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Research Article

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Application of the Chebyshev polynomials to coefficient estimates of analytic functions

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Abstract In this paper, making use of the Chebyshev polynomials, we introduce and investigate new subclasses of the analytic functions in the open unit disk in the complex plane. Here, obtained upper bound estimates for the initial second coefficients of the functions belonging to these classes.

Keywords Analytic function, univalent function, coefficient bound, Chebyshev polynomials
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1. Introduction and preliminaries

Let A denote the class of all complex valued functions $f(z)$ normalized by $f(0)=0=f'(0)-1$ and given by

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ \ a_n\in\mathbb{C} \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}\ :\ |z|<1\}$ in the complex plane .

Furthermore, let S be the class of all functions in A which are univalent in U .

Some of the important and well-investigated subclasses of S are classes S^* and C given below (see also [2, 3, 6]) such that S^* is the class of starlike functions and C is the class of convex functions

$$S^*=\left\{f\in S:\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0, z\in U\right\}$$

and

$$C=\left\{f\in S:\ \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0, z\in U\right\}.$$

An analytic function f is subordinate to an analytic function ϕ and written $f(z)\prec\phi(z)$, provided that there is an analytic function (that is, Schwarz function) ω defined on U with $\omega(0)=0$ and $|\omega(z)|<1$ satisfying $f(z)=\phi(\omega(z))$. Ma and Minda [4] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general function. For this purpose, they considered an analytic function ϕ with positive real part in U, $\phi(0)=1$, $\phi'(0)>0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-

Minda starlike and Ma-Minda convex functions consists of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$, respectively.

Chebyshev polynomials, which are used by us in this paper, play a considerable act in numerical analysis and mathematical physics. It is well-known that the Chebyshev polynomials are four kinds. The most of research articles related to specific orthogonal polynomials of Chebyshev family, contain essentially results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$, and their numerous uses in different applications (see [1, 5]).

The well-known kinds of the Chebyshev polynomials are the first and second kinds. In this paper, we will use second kind Chebyshev polynomials to investigation of the coefficient estimates of the analytic functions.

It is well-known that, in the case of real variable x on $(-1,1)$, the second kinds of the Chebyshev polynomials are defined by

$$U_n(x)=\frac{\sin\left[(n+1)\arccos x\right]}{\sin(\arccos x)}=\frac{\sin\left[(n+1)\arccos x\right]}{\sqrt{1-x^2}}.$$

We consider the function

$$G(t,z)=\frac{1}{1-2tz+z^2},\ t\in(1/2,1), z\in U.$$

It is well-known that if $t=\cos\alpha,\ \alpha\in(0,\pi/3)$, then

$$G(t,z)=1+\sum_{n=1}^{\infty}\frac{\sin\left[(n+1)\alpha\right]}{\sin\alpha}z^n =$$
$$1+2\cos\alpha z+\left(3\cos^2\alpha-\sin^2\alpha\right)z^2+\left(8\cos^3\alpha-4\cos\alpha\right)z^3+\cdots,\ z\in U.$$

That is,

$$G(t,z)=1+U_1(t)z+U_2(t)z^2+U_3(t)z^3+\cdots,\ t\in(1/2,1), z\in U, \tag{1.2}$$

where $U_n(t),\ n\in\mathbb{N}$ are the second kind Chebyshev polynomials.

It is clear that $G(t,0)=0$ and $G_z'(t,0)>0$

From the definition of the second kind Chebyshev polynomials, we easily obtain that $U_0(t)=1$, $U_1(t)=2t$, $U_2(t)=4t^2-1$. Also, it is well-known that

$$U_{n+1}(t)=2tU_n(t)-U_{n-1}(t) \tag{1.3}$$

for all $n\in\mathbb{N}-\{1\}$.

Inspired by the aforementioned works, making use of the Chebyshev polynomials, we define a subclass of univalent functions as follows.

Definition 1.1. *A function $f\in S$ given by (1.1) is said to be in the class $\aleph(G;\beta,t)$, $\beta\geq 0, t\in(1/2,1)$, where G is an analytic function given by (1.2), if the following condition is satisfied*

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\left[1+\frac{zf''(z)}{f'(z)}\right]\prec G(t,z),\ z\in U.$$

Remark 1.1. *Taking $\beta=0$, we have the function class $\aleph(G;0,t)\equiv S^*(G;t),\ t\in(1/2,1)$; that is,*

$$f \in S^*(G;t) \Leftrightarrow \frac{zf'(z)}{f(z)} \prec G(t,z),\ z \in U.$$

Remark 1.2. Taking $\beta = 1$, we have the function class $\aleph(G;1,t) \equiv C(G;t),\ t \in (1/2,1)$; that is,

$$f \in C(G;t) \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} \prec G(t,z),\ z \in U.$$

