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# Coefficient bound estimates for alpha-convex functions of order beta

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**Abstract** In this paper, we give sharp estimates for some initial coefficients for so-called alpha-convex of order beta functions, analytic and univalent in the open unit disk. Also, we give sharp estimates for three initial coefficients for the inverse function. In this study sharp estimates for some initial coefficient of the function $\ln\left(f(z)/z\right)$ are also obtained.

**Keywords** Convex functions, Alpha-convex functions, Coefficient problem, Logarithmic coefficient
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## 1. Introduction and preliminaries

Denote by $A$ the class of the functions normalized by $f(0)=0=f'(0)-1$ with expansion series

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $U=\left\{z\in\mathbb{C}:|z|<1\right\}$ in the complex plane. It is well-known that an analytic function $f:\mathbb{C}\to\mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1=z_2$ if $f(z_1)=f(z_2)$ or $f(z_1)\neq f(z_2)$ if $z_1\neq z_2$ .

It is well-known that (see, for example, [2]) every function $f\in S$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z))=z, z\in U,\ f(f^{-1}(w))=w, w\in D=\left\{w:|w|<r_0(f)\right\}, r_0(f)\geq\frac{1}{4},$$

where $f^{-1}(w)=w+A_2w^2+A_3w^3+A_4w^4\cdots,\ w\in D,$

$$A_2=-a_2,\ \ A_3=2a_2^2-a_3,\ \ A_4=-5a_2^3+5a_2a_3-a_4.$$

Let $S$ be the subclass of $A$ consisting of univalent functions. Some of the important subclasses of $S$ are $S^*(\beta)$ and $C(\beta)$, respectively, starlike and convex functions of order $\beta\geq 0$. By definition (see for details, [1, 3], also [4])

$$S^*(\beta)=\left\{f\in S:\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\beta,\ z\in U\right\} \tag{1.2}$$

and

$$C(\beta)=\left\{f\in S:\ \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta,\ z\in U\right\}. \tag{1.3}$$

For convenience, $S^{*}=S^{*}(0)$ and $C=C(0)$ are, respectively, starlike and convex functions in $U$. It is well known that a function $f\in A$ is called starlike in $U$ if $f(U)$ is starlike domain in the complex plane (with respect to the origin), and convex in $U$ if $f(U)$ is convex domain. Thus it is easy to see that $f\in C$ if an only if $zf'\in S^{*}$. Also, it is easy to verify that $C\subset S^{*}\subset S$. For details on these classes, one could refer to the monograph by Goodman [3].

For $\alpha\in$ , the class $M_{\alpha}$ of alpha-convex functions defined as follows

$$\operatorname{Re}\left\{\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]+(1-\alpha)\frac{zf'(z)}{f(z)}\right\}>0$$

is well-known, and contains a great many interesting properties, the most basic being that for all $\alpha\in$ $M_{\alpha}\subset S^{*}$ [5-8]. Thus $M_{\alpha}$ is natural subset of $S$, with $M_{0}=S^{*}$ and $M_{1}=C$.

We will define so-called alpha-convex functions of order beta as follows.

**Definition 1.1.** *A function $f\in S$ given by (1.1) is said to be in the class $M_{\alpha}(\beta)$, $\alpha,\beta\geq 0$ if the following conditions are satisfied*

$$\operatorname{Re}\left\{\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]+(1-\alpha)\frac{zf'(z)}{f(z)}\right\}>\beta,\ z\in U.$$

For convenience, $M_{\alpha}(0)=M_{\alpha}$ is the class of alpha-convex functions in $U$.

**Remark 1.1.** *Choose $\alpha=0$ in Definition 1.1, we have function class $M_{0}(\beta)=S^{*}(\beta)$, $\beta\geq 0$; that is,*

$$f\in S^{*}(\beta)\Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\beta, z\in U.$$

**Remark 1.2.** *Choose $\alpha=1$ in Definition 1.1, we have function class $M_{1}(\beta)=C(\beta)$, $\beta\geq 0$; that is,*

$$f\in C(\beta)\Leftrightarrow \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta, z\in U.$$

The object of the present paper is to give a series of sharp inequalities involving the initial coefficients of functions in $M_{\alpha}(\beta)$ and its special cases. Also, in this paper we give sharp estimates for three initial coefficients for the inverse function. In this paper sharp estimates for some initial coefficient of the function $\ln\left(f(z)/z\right)$ are also obtained.

