×10^{-3}
$x = y = 1.0$	9.308×10^{-3}	6.904×10^{-3}

Tab. 6: TResults of Example 1, with $m = n = 4$ and $p = 5$.

(x, y)	Absolute error	Relative error
$x = y = 0.0$	0	0
$x = y = 0.1$	1.056×10^{-4}	7.864×10^{-5}
$x = y = 0.2$	4.205×10^{-4}	2.355×10^{-4}
$x = y = 0.3$	9.388×10^{-4}	3.995×10^{-4}
$x = y = 0.4$	1.650×10^{-3}	5.396×10^{-4}
$x = y = 0.5$	2.540×10^{-3}	6.460×10^{-4}
$x = y = 0.6$	3.592×10^{-3}	7.195×10^{-4}
$x = y = 0.7$	4.784×10^{-3}	7.660×10^{-4}
$x = y = 0.8$	6.091×10^{-3}	7.932×10^{-4}
$x = y = 0.9$	7.487×10^{-3}	8.095×10^{-4}
$x = y = 1.0$	8.941×10^{-3}	8.240×10^{-4}

Tab. 7: Results of Example 1, with $m = n = 6$ and $p = 5$.

(x, y)	Absolute error	Relative error
$x = y = 0.0$	0	0
$x = y = 0.1$	6.366×10^{-5}	4.703×10^{-5}
$x = y = 0.2$	2.554×10^{-4}	1.408×10^{-4}
$x = y = 0.3$	5.746×10^{-4}	2.389×10^{-4}
$x = y = 0.4$	1.017×10^{-3}	3.227×10^{-4}
$x = y = 0.5$	1.578×10^{-3}	3.864×10^{-4}
$x = y = 0.6$	2.248×10^{-3}	4.303×10^{-4}
$x = y = 0.7$	3.016×10^{-3}	4.582×10^{-4}
$x = y = 0.8$	3.867×10^{-3}	4.745×10^{-4}
$x = y = 0.9$	4.785×10^{-3}	4.843×10^{-4}
$x = y = 1.0$	5.753×10^{-3}	4.930×10^{-4}

Tab. 8: The absolute error function $E_p(x, y)$ of numerical results for Example 2 with three method for $m = n = 4$.

(x, y)	Iterative method via Bernstein polynomials $m = n = 4$ and$p = 8$	Operational matrices via Bernstein polynomials $m = n = 4$	Operational matrices via triangular functions $m = n = 4$
$x = y = 0.0$	0	3.768×10^{-8}	0
$x = y = 0.1$	9.655×10^{-5}	5.563×10^{-4}	5.432×10^{-3}
$x = y = 0.2$	4.124×10^{-4}	5.946×10^{-4}	2.245×10^{-3}
$x = y = 0.3$	8.874×10^{-4}	8.083×10^{-4}	1.789×10^{-2}
$x = y = 0.4$	1.921×10^{-4}	7.098×10^{-3}	1.043×10^{-2}
$x = y = 0.5$	2.021×10^{-3}	8.891×10^{-3}	5.297×10^{-3}
$x = y = 0.6$	2.934×10^{-3}	3.198×10^{-3}	3.390×10^{-2}
$x = y = 0.7$	3.984×10^{-3}	2.679×10^{-3}	3.287×10^{-2}
$x = y = 0.8$	6.654×10^{-3}	5.045×10^{-3}	2.057×10^{-2}
$x = y = 0.9$	6.132×10^{-3}	4.487×10^{-3}	1.894×10^{-2}
$x = y = 1.0$	8.109×10^{-3}	7.097×10^{-3}	1.721×10^{-2}

accurate results and it is easy to performance. The method is so simple without any complexity and has less error than the methods proposed in [4] ans [1]. Furthermore, only a small number of iterations and bases are needed to achieve a satisfactory solution. We applied the proposed method on two test examples and compared the results with their exact solution in order to demonstrate the validity and procedure of the method. The numerical results represent that the accuracy of the obtained solutions is good and support the claims.

References

[1] B.G. Pachpatte. On mixed Volterra-Fredholm type integral equations. *Indian journal pure and applied mathematics*, 17(4):488–496, 1986. http://www.scirp.org/(S(czeh2tfqyw2orz553k1w0r45))/reference/ReferencesPapers.aspx?ReferenceID=1984980.

[2] H. Brunner. On the numerical solution of nonlinear Volterra-Fredholm integral equations by collocation methods. *SIAM j. Numer. Anal.*, 27(4):987–1000, 1990. doi: 10.1137/0727057.

[3] F. Hosseini Shekarabi, K. Maleknejad, and R. Ezzati. Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations. *Afrika Mathematika*, 26(8):1237–1251, 2014. doi: 10.1007/s13370-014-0283-6.

[4] K. Maleknejad and Z. Jafari-Behbahani. Applications of two-dimensional triangular functions for solving nonlinear class of mixed Volterra-Fredholm integral equations. *Mathematical and Computer Modelling*, 55(5-6):1833–1844, 2012. doi: 10.1016/j.mcm.2011.11.041.

[5] E.H. Doha, A.H. Bhrawy, and M.A. Saker. Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations. *Appl. Math. Lett.*, 24:559–565, 2011. doi: 10.1016/j.aml.2010.11.013.

[6] K. Maleknejad, E. Hashemizadeh, and R. Ezzati. A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation. *Commun. Nonlinear Sci. Numer. Simul.*, 16(2):647–655, 2011. doi: 10.1016/j.cnsns.2010.05.006.

[7] S.A. Yousefi and M. Behroozifar. Operational matrices of bernstein polynomials and their applications, internat. *J. Systems Sci.*, 41(6):709–716, 2010. doi: 10.1080/00207720903154783.

[8] G.G. Lorentz. *Bernstein Polynomials*. Mathematical expositions, Vol. 8. University of Toronto Press, 1953. ASIN: B0007IV734.

[9] T. Popoviciu. Sur i'approximation des fonctions convexes d'ordre superieur. *Mathematica*, 10:49–54, 1935. https://www.scopus.com/inward/record.url?eid=2-s2.0-0002859272&partnerID=10&rel=R3.0.0.

[10] A. Pallini. Bernstein-type approximation of smooth functions. *STATISTICA*, 65(2):169–191, 2005. https://rivista-statistica.unibo.it/article/view/84.

[11] H. Heitzinger. *Simulation and Inverse Modeling of Semiconductor Manufacturing Processes, Multivariate Bernstein Polynomials*. PhD thesis, Fakultat fur Elektrotechnik und Informationstechnik, Technischen Universitat Wien, Luftbadgasse 11, A-1060 Wien, Austria, 12 2002. http://www.iue.tuwien.ac.at/phd/heitzinger/node130.html.