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Total dominator chromatic number of some operations on a graph

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Abstract

Let G be a simple graph. A total dominator coloring of G is a proper coloring of the vertices of G in which each vertex of the graph is adjacent to every vertex of some color class. The total dominator chromatic number $\chi_d^t(G)$ of G is the minimum number of colors among all total dominator coloring of G. In this paper, we examine the effects on $\chi_d^t(G)$ when G is modified by operations on vertex and edge of G.

Keywords: Total dominator chromatic number, contraction, graph.

1 Introduction

In this paper, we consider simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G = (V, E)$ be such a graph and $k \in \mathbb{N}$. A mapping $f : V(G) \longrightarrow \{1, 2, ..., k\}$ is called a k-proper coloring of G if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G. A color class of this coloring is a set consisting of all those vertices assigned the same color. If f is a proper coloring of G with the coloring classes $V_1, V_2, ..., V_k$ such that every vertex in V_i has color i, then sometimes write simply $f = (V_1, V_2, ..., V_k)$.

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The chromatic number $\chi(G)$ of G is the minimum number of colors needed in a proper coloring of a graph. The chromatic number is perhaps the most studied of all graph theoretic parameters. A dominator coloring of G is a proper coloring of G such that every vertex of G dominates all vertices of at least one color class (possibly its own class), i.e., every vertex of G is adjacent to all vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. Kazemi [1, 2] studied a total dominator coloring, abbreviated TD-coloring. Let G be a graph with no isolated vertex, a total dominator coloring is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TD-chromatic number, $\chi_d^t(G)$ of G is the minimum number of color classes in a TD-coloring of G. The TD-chromatic number of a graph is related to its total domination number. Recall that a total dominating set of G is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in S and the total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. The literature on the subject on total domination in graphs has been surveyed and detailed in the book [3]. It has been proved that the computation of the TD-chromatic number is NP-complete ([1]). The TD-chromatic number of some graphs, such as paths, cycles, wheels and the complement of paths and cycles has been computed in [1]. Henning in [4] established the lower and upper bounds on the TD-chromatic number of a graph in terms of its total domination number. Henning has shown that, for every graph G with no isolated vertex satisfies $\gamma_t(G) \leq \chi_d^t(G) \leq \gamma_t(G) + \chi(G)$. The properties of TD-colorings in trees has been studied in [1, 4]. Trees T with $\gamma_t(T) = \chi_d^t(T)$ has been characterized in [4]. In [5] the TD-chromatic number of graphs with specific construction has been studied.

The join $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H. In the study of TD-chromatic number of graphs, this naturally raises the question: What happens to the TD-chromatic number, when we consider some operations on the vertices and the edges of a graph? In this paper we would like to answer this question.

In the next section, we examine the effects on $\chi_d^t(G)$ when G is modified by deleting a vertex or deleting an edge. In Section 3, we study the effects on $\chi_d^t(G)$, when G is modified by contracting a vertex and contracting an edge. Also we consider another graph obtained by operation on a vertex v denoted by $G \odot v$ which is a graph obtained from G by the removal of all edges between any pair of neighbors of v in Section 3 and study $\chi_d^t(G \odot v)$.

2 Vertex and edge removal

The graph $G - v$ is a graph that is made by deleting the vertex v and all edges connected to v from the graph G and the graph $G - e$ is a graph that obtained from G by simply removing the edge e. Our main results in this section are in obtaining a bound for TD-chromatic number of $G - v$ and $G - e$. To do this, we need to consider some preliminaries.

Theorem 1. ([1])

(i) Let P_n be a path of order $n \geq 2$. Then

$$\chi_d^t(P_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil - 1 & \text{if } n \equiv 1 \ (mod\, 3), \\ 2\lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

(ii) Let C_n be a cycle of order $n \geq 3$. Then

$$\chi_d^t(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ 4\lfloor \frac{n}{6} \rfloor + r & \text{if } n \neq 4,\ n \equiv r \ (mod\, 6),\ r = 0,1,2,4, \\ 4\lfloor \frac{n}{6} \rfloor + r - 1 & \text{if } n \equiv r \ (mod\, 6),\ r = 3,5. \end{cases}$$

The following theorem gives an upper bound and a lower bound for $\chi_d^t(G - e)$.

