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Dark parameterization approach to Ito equation

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Abstract The novel coupling Ito systems are obtained with the dark parameterization approach. By solving the coupling equations, the traveling wave solutions are constructed with the mapping and deformation method. Some novel types of exact solutions are constructed with the solutions and symmetries of the usual Ito equation. In the meanwhile, the similarity reduction solutions of the model are also studied with the Lie point symmetry theory.

Keywords Ito equation; Dark parameterization approach; Symmetry reduction.
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1. Introduction

Various astronomical and cosmological observations show that there is dark matter (DM) and dark energy (DE) making up about 20% and 70% of the energy budget of our Universe respectively [1]. Although the evidence for DM and DE has been established for many decades, the identity of its basic constituents has so far remained elusive. Supersymmetry, a new symmetry that transforms bosons to fermions and vice versa in particle physics, still escapes observation [2]. A natural extension to the standard model is the addition of a fourth generation of fermions with masses much larger than those of the three known generations [3]. These additional quarks can offer solutions to some outstanding theoretical questions, such as baryon asymmetry of the universe, Higgs naturalness and fermion mass hierarchy [4]. However, up to now, no evidence for fourth generation quarks is observed at the Larger Hadron Collider. To explain some kinds of indirectly detectable dark things, the dark parameters of the physical models have been proposed [5]. Using this ideal to integrable systems, some novel coupled KdV and KP systems are obtained by the original usual field and partner fields. [5, 6]. It is proved that the dark parameterization method can be successfully to find and solve new integrable systems, new localized excitation modes and new interaction phenomena among soliton or solitary waves.

In this letter, we shall use the dark parameterization method to the Ito equation. The usual Ito equation is

$$u_{tt}+6u_{xx}u_t+6u_xu_{xt}+u_{xxxt}=0, \tag{1}$$

which was first proposed by Ito, and its bilinear Bäcklund transformation, Lax pair and multi-soliton solutions were obtained by means of Hirota's bilinear method [7]. The Ito equation reappeared several times in literature. The properties of the equation such as the nonlinear superposition formula, Kac-Moody algebra, bi-Hamiltonian structure and supersymmetric version have been further proposed [8- 10].

The structure of this paper is organized as follows. In section 2, we present the general dark parameterization approach to integrable systems. The novel coupling Ito systems are given with the approach. In section 3 and 4, we shall use one and two dark parameters as special examples. The exact solutions of the coupling systems are found using the mapping and deformation method and Lie point symmetry. The last section is a simple summary and discussion.

2 General dark parameterization approach

In this section, we consider the generalized dark parameterization approach to integrable system. We assume that the real physical quantity H include the usual quantity H_0 and the indirectly observed quantity H_i [5]

$$H = H_0 + H_i(\alpha_1, \alpha_2, \cdots, \alpha_i) = H_0 + H_i\alpha_i, \tag{2}$$

where α_i are dark parameters and H_0 is independent of α_i. H_i represents can not be directly observed quantities, such as DM, DE and supersymmetry. After introducing dark parameters into the traditional physical models, some types of partner fields may be introduced. We use this ideal to the nonlinear integrable systems, i.e., $\{H, H_i\} \to \{u, u_i\}$, one may obtain infinitely many new coupled integrable systems which are constituted by the original usual field and partner fields.

With the dark parameterization procedure, the special type of the coupling Ito systems write as the following form combination the usual Ito equation (1)

$$\sum_{i=0}^{n}(u_{i,tt} + u_{i,xxxt} + 6\sum_{j=0}^{i}u_{i-j,xx}u_{j,t} + 6\sum_{j=0}^{i}u_{i-j,x}u_{j,xt})\alpha_i = 0. \tag{3}$$

The function u_0 is exactly the usual Ito equation which has been widely studied [8, 9, 10]. The partner fields $u_i(i \geq 1)$ are linear partial differential equations while the previous functions $u_j(j < i)$ are known. We can theoretically solve the equivalent partner fields one after another. It should be mentioned that the system (3) without dark parameter α is completely equivalent to the n-anyon Ito system. Therefore, if we know the solution of u_i, then we can construct a solution of the n-anyon system via $u = u_i\alpha_i$. In the next two sections, we discuss two explicit examples for $n = 1$ and $n = 2$ respectively.

