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Mathematical analysis of two epidemic models with temporary immunity

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Abstract The present paper present a nonlinear mathematical model, which analyzes the spread and stability of the model epidemic. In the first model a population of size $N(t)$ at time t, is divided into three subclasses, where $S(t)$, $I(t)$, and $Q(t)$ denote the sizes of the population susceptible to disease, and infectious members, quarantine members, then the second model we introduce two classes $I_1(t)$, and $I_2(t)$.
This paper deals with the equilibrium and stability, for the two models.
Subject classification: 34D23, 92D30

Keywords Basic reproduction number, endemic equilibrium, epidemic model, temporary immunity

1. Introduction

This paper, discuss the equilibrium and stability of two non linear models with temporary immunity and the different positives parameters. We have made the following contributions:

- The equilibrium and stability of the firts model, we obtain a disease-free equilibrium in the absence of infection but in the presence of infection, it was a unique positive endemic equilibrium and we define the basic reproduction number of the infection R_0.
- The equilibrium and stability of the first model with age, we obtain a the unique positive equilibrium point.
- Next, we modified the previous model, then we have the population is divided into five subclasses, we study the equilibrium and stability of the model and we define the basic reproduction number of the infection $(R_0)_1$.
- Finally we find the relationship betwen the basic reproduction number of both epidemic models.

2. SIQ Model

This paper considers the following epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) = \rho + \mu - \nu - (\mu_1 + d)S(t) - (\beta + k)S(t)Q(t), \\ \dot{I}(t) = \beta S(t)Q(t) - (\mu_2 + d)I(t) - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau), \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d)Q(t), \end{cases} \quad (1.1)$$

Consider a population of size $N(t)$ at time t, this population is divided into three subclasses, with $N(t) = S(t) + I(t) + Q(t)$; where $S(t)$, $I(t)$, and $Q(t)$ denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary

immunity, respectively. The positive constants μ_1, μ_2, and μ_3 represent the death rates of susceptible, infectious and quarantine. Biologically, it is natural to assume that $\mu_1 \le \min\{\mu_2, \mu_3\}$. The positive constant d is natural mortality rate. The positive constants μ represent rate of insidence. The positive constant γ represent the recovery rate of infection. The positive constant β is the average numbers of contacts infective for S and I. k the rate of unknown persons infected with are detected by the system. ρ the positive constant is the parameter of immigration. ν the positive constant is the parameter of emigration. The term $\gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau)$ reflects the fact that an individual has recovered from infection and still are alive after infectious period τ, where τ is the length of immunity period.

The initial condition of (1.1) is given as.

$$S(\eta) = \Phi_1(\eta), I(\eta) = \Phi_2(\eta), Q(\eta) = \Phi_3(\eta), \qquad -\tau \le \eta \le 0, \tag{1.2}$$

Where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \in C$ such that $S(\eta) = \Phi_1(\eta) = \Phi_1(0) \ge 0$, $I(\eta) = \Phi_2(\eta) = \Phi_2(0) \ge 0$, $Q(\eta) = \Phi_3(\eta) = \Phi_3(0) \ge 0$.

Let C denote the Banach space $C([-\tau, 0], \mathbb{R}^3)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3$. With a biological meaning, we further assume that $\Phi_i(\eta) = \Phi_i(0) \ge 0$ for $i = 1,2,3$.

$$\begin{cases} \dot{S}(t) = \rho + \mu - \nu - (\mu_1 + d)S - (\beta + k)SQ, \\ \dot{I}(t) = (\beta + k)SQ - (\mu_2 + d)I - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau), \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d)Q, \end{cases} \tag{1.3}$$

With the initial conditions.

$$S(\eta) = \Phi_1(\eta), I(\eta) = \Phi_2(\eta), Q(\eta) = \Phi_3(\eta), \qquad -\tau \le \eta \le 0, \tag{1.4}$$

Where $\Phi_1(0) \ge 0, \Phi_2(0) \ge 0, \Phi_3(0) \ge 0, \qquad -\tau \le \eta \prec 0.$

The region $\Omega = \{(S, I, Q) \in \mathbb{R}^3_+, \ S + I + Q \le N < \dfrac{\rho + \mu - \nu}{\mu_1 + d}\}$ is positively invariant set of (3).