Theorem 2. *Let G be a connected graph, and $e = vw \in E(G)$ is not a bridge of G. Then we have:*

$$\chi_d^t(G) - 1 \leq \chi_d^t(G - e) \leq \chi_d^t(G) + 2.$$

Proof. First we prove the left inequality. We shall present a TD-coloring for $G-e$. If we add the edge e to $G-e$, then we have two cases. If two vertices v and w have the same color in the TD-coloring of $G-e$, then in this case we add a new color, like i, to one of them. Since every vertex use the old class for TD-coloring then this is a TD-coloring for G. So we have $\chi_d^t(G) \leq \chi_d^t(G-e)+1$. If two vertices v and w do not have the same color in the TD-coloring of $G-e$, then the TD-coloring of $G-e$ can be a TD-coloring for G. So $\chi_d^t(G) \leq \chi_d^t(G-e)$ and therefore we have $\chi_d^t(G)-1 \leq \chi_d^t(G-e)$.

Now we prove $\chi_d^t(G-e) \leq \chi_d^t(G)+2$. Suppose that the vertex v has color i and w has color j. We have the following cases:

Case 1) The vertex v does not use the color class j and w does not use the color class i in the TD-coloring of G. So the TD-coloring of G gives a TD-coloring of $G-e$ and in this case $\chi_d^t(G-e)=\chi_d^t(G)$.

Case 2) The vertex v uses the color class j but w does not use the color class i in the TD-coloring of G. Since v used the color class j for the TD-coloring then we have two cases:

(i) If v has some adjacent vertices which have color j, then we give the new color l to all of these vertices and this coloring is a TD-coloring for $G-e$.

(ii) If any other vertex does not have color j, since $G-e$ is a connected graph, then exists vertex s which is adjacent to v. Now we give to s the new color l and this coloring is a TD-coloring for $G-e$.

So for this case, we have $\chi_d^t(G-e)=\chi_d^t(G)+1$.

Case 3) The vertex v uses the color class j and w uses the color class i in the TD-coloring of G. We have three cases:

(i) There are some vertices which are adjacent to v and have color j. Then we color all of them with color l. And there are some vertices which are adjacent to w and have color i. We color all of them with color k. So this is a TD-coloring for $G-e$.

(ii) Any other vertex does not have color j. Then we do the same as Case 2 (ii) and there are some vertices which are adjacent to w and have color i. Then we do the same as Case 3 (i).

(iii) Any other vertex does not have colors i and j. Then we do the same as Case 2 (ii) and use two new colors l and k.

So we have $\chi_d^t(G-e) \leq \chi_d^t(G)+2$. □

Now we consider the graph $G-v$, and present a lower bound and an upper bound for the TD-chromatic number of $G-v$.

Theorem 3. *Let G be a connected graph, and $v \in V(G)$ is not a cut vertex of G. Then we have:*

$$\chi_d^t(G)-2 \leq \chi_d^t(G-v) \leq \chi_d^t(G)+deg(v)-1.$$

Proof. First we prove $\chi_d^t(G)-2 \leq \chi_d^t(G-v)$. We shall present a TD-coloring for $G-v$. If we add vertex v and all the corresponding edges to $G-v$, then it suffices to give the new color i to vertex v and the new color j only to one of the adjacent vertices of v like w and do not change all the other colors. Since every vertices except v and w use the old classes for TD-coloring and v uses the color class j and w uses the color class i so we have a TD-coloring of G. Therefore we have $\chi_d^t(G) \leq \chi_d^t(G-v)+2$ and we have the result.

Now we prove $\chi_d^t(G-v) \leq \chi_d^t(G)+deg(v)-1$. First we give a TD-coloring to G. Suppose that the vertex v has the color i. So we have the following cases:

Case 1) There is another vertex with color i. In this case every vertex uses the old class for TD-coloring and then this is a TD-coloring for $G-v$. So $\chi_d^t(G-v) \leq \chi_d^t(G)$.

Case 2) There is no other vertex with color i. In this case we give the new colors $i, a_1, a_2, \ldots, a_{deg(v)-1}$ to all the adjacent vertices of v. Obviously, this is a TD-coloring for $G-v$. Therefore $\chi_d^t(G-v) \leq \chi_d^t(G)+deg(v)-1$. □

Remark 1. *The lower bound in Theorem 3 is sharp. Consider the cycle C_{10}, as G. For every $v \in V(C_{10})$ we have $C_{10}-v = P_9$ which is a path graph of order 9. Then by the Theorem 1 we have $\chi_d^t(C_{10}) = 8$ and $\chi_d^t(P_9) = 6$.*

To obtain more results, we consider the corona of P_n and C_n with K_1. The following theorem gives the TD-chromatic number of these kind of graphs:

Theorem 4. *(i) For every $n \geq 2$, $\chi_d^t(P_n \circ K_1) = n+1$.*

(ii) For every $n \geq 3$, $\chi_d^t(C_n \circ K_1) = n+1$.