3. One anyon Ito System and its solutions

For $n = 1$, (3) becomes

$$u_{0,tt} + u_{0,xxxt} + 6u_{0,xx}u_{0,t} + 6u_{0,x}u_{0,xt} = 0, \tag{4a}$$

$$u_{1,tt} + u_{1,xxxt} + 6u_{1,xx}u_{0,t} + 6u_{0,xx}u_{1,t} + 6u_{1,x}u_{0,xt} + 6u_{0,x}u_{1,xt} = 0. \tag{4b}$$

The above system may has the different meanings with selecting different kinds of parameter α. If we simply take α as the Grassmann number, then u_1 is fermionic and the model is just the supersymmetric Ito version for fermionic component field [10]. If we take the dark parameter $\alpha = \zeta_1\zeta_2$ as a multiplication of two Grassmann numbers, then both u_0 and u_1 are bosonic field [11]. The solution of 1-anyon Ito system (3) can be constructed via $u = u_0 + \alpha_1 u_1$ by means of (4).

3.1. Traveling wave solutions with mapping and deformation method

Introducing the traveling wave variable $X = kx + \omega t + c_0$ with constants k, ω and c_0, (4) is transformed to the ordinary differential equations (ODEs) and directly integrate once

$$k^3u_{0,XXX} + \omega u_{0,X} + 6k^2u_{0,X}^2 = 0, \tag{5a}$$

$$k^3u_{1,XXX} + \omega u_{1,X} + 12k^2u_{1,X}u_{0,X} = 0. \tag{5b}$$

As the well known exact solutions of (5a), we try to build the mapping and deformation relationship between u_0 and u_1. We get $u_{0,X}$ with (5a)

$$u_{0,X} = -\frac{u_0^2}{k} - \frac{\omega}{4k^2}. \tag{6}$$

In order to get the mapping relationship between u_1 and u_0, we introduce the variable transformation

$$u_1(X) = U_1(u_0(X)). \tag{7}$$

Using the transformation (7) and vanishing $u_{0,X}$ via (6), the linear ODEs (5b) becomes

$$(4ku_0^2+\omega)\frac{\mathrm{d}^3U_1}{\mathrm{d}u_0^3}+24ku_0\frac{\mathrm{d}^2U_1}{\mathrm{d}u_0^2}-24k\frac{\mathrm{d}U_1}{\mathrm{d}u_0}=0. \tag{8}$$

The mapping and deformation relations are constructed via (8)

$$U_1=c_1+c_2(20ku_0^2+\omega)+c_3((20ku_0^2+\omega)\int^{u_o}\frac{6\sqrt{k\omega}}{(4ky^2+\omega)}dy+\frac{2\sqrt{k\omega}u_0(60ku_0^2+13\omega)}{4ku_0^2+\omega}), \tag{9}$$

where $c_i(i=1,2,3)$ are arbitrary constants. If we know the solution of u_0, the traveling wave solution u_1 will be given with considering (7) and (9). Here, we list one solution as an example. The solution of u_0 can be expressed as the following form using (6)

$$u_0=-\frac{1}{2}\sqrt{\frac{\omega}{k}}\tan(\frac{\sqrt{k\omega}(X+c_4)}{2k^2}). \tag{10}$$

With (11) and (12), we can get the solution of 1-anyon Ito system (3)

$$u=-\frac{1}{2}\sqrt{\frac{\omega}{k}}\tan\xi+\alpha_1[c_1-c_2\omega(4-5\sec^2\xi)+ \tag{11}$$
$$c_3\omega(\sin 2\xi-15\tan\xi+12\arctan(\tan\xi)-15\arctan(\tan\xi)\sec^2\xi)],$$

where $\xi=\dfrac{\sqrt{k\omega}(X+c_4)}{2k^2}$.

Besides, u_1 of (4b) exactly satisfies the symmetry equation of the usual Ito system. For any given a solution u_0 of the usual Ito equation, a certain type solutions (4) can be constructed

$$u=u_0+\alpha\sigma(u_0), \tag{12}$$

where $\sigma(u_0)$ represents the symmetry of the usual Ito equation (4a). It means that we have much freedom to choose u_0 so as to construct solutions of the (4). The solution u_0 is not restricted to the traveling wave solutions. All in all, we can construct not only traveling wave solutions but also some novel types of solutions of (4) with the solutions and symmetries of the Ito equation. As illustrative example, the N -soliton solution of the Ito equation reads [7]

$$u_{Ito}=2[\log(1+\sum_{k=1}^{N}\sum_{i_1>i_2>\cdots>i_k}\prod_{m>n}A_{i_m i_n}\exp\sum_{i=1}^{k}\eta_i)]_{xx}, \tag{13}$$

