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Research Article

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Global Stability of Disease: Free Equilibrium of a Model of Tuberculosis

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Abstract The global asymptotic stability for the disease-free equilibrium (DFE) of a mathematical model for tuberculosis (TB) infection is obtained by constructing a suitable Lyapunov function and LaSalle's invariance principle. We define a number called the basic reproduction number (R_O) and show that this number determines the global dynamics of the system. In particular, it was shown that the DFE is globally asymptotically stable (GAS) if $R_O < 1$. Numerical simulation was provided to illustrate the result.

Keywords tuberculosis, global stability, Lyapunov function, numerical simulation, disease- free equilibrium points

1. Introduction

Tuberculosis (TB) is an infectious disease of the lung caused by a bacteria known as Mycobacterium tuberculosis. Symptoms of TB include loss of appetite, fever, loss of weight, night sweats and chest pain [1]. Although, TB is currently well controlled in most countries, recent studies show that TB is rising in Africa, Eastern Europe and Asia due to emergence of multi-drug resistance TB, improper use of anti-biotics and HIV/TB co-infection [2].

Global stability for the disease-free and endemic equilibrium of mathematical models for infectious diseases have been reported in the literature [3-6].

Usually, the disease-free equilibrium is globally asymptotically stable when $R_O < 1$ and the endemic equilibrium is global asymptotically stable when $R_0 > 1$ [7].

The aim of this paper is to prove the global asymptotic stability for the disease-free equilibrium of a tuberculosis model using a linear Lyapunov function and LaSalle's invariance principle [8].

2. The Model and Preliminaries

The model considered for the transmission dynamics of tuberculosis in this paper is given by

$$\left.\begin{aligned} S' &= (1-\gamma)\pi - \beta IS - \mu S \\ E' &= (1-\rho)\beta IS - (\mu + v)E \\ I' &= d\rho\beta IS - (\mu + \mu T + s)I \\ R' &= sI - \beta IR - \mu R \end{aligned}\right\} \tag{1}$$

The model description of variables and parameters are given in Table 1

Table 1: Model Variables and Parameters

Variables/Parameters	Definitions
S	number of susceptible who do not have the disease yet but could get it
E	Number of exposed who have the disease but are yet to show any sign of symptoms
I	number of infected who have the disease and could transmit it to others

R	number of recovered or removed who cannot get the disease or transmit
π	recruitment rate of susceptible individuals
β	transmission rate of TB
μ	natural death rate
μT	death rate due to TB
ρ	rate of fast progression
v	rate of slow progression
d	detection rate of TB
s	treatment rate of TB
γ	proportion of recruitment due to migration

By using the next generation matrix approach formulated by Diekmann et al., [9], the basic reproduction number of our model is

$$R_0 = \frac{d\beta\pi(1-\gamma)}{\mu(\mu+\mu_T+s)} \quad (2)$$

The disease-free equilibrium of the system is obtained as follows:
In a DFE, there is no infection in the population, so equation (1) becomes

$$\left.\begin{aligned} S' &= (1-\gamma)\pi - \mu S \\ R' &= -\mu R \end{aligned}\right\} \quad (3)$$

Since $S' = R' = 0$ is necessary for an equilibrium. Then, solving, we obtain

$$P_0 = \left(\frac{(1-\gamma)\pi}{\mu}, 0, 0, 0\right) \quad (4)$$

In the absence of tuberculosis disease, the population size converges to the $DFE\,\frac{(1-\gamma)\pi}{\mu}$.

We thus study the model in the following region for stability of the DFE

$$\Omega = \left\{ \begin{aligned} &(S,E,I,R) \in \Re_+^4 : S \geq 0, E \geq 0, I \geq 0, \\ &R \geq 0, S+E+I+R \leq \frac{(1-\gamma)\pi}{\mu} \end{aligned} \right\} \quad (5)$$

3. Global Stability of the DFE

Theorem 1: If $R_0 < 1$, then the DFE P_0 of the model is globally asymptotically stable (GAS) in Ω.

Proof

The variable S does not appear in the 1st term of susceptible compartment. By dropping this term, equation (1) reduces to

$$\left.\begin{aligned} S' &= -\beta IS - \mu S \\ E' &= (1-\rho)\beta IS - (\mu+v)E \\ I' &= d\rho\beta IS - (\mu+\mu_T+s)I \\ R' &= sI - \beta IR - \mu R \end{aligned}\right\} \quad (6)$$

We analyse the following reduced system for stability of the DFE.
Define a Lyapunov function

$$W(S,E,I,R) = fS + gE + hI + iR \quad (7)$$

where *f*, *g*, *h* and *i* are all positive constants.
Taking the derivative of *W* and substituting (6) gives

$$\frac{dW}{dt} = fS' + gE' + hI' + iR' \quad (8)$$

$$\begin{aligned} &= f(-\beta IS - \mu S) + g[(1-\rho)\beta IS - (\mu+v)E] \\ &\quad + h[d\rho\beta IS - (\mu+\mu_T+s)I \\ &\quad + i(sI - \beta IR - \mu R) \end{aligned} \quad (9)$$

$$\begin{aligned} &= -(f\mu)S - (f\beta + g\beta + g\rho\beta + hd\rho\beta)IS \\ &\quad - g(\mu+v)E - [h(\mu+\mu T+s) - iS]I \\ &\quad - i\beta IR - i\mu R \end{aligned} \quad (10)$$

$$\begin{aligned} &-(f\mu)S - (f\beta + g\beta + g\rho\beta + hd\rho\beta)IS \\ &\quad - g(\mu+v)E - hR_0\left(\frac{d\beta\pi(1-\gamma)}{\mu} - \frac{1}{R_0}is\right)I \\ &\quad - i\beta IR - i\mu R \end{aligned} \quad (11)$$

Since the model monitors human population, $\mu > 0, \beta > 0, d > 0, v > 0, \rho > 0, \pi > 0, \gamma > 0$. It follows then that

$$\left.\begin{aligned} &-f\mu < 0 \\ &-(f\beta + g\beta + g\rho\beta + hd\rho\beta) < 0 \\ &-g(\mu+v) < 0 \\ &-i\beta < 0 \\ &-i\mu < 0 \end{aligned}\right\} \quad (12)$$

Since $R_0 < 1$ in (11) implies that $-hR_0\left(\frac{d\beta\pi(1-\gamma)}{\mu} - \frac{1}{R_0}is\right) < 0$. Then, from (12), it is evident that $\frac{dW}{dt} < 0 \text{ in } \Omega$. Furthermore, $\frac{dW}{dt} = 0$ only if $S = E = I = R = 0$. Hence, the maximum invariant set in $\left\{(S,E,I,R) : \frac{dW}{dt} = 0\right\}$ is the single t on $\{P_0\}$. By LaSalle's invariance principle, every solution of equation (6) with initial conditions in Ω tends to DFE P_0 as t→ ∞. Hence, P_0 is globally asymptotically stable in the invariant region Ω if $R_0 < 1$.

4. Numerical Simulation

To validate the result in section 3, we calculate the basic reproduction number using Table 2 and show that it is less than one.

Table 2: Parameter/Variable and their Assigned Values

Variables/Parameters	Assigned Values
S	5
I	2
γ	0.14
π	0.1
μ	0.1
ρ	0.01
β	0.002
d	0.24
s	015

Using equation (2), we calculate $R_0 = 0.4357 < 1$. Consequently, the disease dies out with time and infection is cleared in the population.

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