To prove our main results, we need the following lemmas concerning functions with positive real part (see e. g. [9, 10]).

Denote by $\mathrm{P}$ the set of functions $p(z)$ analytic in $U$ and satisfying $\operatorname{Re}(p(z))>0$ with expansion series

$$p(z)=1+\sum_{n=1}^{\infty}p_{n}z^{n}, z\in U.$$

**Lemma 1.1.** *Let $p\in\mathrm{P}$. Then, $|p_{n}|\leq 2$ for all $n=1,2,\ldots$ and*

$$\left|p_{2}-\frac{\mu}{2}p_{1}^{2}\right|\leq \max\{2,2|\mu-1|\}=\begin{cases}2 & \text{if } \mu\in[0,2],\\ 2|\mu-1|, & \text{elsewhere.}\end{cases}$$

**Lemma 1.2.** *Let* $p \in \mathrm{P}$, $B \in [0,1]$ *and* $B(2B-1) \le D \le B$ . *Then,*

$$\left| p_3 - 2Bp_1p_2 + Dp_3 \right| \le 2 .$$

## 2. Coefficient estimates for the function class $M_\alpha(\beta)$

In this section, we give the following theorem on the sharp estimates for initial three coefficients of the function class $M_\alpha(\beta)$.

**Theorem 2.1.** *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\alpha(\beta)$, $\alpha \in [0,1]$, $\beta \in [0,1)$. *Then,*

$$|a_2| \le \frac{2(1-\beta)}{1+\alpha}, \quad |a_3| \le \frac{1-\beta}{1+2\alpha}\left[1+\frac{2(1+3\alpha)(1-\beta)}{(1+\alpha)^2}\right]$$

*and*

$$|a_4| \le \frac{2(1-\beta)}{3(1+3\alpha)}\left[1+\frac{3(1+5\alpha)(1-\beta)}{(1+\alpha)(1+2\alpha)}+\frac{2(17\alpha^2+6\alpha+1)(1-\beta)^2}{(1+\alpha)^3(1+2\alpha)}\right].$$

*All the inequalities obtained here are sharp.*

**Proof.** Let $f \in M_\alpha(\beta), \alpha \in [0,1], \beta \in [0,1)$. Then, from the Definition 1.1 we have

$$\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]+(1-\alpha)\frac{zf'(z)}{f(z)} = \beta+(1-\beta)\,p(z), \tag{2.1}$$

where $p \in \mathrm{P}$ and $p(z) = 1+\sum_1 p_n z^n = 1+p_1z+p_2z^2+p_3z^3+\cdots$ .

As a result of simple simplification from (2.1) we write

$$\begin{aligned} &1+(1+\alpha)a_2z+\left[2(1+2\alpha)a_3-(1+3\alpha)a_2^{\,2}\right]z^2+ \\ &\left[3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^{\,3}\right]z^3+\cdots \\ &=1+(1-\beta)\,p_1z+(1-\beta)\,p_2z^2+(1-\beta)\,p_3z^3+\cdots \quad . \end{aligned} \tag{2.2}$$

Comparing the coefficients of the like power of $z$ in the both sides (2.2) we get

$$(1+\alpha)a_2 = (1-\beta)\,p_1, \tag{2.3}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^{\,2} = (1-\beta)\,p_2, \tag{2.4}$$

$$3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^{\,3} = (1-\beta)\,p_3. \tag{2.5}$$

From (2.3) – (2.5), we obtain

$$a_2 = \frac{1-\beta}{1+\alpha}\,p_1, \tag{2.6}$$

$$a_3 = \frac{1-\beta}{2(1+2\alpha)}\,p_2+\frac{(1+3\alpha)(1-\beta)^2}{2(1+2\alpha)(1+\alpha)^2}\,p_1^2 \tag{2.7}$$

and

$$a_4 = \frac{1-\beta}{3(1+3\alpha)}\,p_3+\frac{(1+5\alpha)(1-\beta)^2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)}\,p_1p_2+\frac{(17\alpha^2+6\alpha+1)(1-\beta)^3}{6(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\,p_1^3. \tag{2.8}$$

Since the coefficients of $p_1, p_2, p_3, p_1p_2, p_1^2$ and $p_1^3$ in the equations (2.6) – (2.8) are positive for $\alpha \in [0,1]$ and $\beta \in [0,1)$, the required inequalities are valid on using the inequalities from Lemma 1.1 $|p_n| \le 2$ for $n = 1, 2, 3$.