2.1. Equilibrium and stability

An equilibrium point of system (1.3) satisfies

$$\begin{cases} \rho + \mu - \nu - (\mu_1 + d)S - (\beta + k)SQ = 0, \\ (\beta + k)SQ - (\mu_2 + d)I - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) = 0, \\ \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d)Q = 0, \end{cases} \tag{1.5}$$

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection $I = 0$, the system (1.5) has a disease-free equilibrium E_0:

$$E_0 = (\hat{S}, \hat{I}, \hat{Q})^T = \left(\frac{\rho + \mu - \nu}{\mu_1 + d}, 0, 0 \right)^T.$$

The eigenvalues can be determined by solving the characteristic equation of the linearization of (1.3) near E_0 is

$$\det \begin{pmatrix} ccc -(\mu_1 + d) - A & 0 & -\dfrac{(\beta + k)(\rho + \mu - \nu)}{\mu_1 + d} \\ 0 & -(\mu_2 + d) - A & \dfrac{(\rho + \mu - \nu)(\beta + k - \gamma e^{-\mu_2 \tau})}{\mu_1 + d} \\ 0 & 0 & \dfrac{(\rho + \mu - \nu)\gamma e^{-\mu_2 \tau}}{\mu_1 + d} - (\mu_3 + d) - A \end{pmatrix} = 0 \tag{1.6}$$

So the eigenvalues are

$$A_1 = -(\mu_1 + d), A_2 = -(\mu_2 + d).$$

In order for A_1, A_2, to be negative, it is required that.

$$\frac{(\rho + \mu - \nu)\gamma e^{-\mu_2 \tau}}{\mu_1 + d} < \mu_3 + d \tag{1.7}$$

Then the basic reproduction number of the infection R_0 as follows.

$$R_0 = \frac{(\rho + \mu - \nu)\gamma e^{-\mu_2 \tau}}{(\mu_1 + d)(\mu_3 + d)} \tag{1.8}$$

In the presence of infection $I \neq 0$, substituting in the system, Ω also contains a unique positive, endemic equilibrium $E_\tau^* = (S_\tau^*, I_\tau^*, Q_\tau^*)^T$ where

$$\begin{cases} S_\tau^* = \dfrac{\rho + \mu - \nu}{\mu_1 + d} \times \dfrac{1}{R_0}, \\ I_\tau^* = \dfrac{R_0 - 1}{\mu_2 + d}\left[\dfrac{\rho + \mu - \nu}{R_0} - \dfrac{(\mu_1 + d)(\mu_3 + d)}{\beta}\right], \\ Q_\tau^* = \dfrac{\mu_1 + d}{\beta + k}(R_0 - 1) \end{cases} \tag{1.9}$$

So $E_\tau^* = (S_\tau^*, I_\tau^*, Q_\tau^*)^T$ is the unique positive endemic equilibrium point which exists if $R_0 > 1.$

Theorem 1 *The disease-free equilibrium* E_0 *is locally asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1.$

Theorem 2 *With* $R_0 > 1,$ *system* (1.3) *has a unique non-trivial equilibrium* E_τ^* *is locally asymptotically stable.*

3. SIR Model with Age

The age distributions of the numbers in the classes are denoted by $S(a,t)$, $I(a,t)$, and $Q(a,t)$, denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively of age $a,$ at time t, $d(a)$ is the age-specific death rate, The system of partial equations for the age distributions is

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{\partial S}{\partial a} = -(\mu_1 + d(a))S(a,t) + \beta_1(t)S(a,t), \\ \dfrac{\partial I}{\partial t} + \dfrac{\partial I}{\partial a} = -\beta_1(t)S(a,t) - (\mu_2 + d(a))I(a,t) + \gamma_1(t-\tau)S(a,t-\tau), \\ \dfrac{\partial Q}{\partial t} + \dfrac{\partial Q}{\partial a} = -\gamma_1(t-\tau)S(a,t-\tau) - (\mu_3 + d(a))Q(a,t), \end{cases} \tag{1.10}$$

With

$$\beta_1(t) = -(\beta + k)Q(a,t)da1.11 \tag{1}$$

$$\gamma_1(t-\tau) = -\gamma e^{-\mu_2 \tau} Q(a,t-\tau)da$$

3.1. Equilibrium and stability

Assume that sub population does not depend on the time when the system (1.10) is written as follows

$$\begin{cases} \dfrac{dS}{da} = (\beta_1 - \mu_1 - d(a))S(a), \\ \dfrac{dI}{da} = (\gamma_1 - \beta_1)S(a) - (\mu_2 + d(a))I(a), \\ \dfrac{dQ}{da} = -\gamma_1 S(a) - (\mu_3 + d(a))Q(a), \end{cases} \tag{1.12}$$

The initial condition of (1.12) is given as

$$S(0) = S_1,\ I(0) = I_1,\ Q(0) = Q_1 \tag{1.13}$$

Differential equations of the system (1.12) are solved with different methods of resolutions and with (1.13), so

$$S(a) = S_1 e^{-(\mu_1 - \beta_1)a}\Phi(a), \tag{1.14}$$

$$I(a) = I_1\Phi(a)e^{-\mu_2 a} - \frac{(\gamma_1 - \beta_1)S_1\Phi(a)}{\mu_1 - \beta_1 - \mu_2}\left(e^{-(\mu_1 - \beta_1)a} - e^{-\mu_2 a}\right), \tag{1.15}$$

$$Q(a) = Q_1\Phi(a)e^{-\mu_3 a} - \frac{\gamma_1 S_1\Phi(a)}{\mu_1 - \beta_1 - \mu_3}\left(e^{-(\mu_1 - \beta_1)a} - e^{-\mu_3 a}\right) \tag{1.16}$$