where $A_{ij}=\dfrac{(k_i-k_j)(k_i^3-k_j^3)}{(k_i+k_j)(k_i^3+k_j^3)}$ which is the phase shift from the interaction of the soliton ``$i$" with the soliton ``j", $\eta_i=k_ix-k_i^3t+\eta_i^0$ and arbitrary constants $(k_i,\eta_i^0,\eta_i,i=1,2,\cdots,N)$. Correspondingly, a special type of multiple soliton solutions of (4) reads

$$u=u_{Ito}+\alpha(xu_{Ito,x}+3tu_{Ito,t}+u_{Ito}). \tag{14}$$

3.2. Similarity reduction solutions with symmetry reduction approach

The symmetry study plays a prominent role in nonlinear partial differential systems for the existence of infinitely many symmetries [12, 13]. The classical Lie group [14], the nonclassical approach [15], Clarkson and Kruskal (CK) direct method [16] are effective methods to obtain the explicit exact solutions. Now, we shall use the Lie point symmetry approach to study (4).

A Lie point symmetry vector field is given

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial u_0} + U_1\frac{\partial}{\partial u_1}, \tag{15}$$

where X, T, U_0 and U_1 are functions of x, t, u_0 and u_1. It means the system of (4) is invariant under

$$\{x, t, u_0, u_1\} \to \{x + \varepsilon X, t + \varepsilon T, u_0 + \varepsilon U_0, u_1 + \varepsilon U_1\}, \tag{16}$$

with an infinitesimal parameter ε. The corresponding symmetry can be supposed

$$\sigma_0 = Xu_{0,x} + Tu_{0,t} - U_0,\ \ \sigma_1 = Xu_{1,x} + Tu_{1,t} - U_1. \tag{17}$$

Considering the notation (17), $\sigma_{0;1}$ is the solution of the linearized (3)

$$\sigma_{0,tt} + \sigma_{0,xxxt} + 6\sigma_{0,t}u_{0,xx} + 6\sigma_{0,xx}u_{0,t} + 6(\sigma_{0,x}u_{0,x})_t = 0, \tag{18a}$$

$$\sigma_{1,tt} + \sigma_{1,xxxt} + 6\sigma_{0,t}u_{1,xx} + 6\sigma_{1,xx}u_{0,t} + 6\sigma_{1,t}u_{0,xx} + 6\sigma_{0,xx}u_{1,t} + 6(\sigma_{0,x}u_{1,x})_t + 6(\sigma_{1,x}u_{0,x})_t = 0. \tag{18b}$$

Substituting (17) into the symmetry equations (18) with u_0 and u_1 satisfying (4), we obtain the determining equations by identifying all coefficients of derivatives of u_0 and u_1. The solutions of the functions X, T, U_0 and U_1 can be concluded using the determining equations

$$T = C_1 t + C_2,\quad X = \frac{C_1}{3}x + C_3,\quad U_0 = -\frac{C_1}{3}u_0 + C_5,\quad U_1 = C_4 u_1 + C_6, \tag{19}$$

where $C_i (i = 1,2,...,6)$ are arbitrary constants. Then, one can solve the characteristic equations to obtain similarity solutions

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}t}{T},\quad \frac{\mathrm{d}u_0}{U_0} = \frac{\mathrm{d}t}{T},\quad \frac{\mathrm{d}u_1}{U_1} = \frac{\mathrm{d}t}{T}, \tag{20}$$

where X, T, U_0 and U_1 are given by (19). An simple example is listed concerning the solutions of (4) in the following.

With $C_1 = C_4 = C_5 = C_6 = 0$, we can find the similarity solutions after solving out the characteristic equations

$$u_0 = U_0(\xi),\quad u_1 = U_1(\xi), \tag{21}$$

with the similarity variable $\xi = t - (C_2/C_3)x$. We redefine the similarity variable as $\xi = x + ct$ with c an arbitrary velocity constant. Substituting (21) into (18), the invariant functions U_0 and U_1 satisfy the following reduction systems

$$U_{0,\xi\xi\xi\xi} + cU_{0,\xi\xi} + 12U_{0,\xi\xi}U_{0,\xi} = 0, \tag{22a}$$

$$U_{1,\xi\xi\xi\xi} + cU_{1,\xi\xi} + 12(U_{1,\xi}U_{0,\xi})_\xi = 0. \tag{22b}$$

These reduction equations are linear ODEs while the previous functions are known, we can theoretically solve (22) one after another.