To see that the inequalities obtained in the theorem are sharp, we note that equality is attained in the inequalities when $p_1 = p_2 = p_3 = 2$.

Moreover, we can easily show that all inequalities obtained in the theorem are sharp for the particular solution of the following nonlinear homogeneous differential equation

$$\alpha(1-z)zy'' + (1-\alpha)(1-z)z(y')^2 - [(1+\alpha-2\beta)z + 1-\alpha]yy' = 0.$$

Thus, the proof of Theorem 2.1 is completed.

From Theorem 2.1 we arrive at the following results.

**Corollary 2.1.** *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(\beta)$, $\beta \in [0,1)$. *Then,*

$$|a_2| \le 2(1-\beta), \ |a_3| \le (1-\beta)(3-2\beta)$$

*and*

$$|a_4| \le |a_4| \le \frac{2(1-\beta)}{3}[1+(1-\beta)(5-2\beta)].$$

*All the inequalities obtained here are sharp.*

**Corollary 2.2.** *Let the function* $f(z)$ *given by (1.1) be in the class* $C(\beta)$, $\beta \in [0,1)$. *Then,*

$$|a_2| \le 1-\beta, \ |a_3| \le \frac{(1-\beta)(3-2\beta)}{3}$$

*and*

$$|a_4| \le \frac{1-\beta}{6}[1+(1-\beta)(5-2\beta)].$$

*All the inequalities obtained here are sharp.*

**Corollary 2.3.** Let *the function* $f(z)$ *given by (1.1) be in the class* $M_\alpha$, $\alpha \in [0,1]$. *Then,*

$$|a_2| \le \frac{2}{1+\alpha}, \ |a_3| \le \frac{\alpha^2 + 8\alpha + 3}{(1+\alpha)^2(1+2\alpha)}$$

*and*

$$|a_4| \le \frac{2}{3(1+3\alpha)}\left[1 + \frac{3(1+5\alpha)}{(1+\alpha)(1+2\alpha)} + \frac{2(17\alpha^2 + 6\alpha + 1)}{(1+\alpha)^3(1+2\alpha)}\right].$$

*All the inequalities obtained here are sharp.*

**Corollary 2.4.** *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*$. *Then,*

$$|a_n| \le n, \ n = 2, 3, 4.$$

*All the inequalities obtained here are sharp.*

**Corollary 2.5.** *Let the function* $f(z)$ *given by (1.1) be in the class* $C$. *Then,*

$$|a_n| \le 1, \ n = 2, 3, 4.$$

*All the inequalities obtained here are sharp.*

**3. Coefficient estimates for the inverse function**

In this section, we give the following theorem on the sharp estimates for some initial coefficients of the inverse function of the function $f\in M_\alpha(\beta)$.

**Theorem 3.1.** *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\alpha(\beta)$, $\alpha\in[0,1]$, $\beta\in[0,1)$ *and* $f^{-1}(w)$ *is the inverse function of* $f(z)$ *. Then, for the initial three coefficients of the inverse function* $f^{-1}$, *we have*

$$|A_2|\le\frac{2(1-\beta)}{1+\alpha},\quad |A_3|\le\begin{cases}\dfrac{1-\beta}{1+2\alpha}\left[\dfrac{2(3+5\alpha)(1-\beta)}{(1+\alpha)^2}-1\right], & \beta\in\left[0,\dfrac{2}{3}\right],\\[2ex] \dfrac{1-\beta}{1+2\alpha}, & \beta\in\left(\dfrac{2}{3},1\right)\end{cases}$$

*and*

$$|A_4|\le\frac{2(1-\beta)}{3(1+3\alpha)}.$$

*All the inequalities obtained here are sharp.*

**Proof.** Let $f\in M_\alpha(\beta),\alpha\in[0,1],\beta\in[0,1)$. Then, since $M_\alpha(\beta)\subset S$, inverse function $f^{-1}(w)$ exist defined in some open disk $D=\{w:|w|<r_0(f)\},r_0(f)\ge 1/4$, and

$$f^{-1}(w)=w+A_2w^2+A_3w^3+A_4w^4\cdots,\ w\in D$$

with the initial three coefficients (see, for example [2])

$$A_2=-a_2,\ \ A_3=2a_2^2-a_3,\ \ A_4=-5a_2^3+5a_2a_3-a_4. \tag{3.1}$$

From Theorem 2.1 and the first equality of (3.1) the inequality of $|A_2|$ is obvious.