Note 1.1. As you can see that the class $\aleph(G;\beta,t)$, defined by Definition 1.1, is a generalization of the Ma-Minda starlike and Ma-Minda convex functions such that in the special case for $\beta = 0$ and $\beta = 1$ the classes $\aleph(G;0,t)$ and $\aleph(G;1,t)$ are Ma-Minda starlike and Ma-Minda convex functions, respectively, when there function $\phi(z)$ is $G(t,z)$ for fixed value $t \in (1/2,1)$.

In this paper, making use of the Chebyshev polynomials, we introduce and investigate new subclasses $\aleph(G;\beta,t), \beta \ge 0, t \in (1/2,1)$, $S^*(G;t), t \in (1/2,1)$ and $C(G;t), t \in (1/2,1)$ of the analytic functions in the open unit disk in the complex plane. Here, we will obtain upper bound estimates for the initial second coefficients of the functions belonging to these classes.

To prove our main results, we shall require the following well known lemma.

Lemma 1.1. ([2]) *Let* P *be the class of all analytic functions* $p(z)$ *of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

satisfying $\mathrm{Re}(p(z)) > 0,\ z \in U$ and $p(0) = 1$. *Then,* $|p_n| \le 2,$ for every $n = 1, 2, 3, \ldots$. *These inequalities is sharp for each* n.

Moreover,

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some $x,\ z$ with $|x| \le 1,\ |z| \le 1$.

2. Upper bound estimates for the coefficients

In this section, we prove the following theorem on upper bound estimates for the coefficients of the functions belonging to the class $\aleph(G;\beta,t)$.

Theorem 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $\aleph(G;\beta,t),\ \beta \in [0,1], t \in (1/2,1)$. *Then,*

$$|a_2| \le \frac{2t}{1+\beta} \quad and \quad |a_3| \le \frac{4(\beta^2 + 5\beta + 2)t^2 - (1+\beta)^2}{2(1+\beta)^2(1+2\beta)}.$$

Proof. Let $f \in \aleph(G;\beta,t),\ \beta \ge 0, t \in (1/2,1)$. Then, according to Definition 1.1, there is an analytic function $\omega : U \to U$ with $\omega(0) = 0,\ |\omega(z)| < 1$ satisfying the following condition

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\left[1+\frac{zf''(z)}{f'(z)}\right]=G(t,\omega(z)),\ z\in U. \tag{2.1}$$

It follows that

$$1+(1+\beta)a_2z+\left[2(1+2\beta)a_3-(1+3\beta)a_2^2\right]z^2+\cdots=G(t,\omega(z)). \tag{2.2}$$

Let the function $p\in \mathrm{P}$ be define as follows

$$p(z):=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+\cdots=1+\sum_{n=1}^{\infty}p_nz^n.$$

From this, we have

$$\omega(z):=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+\cdots\right]. \tag{2.3}$$

Taking $z\equiv\omega(z)$ in (1.2), we get

$$G(t,\omega(z))=1+\frac{U_1(t)}{2}p_1z+\left[\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2\right]z^2+\cdots. \tag{2.4}$$

Thus, by substituting the expression $G(t,\omega(z))$ in (2.2), we can easily write

$$\begin{aligned}&1+(1+\beta)a_2z+\left[2(1+2\beta)a_3-(1+3\beta)a_2^2\right]z^2+\cdots=\\&1+\frac{U_1(t)}{2}p_1z+\left[\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2\right]z^2+\cdots.\end{aligned} \tag{2.5}$$

Comparing the coefficients of the like power of z in both sides of (2.5), we have

$$(1+\beta)a_2=\frac{U_1(t)}{2}p_1, \tag{2.6}$$

$$2(1+2\beta)a_3-(1+3\beta)a_2^2=\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2. \tag{2.7}$$

It follows that

$$a_2=\frac{U_1(t)}{2(1+\beta)}p_1, \tag{2.8}$$

$$a_3=\frac{(1+3\beta)U_1^2(t)+(1+\beta)^2U_2(t)}{8(1+\beta)^2(1+2\beta)}p_1^2+\frac{U_1(t)}{4(1+2\beta)}\left(p_2-\frac{p_1^2}{2}\right). \tag{2.9}$$

Since $|p_1|\le 2$, from (2.8) we have

$$a_2\le\frac{2t}{1+\beta}. \tag{2.10}$$

From the Lemma 1.1, we write

$$p_2-\frac{p_1^2}{2}=\frac{(4-p_1^2)x}{2} \tag{2.11}$$

for some x with $|x|\le 1$.