Proof. (i) We color the $P_n \circ K_1$ with numbers $1, 2, ..., n+1$, as shown in the Figure 1. Observe that, we need $n+1$ color for TD-coloring. We shall show that we are not able to have TD-coloring with less colors.

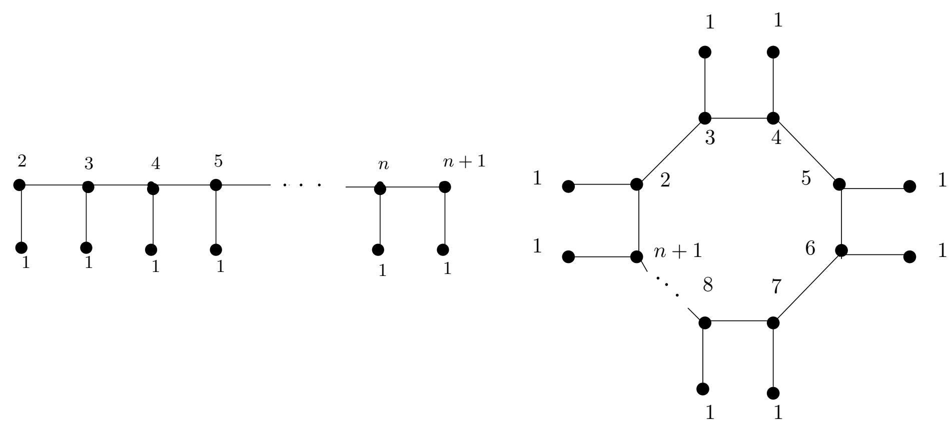

Fig. 1: Total dominator coloring of $P_n \circ K_1$ and $C_n \circ K_1$, respectively.

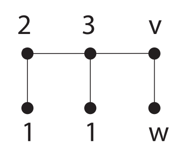

Fig. 2: $P_3 \circ K_1$

Obviously we have $\chi_d^t(P_2 \circ K_1) = 3$. Now we consider $P_3 \circ K_1$. As we see in Figure 2, we can not give number 1 to vertex v, because there is no number to color vertex w. Also we can't consider number 2 for vertex v since the vertex which has color 1 and is adjacent to vertex with number 2, is not adjacent with v. Since the coloring is proper, we cannot use color 3 too for this vertex. So we give number 4 to vertex v. Between used colors, we can use only number 1 for vertex w. Therefore $\chi_d^t(P_3 \circ K_1) = 4$. Similarly, we color $P_i \circ K_1$ from $P_{i-1} \circ K_1$ when $i \geq 3$. Any other kinds of coloring of this graph needs more colors. So we have the result.

(ii) It is similar to the part (i). □

We end this section with the following theorem:

Theorem 5. *There is a connected graph G, and a vertex $v \in V(G)$ which is not a cut vertex of G such that $|\chi_d^t(G) - \chi_d^t(G - v)|$ can be arbitrarily large.*

Proof. Consider the graph G in Figure 3. We color the vertices $a_1, a_2, \ldots, a_n$ with $\chi_d^t(P_n)$ colors. Then we give the new color $\chi_d^t(P_n)+1$ to all the adjacent

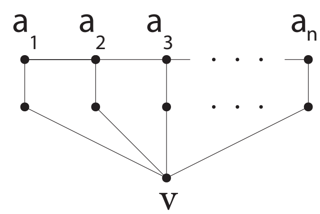

Fig. 3: Graph G in the proof of Theorem 5

vertices of v and $\chi_d^t(P_n) + 2$ to v. Obviously this is a TD-coloring for G. So we have:

$$\chi_d^t(G) = 2 + \chi_d^t(P_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \ (mod\, 3), \\ 2\lceil \frac{n}{3} \rceil + 2 & \text{otherwise.} \end{cases}$$

Now by removing the vertex v, we have $G - v = P_n \circ K_1$ and by Theorem 4 we have $\chi_d^t(G - v) = n + 1$. So we conclude that $|\chi_d^t(G) - \chi_d^t(G - v)|$ can be arbitrarily large. □