From the second equality of (3.1), using (2.6) and (2.7), we can write expression for $A_3$ as

$$A_3=-\frac{1-\beta}{2(1+2\alpha)}\left[p_2-\frac{(3+5\alpha)(1-\beta)}{(1+\alpha)^2}p_1^2\right].$$

Then, a simple application of Lemma 1.1 with $\mu=\dfrac{2(3+5\alpha)(1-\beta)}{(1+\alpha)^2}$, gives the inequalities for $|A_3|$.

From the third equality of (3.1), using (2.6), (2.7) and (2.8), we can write expression for $A_4$ as

$$A_4=-\frac{1-\beta}{3(1+3\alpha)}\left[p_3-2Bp_1p_2+Dp_1^3\right],$$

where

$$B=\frac{3(2+5\alpha)(1-\beta)}{2(1+\alpha)(1+2\alpha)}\ \text{ and }\ D=\frac{(62\alpha^2+66\alpha+15)(1-\beta)^2}{2(1+\alpha)^3(1+2\alpha)}.$$

It is easily shown that $0\le B\le 1$ and $B(2B-1)\le D\le B$ for $\alpha\in[0,1]$ and $\beta\in[0,1)$. Then, a simple application of Lemma 1.2 gives the inequality for $|A_4|$.

To see that the inequalities obtained in the theorem are sharp, we note that equality is attained in the inequality for $|A_2|$ when $p_1 = 2$, in the first inequality for $|A_3|$ when $p_1 = p_2 = 2$, in the second inequality for $|A_3|$ when $p_1 = 0 = p_2 - 2$, and in the inequality for $|A_4|$ when $p_1 = p_2 = 0 = p_3 - 2$.

Thus, the proof of Theorem 3.1 is completed.

From Theorem 2.1 we arrive at the following results.

**Corollary 3.1.** *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(\beta)$, $\beta \in [0,1)$ *and* $f^{-1}(w)$ *is the inverse function of* $f(z)$ *. Then, for the initial three coefficients of the inverse function* $f^{-1}$*, we have*

$$|A_2| \le 2(1-\beta), \quad |A_3| \le \begin{cases} (1-\beta)(5-6\beta), & \beta \in \left[0, \frac{2}{3}\right], \\ 1-\beta, & \beta \in \left(\frac{2}{3}, 1\right) \end{cases}$$

*and*

$$|A_4| \le \frac{2(1-\beta)}{3}.$$

*All the inequalities obtained here are sharp.*

**Corollary 3.2.** *Let the function* $f(z)$ *given by (1.1) be in the class* $C(\beta)$, $\beta \in [0,1)$ *and* $f^{-1}(w)$ *is the inverse function of* $f(z)$ *. Then, for the initial three coefficients of the inverse function* $f^{-1}$*, we have*

$$|A_2| \le 1-\beta, \quad |A_3| \le \begin{cases} \frac{(1-\beta)(3-4\beta)}{3}, & \beta \in \left[0, \frac{2}{3}\right], \\ \frac{1-\beta}{3}, & \beta \in \left(\frac{2}{3}, 1\right) \end{cases}$$

*and*

$$|A_4| \le \frac{1-\beta}{6}.$$

*All the inequalities obtained here are sharp.*

**Corollary 3.3.** *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\alpha$, $\alpha \in [0,1]$ *and* $f^{-1}(w)$ *is the inverse function of* $f(z)$ *. Then, for the initial three coefficients of the inverse function* $f^{-1}$*, we have*

$$|A_2| \le \frac{2}{1+\alpha}, \quad |A_3| \le \frac{5+8\alpha-\alpha^2}{(1+2\alpha)(1+\alpha)^2}$$