Substituting the expression (2.11) in (2.9) and using triangle inequality, letting $|x| = \xi$ and $\tau = |p_1|$ for the upper bound estimate of the $|a_3|$ we obtain

$$|a_3| \le c_1(t,\tau)\xi + c_2(t,\tau), \tag{2.12}$$

where

$$c_1(t,\tau) = \frac{U_1(t)(4-\tau^2)}{8(1+2\beta)},\ c_2(t,\tau) = \frac{(1+3\beta)U_1^2(t) + (1+\beta)^2 U_2(t)}{8(1+\beta)^2(1+2\beta)}\tau^2,$$
$$\xi \in [0,1], t \in (1/2,1), \tau \in [0,2].$$

Since $c_1(t,\tau) \ge 0$ and $c_2(t,\tau) \ge 0$ for all $t \in (1/2,1)$ and $\tau \in [0,2]$, from (2.12) we have

$$|a_3| \le c_1(t,\tau) + c_2(t,\tau);$$

that is,

$$|a_3| \le A(t,\beta)\tau^2 + B(t,\beta), \tag{2.13}$$

where

$$A(t,\beta) = \frac{(1+3\beta)U_1^2(t)}{8(1+\beta)^2(1+2\beta)} + \frac{U_2(t)}{8(1+2\beta)} - \frac{U_1(t)}{8(1+2\beta)},\ B(t,\beta) = \frac{U_1(t)}{2(1+2\beta)}.$$

Let the function $\varphi : [0,2] \to \mathbb{R}$ defined as follows:

$$\varphi(\tau) = A(t,\beta)\tau^2 + B(t,\beta), \tau \in [0,2] \tag{2.14}$$

for some $t \in (1/2,1)$ and $\beta \in [0,1]$. Since $1+3\beta \ge (1+\beta)^2$ and $U_1^2(t) + U_2(t) - U_1(t) = 8t^2 - 2t - 1 > 0$ for all $\beta \in [0,1]$ and $t \in (1/2,1)$, respectively, we can easily see that $A(t,\beta) > 0$. Also, it is clear that $B(t,\beta) > 0$ for all $\beta \in [0,1]$ and $t \in (1/2,1)$. Hence, the function $\varphi(\tau)$ is a strictly increasing function on the closed interval $[0,2]$.

Therefore, the maximum of the function $\varphi(\tau)$ occurs at $\tau = 2$, and

$$\max\{\varphi(\tau) : \tau \in [0,2]\} = \varphi(2) = \frac{(1+3\beta)U_1^2(t)}{2(1+\beta)^2(1+2\beta)} + \frac{U_2(t)}{2(1+2\beta)}. \tag{2.15}$$

Thus, from (2.12) - (2.15), we have

$$|a_3| \le \frac{4(\beta^2 + 5\beta + 2)t^2 - (1+\beta)^2}{2(1+\beta)^2(1+2\beta)}. \tag{2.16}$$

With this, the proof of Theorem 2.1 is completed.

Setting $\beta = 0$ and $\beta = 1$ in Theorem 2.1, we can readily deduce the following results, respectively.

Corollary 2.1. *Let the function* $f(z)$ *given by (1.1) be in the function class* $\aleph(G;0,t) \equiv S^{*}(G;t)$, $t \in (1/2,1)$. *Then,*

$$|a_2| \leq 2t \quad \text{and} \quad |a_3| \leq \frac{8t^2-1}{2}.$$

Corollary 2.2. *Let the function* $f(z)$ *given by (1.1) be in the function class* $\aleph(G;1,t) \equiv C(G;t)$, $t \in (1/2,1)$. *Then,*

$$|a_2| \leq t \quad \text{and} \quad |a_3| \leq \frac{8t^2-1}{6}.$$

Remark 2.1. Using the same technique, this work can be done also for all $\beta \geq 0$. Using this work, we can be examined $|a_3 - \mu a_2^2|$ the Fekete - Szegö problem for the coefficients of the function class $\aleph(G;\beta,t)$. Moreover, using the same technique, we can find the coefficient a_4, and thus we can find upper bound estimate for $|a_4|$. So, using this work we can be find $H_2(2) = a_2 a_4 - a_3^2$ and upper bound estimate for the second Hankel determinant $|a_2 a_4 - a_3^2|$ for the functions belonging in the class $\aleph(G;\beta,t)$.

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