3 Vertex and edge contraction

Let v be a vertex in graph G. The contraction of v in G denoted by G/v is the graph obtained by deleting v and putting a clique on the (open) neighbourhood of v. Note that this operation does not create parallel edges; if two neighbours of v are already adjacent, then they remain simply adjacent (see [6]). In a graph G, contraction of an edge e with endpoints u, v is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v. The resulting graph G/e has one less edge than G ([7]). We denote this graph by G/e. In this section we examine the effects on $\chi_d^t(G)$ when G is modified by an edge contraction and vertex contraction. First we consider edge contraction:

Theorem 6. *Let G be a connected graph and $e \in E(G)$. Then we have:*

$$\chi_d^t(G) - 2 \leq \chi_d^t(G/e) \leq \chi_d^t(G) + 1.$$

Proof. First, we find a TD-coloring for G. Suppose that the end points of e are u and v. The vertex u has the color i and the vertex v has the color j. We give all the used colors in the previous coloring to the vertices $E(G) - \{u, v\}$. Now we give the new color k to $u = v$. Every vertices on the edges of $E(G) - \{u, v\}$ can uses the previous color class (or even k) in this coloring. The vertex $u = v$ uses the color class which used for u or v unless u used the color class j and v used the color class i. In this case, if there is another vertex with color i, then $u = v$ uses color class i and if there is another vertex with color j, then $u = v$ uses color class j. If any other vertex does not have the color i and j, then it suffices to give color i to one of the adjacent vertices of u (or v) in G. Then this is a TD-coloring for G/e. So we have $\chi_d^t(G/e) \leq \chi_d^t(G) + 1$.

To find the lower bound, we shall give a TD-coloring to G/e. We add the removed vertex and all the corresponding edges to G/e and keep the old coloring for the new graph. Now we consider the endpoints of e and remove the used color. Now add new colors i and j to these vertices. All the vertices of edges in $E(G) - \{u, v\}$ can use the previous color class and u can use color class j and v can use color class i. So this is a TD-coloring and we have $\chi_d^t(G) \leq \chi_d^t(G/e) + 2$. Therefore $\chi_d^t(G) - 2 \leq \chi_d^t(G/e)$. □

Remark 2. *The bounds in Theorem 6 are sharp. For the upper bound consider the cycle C_4 as G and for the lower bound consider cycle C_5.*

Corollary 1. *Suppose that G is a connected graph and $e \in E(G)$ is not a bridge of G. We have:*

$$\frac{\chi_d^t(G-e) + \chi_d^t(G/e) - 3}{2} \leq \chi_d^t(G) \leq \frac{\chi_d^t(G-e) + \chi_d^t(G/e) + 3}{2}$$

Proof. It follows from Theorems 2 and 6. □

Now we consider the vertex contraction of graph G and examine the effect on $\chi_d^t(G)$ when G is modified by this operation:

Theorem 7. *Let G be a connected graph and $v \in V(G)$. Then we have:*

$$\chi_d^t(G) - 2 \leq \chi_d^t(G/v) \leq \chi_d^t(G) + deg(v) - 1.$$

Proof. First we present a TD-coloring for G. We remove the vertex v and create G/v. We consider one of the adjacent vertices of v like u and do not change its color and give the new colors $i, i+1, \ldots, i + deg(v) - 1$ to other adjacent vertices of v. Now each vertex which was not adjacent to

v can use the previous color class (or if the color class changed, the new color class we give to adjacent vertices of v). Therefore we have $\chi_d^t(G/v) \leq \chi_d^t(G) + deg(v) - 1$.

To find the lower bound, at first we shall give a TD-coloring to G/v. We add the vertex v, add all the removed edges and remove all the added edges. It suffices to give the vertex v the new color i and only to one of its adjacent vertices like w the new color class j. All the vertices which are not adjacent to v can use the previous color classes. All the adjacent vertices of v can use the color class i and v can use the color class j. So we have $\chi_d^t(G) \leq \chi_d^t(G/v) + 2$. Therefore we have the result. □

Remark 3. *The bounds in Theorem 7 are sharp. For the upper bound consider the complete bipartite graph $K_{2,4}$ as G. We have $\chi_d^t(K_{2,4}) = 2$. By choosing a vertex which is adjacent to four vertices as v, we have $K_{2,4}/v = K_5$ which is the complete graph of order 5 and $\chi_d^t(K_5) = 5$. For the lower bound, we consider cycle graph C_5. For every $v \in V(C_5)$ we have $C_5/v = C_4$. Now by Theorem 1 we have the result.*

Corollary 2. *Let G be a connected graph. For every $v \in V(G)$ which is not cut vertex of G, we have:*

$$\frac{\chi_d^t(G-v) + \chi_d^t(G/v)}{2} - deg(v) + 1 \leq \chi_d^t(G) \leq \frac{\chi_d^t(G-v) + \chi_d^t(G/v)}{2} + 2.$$

Proof. It follows from Theorems 3 and 7. □

Here we consider another operation on vertex of a graph G and examine the effects on $\chi_d^t(G)$ when we do this operation. We denote by $G \odot v$ the graph obtained from G by the removal of all edges between any pair of neighbors of v, note v is not removed from the graph [8]. The following theorem gives upper bound and lower bound for $\chi_d^t(G \odot v)$.