*and*

$$|A_4| \le \frac{2}{3(1+3\alpha)}.$$

*All the inequalities obtained here are sharp.*

**4. The coefficient of** $\ln(f(z)/z)$

The logarithmic coefficients $\delta_n$ of a function $f \in S$ are defined as follows

$$\ln(f(z)/z) = 2\sum_{n=1}^{\infty} \delta_n z^n. \tag{4.1}$$

On differentiating both sides of (4.1), it is a trivial consequence of the inequality $|p_n| \le 2$, that for $n \ge 1$, $|\delta_n| \le 1/n$ when $f \in S^*$, and $|\delta_n| \le 1/2n$ when $f \in C$. However when $f \in M_\alpha(\beta)$, the same procedure does not give a convenient expression in terms of $1 + zf''(z)/f'(z)$ or $zf'(z)/f(z)$ unless $\beta = 0$ and $\alpha = 1$ or $\alpha = 0$. We show next that it is however possible to obtain sharp estimates for the modulus of the initial coefficients of $\ln(f(z)/z)$ when $f \in M_\alpha(\beta)$.

We give the following theorem on the sharp estimates for initial two coefficients of $\ln(f(z)/z)$ for the function $f \in M_\alpha(\beta)$.

**Theorem 4.1.** *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\alpha(\beta)$, $\alpha \in [0,1]$, $\beta \in [0,1)$. *Then,*

$$|\delta_1| \le \frac{1-\beta}{1+\alpha}, \quad |\delta_2| \le \frac{1-\beta}{1+2\alpha}\left[\frac{1}{2} + \frac{(1-\beta)\alpha}{(1+\alpha)^2}\right].$$

*The inequalities obtained here are sharp.*

**Proof.** Let $f \in M_\alpha(\beta)$, $\alpha \in [0,1]$, $\beta \in [0,1)$. Differentiating both sides of (4.1) and comparing the coefficients of the like power of $z$ in the both sides, we obtain

$$2\delta_1 = a_2, \tag{4.2}$$

$$4\delta_2 + 2a_2\delta_1 = 2a_3. \tag{4.3}$$

From (4.2), easily write

$$\delta_1 = \frac{a_2}{2}. \tag{4.4}$$

From (4.4), using Theorem 2.1, the inequality for $|\delta_1|$ is obvious.

From (4.3), (4.4), (2.6) and (2.7), we can write the expression for $\delta_2$ as

$$\delta_2 = \frac{1-\beta}{4(1+2\alpha)} p_2 + \frac{(1-\beta)^2 \alpha}{4(1+\alpha)^2(1+2\alpha)} p_1^2. \tag{4.5}$$

Since the coefficients of $p_2$ and $p_1^2$ in the above equation (4.5) are positive for $\alpha \in [0,1]$ and $\beta \in [0,1)$, the required inequalities for $|\delta_2|$ is valid on using the inequalities from Lemma 1.2 $|p_n| \le 2$ for $n = 1,2,3$.

To see that the inequalities obtained in the theorem are sharp, we note that equality is attained in the inequalities when $p_1 = p_2 = 2$.

Thus, the proof of Theorem 4.1 is completed.

From Theorem 4.1 we arrive at the following results.

**Corollary 4.1.** *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(\beta)$, $\beta \in [0,1)$. *Then,*

$$|\delta_1| \le 1-\beta, \quad |\delta_2| \le \frac{1-\beta}{2}$$

*The inequalities obtained here are sharp.*

**Corollary 4.2.** *Let the function* $f(z)$ *given by (1.1) be in the class* $C(\beta)$, $\beta \in [0,1)$. *Then,*

$$|\delta_1| \le \frac{1-\beta}{2}, \quad |\delta_2| \le \frac{(1-\beta)(3-\beta)}{12}$$

*The inequalities obtained here are sharp.*

**Corollary 4.3.** Let *the function* $f(z)$ *given by (1.1) be in the class* $M_\alpha$, $\alpha \in [0,1]$. *Then,*

$$|\delta_1| \le \frac{1}{1+\alpha}, \quad |\delta_2| \le \frac{\alpha^2+4\alpha+1}{2(1+2\alpha)(1+\alpha)^2}$$

*The inequalities obtained here are sharp.*

**Corollary 4.4.** *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*$. *Then,*

$$|\delta_n| \le \frac{1}{n}, \quad n = 1, 2.$$

*Tthe inequalities obtained here are sharp.*

**Corollary 4.5.** *Let the function* $f(z)$ *given by (1.1) be in the class* $C$. *Then,*

$$|\delta_n| \le \frac{1}{2n}, \quad n = 1, 2.$$

*All the inequalities obtained here are sharp.*

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