4. Two anyon Ito System and its solutions

For $n=3$, (3) becomes

$$u_{0,tt}+u_{0,xxxt}+6u_{0,xx}u_{0,t}+6u_{0,x}u_{0,xt}=0, \tag{23a}$$

$$u_{1,tt}+u_{1,xxxt}+6u_{1,xx}u_{0,t}+6u_{0,xx}u_{1,t}+6u_{1,x}u_{0,xt}+6u_{0,x}u_{1,xt}=0, \tag{23b}$$

$$u_{2,tt}+u_{2,xxxt}+6u_{2,xx}u_{0,t}+6u_{0,xx}u_{2,t}+6(u_{1,x}u_{1,t})_x+6(u_{2,x}u_{0,x})_t=0. \tag{23c}$$

The solution of 2-anyon Ito system (3) can be constructed via $u=u_0+\alpha_1u_1+\alpha_2u_2$ with solving (23).

4.1. Traveling wave solutions with mapping and deformation method

With the variable $X=kx+\omega t+c_0$, (23) is transformed to the ODEs and integrate once

$$k^3u_{0,XXX}+\omega u_{0,X}+6k^2u_{0,X}u_{0,X}=0, \tag{24a}$$

$$k^3u_{1,XXX}+\omega u_{1,X}+12k^2u_{1,X}u_{0,X}=0, \tag{24b}$$

$$k^3u_{2,XXX}+\omega u_{2,X}+12k^2u_{2,X}u_{0,X}+6k^2u_{1,X}^2=0. \tag{24c}$$

We consider the variable transformations

$$u_1(X)=U_1(u_0(X)),\quad u_2(X)=U_2(u_0(X)). \tag{25}$$

With the above transformations and eliminating $u_{0,X}$ via (8), the ODEs (24b) and (24c) are changed

$$(4ku_0^2+\omega)\frac{\mathrm{d}^3U_1}{\mathrm{d}u_0^3}+24ku_0\frac{\mathrm{d}^2U_1}{\mathrm{d}u_0^2}-24k\frac{\mathrm{d}U_1}{\mathrm{d}u_0}=0, \tag{26a}$$

$$(4ku_0^2+\omega)\frac{\mathrm{d}^3U_2}{\mathrm{d}u_0^3}+24ku_0\frac{\mathrm{d}^2U_2}{\mathrm{d}u_0^2}-24k\frac{\mathrm{d}U_2}{\mathrm{d}u_0}-24k\left(\frac{\mathrm{d}U_1}{\mathrm{d}u_0}\right)^2=0. \tag{26b}$$

By repeating the processes of the last section, the traveling wave solution can be obtained using (25) and (26). The expression solution of U_2 is very length so that we neglect write it.

4.2. Similarity reduction solutions with symmetry reduction approach

For 2-anyon Ito system (23), the solutions of the functions X, T, U_0, U_1 and U_2 read

$$T=C_1t+C_2,\quad X=\frac{C_1}{3}x+C_3,\quad U_0=-\frac{C_1}{3}u_0+C_4,\quad U_1=\frac{3C_5-C_1}{6}u_1+C_6,$$
$$U_2=C_5u_2+C_7u_1+C_8, \tag{27}$$

where $C_i(i=1,2,...,8)$ are arbitrary constants. The similarity solutions can be obtained by solving the characteristic equations. An explicit example is discussed in the following.

With $C_1=C_4=C_5=C_6=C_7=C_8=0$, the invariant solutions are given

$$u_0=U_0(\xi),\quad u_1=U_1(\xi),\quad u_2=U_2(\xi). \tag{28}$$

The reduction equations lead to

$$U_{0,\xi\xi\xi\xi}+cU_{0,\xi\xi}+12U_{0,\xi\xi}U_{0,\xi}=0, \tag{29a}$$

$$U_{1,\xi\xi\xi\xi}+cU_{1,\xi\xi}+12(U_{1,\xi}U_{0,\xi})_\xi=0, \tag{29b}$$

$$U_{2,\xi\xi\xi\xi}+cU_{2,\xi\xi}+12(U_{0,\xi}U_{0,\xi})_\xi+12U_{1,\xi\xi}U_{1,\xi}=0, \tag{29c}$$

where the similarity variable $\xi=x+ct$. Similar to the one anyon Ito system, (29) can theoretically be solved one after another.

5. Conclusions

In this paper, the coupled Ito systems are obtained with the dark parameterization approach. Using the mapping and deformation method, the traveling wave solutions of the coupled systems are obtained. Besides, some special types of exact solutions can be given straightforwardly through the exact solutions of the Ito equation and its symmetries. In addition, the similarity reduction solutions of the model are derived using the Lie point symmetry theory. The dark parameterization procedure can be applicable all the models including nonlinear integrable or non-integrable systems. The method is worthy of further studying.

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