Theorem 8. *Let G be a connected graph and $v \in V(G)$. Then we have:*

$$\chi_d^t(G) - deg(v) + 1 \leq \chi_d^t(G \odot v) \leq \chi_d^t(G) + 1.$$

Proof. First we prove $\chi_d^t(G \odot v) \leq \chi_d^t(G) + 1$. We give a TD-coloring for the graph G. Suppose that the vertex v has the color i. We have the following cases:

Case 1) The color i uses only for the vertex v. In this case, adjacent vertices of the vertex v, can use the color class i and all the other vertices can use the old color class. So we have $\chi_d^t(G \odot v) \leq \chi_d^t(G)$.

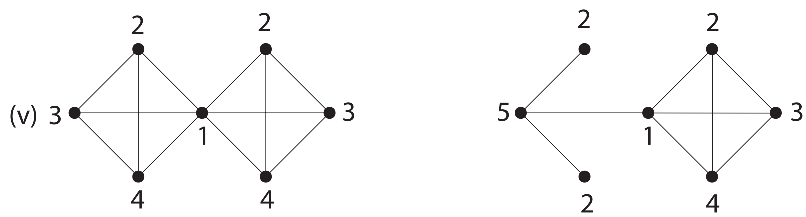

Fig. 4: TD-coloring of the graph G and $G \odot v$.

Case 2) The color i uses for another vertex except v. In this case, we give the new color j to all of these vertices (except v). This is a TD-coloring for $G \odot v$, because if a vertex is adjacent to v, it can use the color class i and all the other vertices can use old color class and if the old color class changes to j can use j as new color class. So we have $\chi_d^t(G \odot v) \leq \chi_d^t(G) + 1$.

Now we prove $\chi_d^t(G) - deg(v) + 1 \leq \chi_d^t(G \odot v)$. Consider the graph $G \odot v$ and shall find a TD-coloring for it. We make G from $G \odot v$ and just change the color of all the adjacent vertices of v except one of them like w to the new colors $a_1, a_2, \ldots, a_{deg(v)} - 1$ and do not change the color of v, w and other vertices. This is a TD-coloring for G, because v can use the the color class a_1. Adjacent vertices of v, can use the old color class of the TD-coloring of $G \odot v$, and other vertices can use old color class and if the old color classes changes to a_1 or a_2 or $\ldots$ or $a_{deg(v)-1}$ can use a_1 or a_2 or $\ldots$ or $a_{deg(v)-1}$ as new color classes. So we have $\chi_d^t(G) \leq \chi_d^t(G \odot v) + deg(v) - 1$. Therefore we have the result. □

Remark 4. *The bounds in Theorem 8 are sharp. For the upper bound consider the graph G in Figure 4. It is easy to see that these colorings are TD-coloring. For the lower bound consider to the complete graph K_n as G ($n \geq 3$). $\chi_d^t(K_n) = n$. Now for every $v \in V(K_n)$, $K_n \odot v$ is the star graph S_n and we have $\chi_d^t(S_n) = 2$. By this example we have the following result:*

Corollary 3. *There is a connected graph G and $v \in V(G)$ such that $\frac{\chi_d^t(G)}{\chi_d^t(G \odot v)}$ can be arbitrarily large.*

4 Conclusion

We examined the effects on the total dominator chromatic number $\chi_d^t(G)$ of G, when G is modified by deleting a vertex or deleting an edge. Theorem 2 shows that the removing an edge (which is not a bridge) decreases $\chi_d^t(G)$ by one and increases it by two. The effects on $\chi_d^t(G)$ when G is modified by deleting a vertex given in Theorem 3.
Theorem 6 shows that the contracting an edge decreases $\chi_d^t(G)$ by two and increases it by one. Also in Theorem 8 the total dominator chromatic number of another graph obtained from G by the removal of all edges between any pair of neighbors of a vertex v has investigated.

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