Where

$$\Phi(a) = \exp\left(-d(a)da\right) \tag{1.17}$$

The system (1.12) has the unique positive equilibrium point P_1,

$$P_1 = \left(\hat{S}_1, \hat{I}_1, \hat{Q}_1\right)^T = (0,0,0)^T.$$

We calculate the Jacobian matrix according to the system (12) with P_1

$$J(P_1) = [ccc\beta_1 - \mu_1 - d(a)00\lambda - \gamma_0 - (\mu_2 + d(a))0 - \gamma_0 0 - (\mu_3 + d(a))]$$

The epidemic is locally asymptotically stable if and only if all eigenvalues of the Jacobian matrix $J(P_1)$ have negative real part. The eigenvalues can be determined by solving the characteristic equation of the linearization of (13) near P_1 is

$$\det(ccc\beta_1 - \mu_1 - d(a) - A00\lambda - \gamma_0 - (\mu_2 + d(a)) - A0 - \gamma_0 0 - (\mu_3 + d(a)) - A) = 0 \tag{1.18}$$

So the eigenvalues are

$$A_1 = \beta_1 - \mu_1 - d(a),\ A_2 = -(\mu_2 + d(a)),\ A_3 = -(\mu_3 + d(a))$$

In order to $A_1, A_2,$ and A_3 will be negative, it is required that

$$\beta_1 < \mu_1 + d(a)$$

The basic reproduction number R_0 is defined as the total number of infected population in the resulting sub-infected population where almost all of the uninfected. The basic reproduction number of the infection R_0 is defined as follows:

$$R_0 = \frac{\beta_1}{\mu_1 + d(a)} \tag{1.19}$$

The time during which people remain infective is defined as

$$T = \frac{1}{\mu_1 + d(a)}$$

The doubling time t_d of the epidemic can be obtained as

$$t_d = \frac{(\ln 2)T}{R_0 - 1} \tag{1.20}$$

Theorem The disease-free equilibrium P_1 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1.$

Proof

Let (14), so if $R_0 < 1$ then $\mu_1 - \beta_1 > 0,$ so $S(a)$ converges to zero.

Let (15), so

$$I(a) \le \left[I_1\Phi(a) - \frac{(\gamma_1 - \beta_1)S_1\Phi(a)}{\mu_1 - \beta_1 - \mu_2} \right] e^{-m_1 a}, m_1 = \min\{\mu_1 - \beta_1, \mu_2\} \tag{1.21}$$

If $R_0 < 1,$ $I(a)$ converges to zero.

Let (17), so

$$Q(a) = \left[Q_1\Phi(a) - \frac{\gamma_1 S_1\Phi(a)}{\mu_1 - \beta_1 - \mu_3} \right] e^{-m_2 a}, m_1 = \min\{\mu_1 - \beta_1, \mu_3\} \tag{1.22}$$

If $R_0 < 1,$ $Q(a)$ converges to zero.

4. Modified SIQ Model

This paper considers the modified epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) = \rho + \mu - \nu - (\mu_1 + d)S(t) - (\beta + k)S(t)Q(t), \\ \dot{I}(t) = \beta S(t)Q(t) - (\mu_2 + d)I(t) - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau), \\ \dot{I}_1(t) = \alpha_1 I(t) - (\mu_2^1 + d)I_1(t) - \gamma e^{-\mu_2^1 \tau} S(t-\tau)Q(t-\tau), \\ \dot{I}_2(t) = \alpha_2 I(t) - (\mu_2^2 + d)I_2(t) - \gamma e^{-\mu_2^2 \tau} S(t-\tau)Q(t-\tau), \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d)Q(t), \end{cases} \tag{2.1}$$

The modified epidemic model is divided into five subclasses, $S(t),$ $I(t),$ $I_1(t),$ $I_2(t),$ and $Q(t)$ denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively. α_1 constant rate from I to I_1; and α_2 constant rate from I to I_2. The positive constants μ_2^1, μ_2^2 represent the death rates of $I_1, I_2; \mu_1 \le \min\{\mu_2, \mu_2^1, \mu_2^2, \mu_3\}$

The initial condition of (2.1) is given as.

$$S(\eta) = \Phi_1(\eta), I(\eta) = \Phi_2(\eta), I_1(\eta) = \Phi_3(\eta), I_2(\eta) = \Phi_4(\eta), Q(\eta) = \Phi_5(\eta), \quad -\tau \le \eta \le 0, \tag{2.2}$$

Where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5)^T \in C$ such that $S(\eta) = \Phi_1(\eta) = \Phi_1(0) \ge 0,$ $I(\eta) = \Phi_2(\eta) = \Phi_2(0) \ge 0, I_1(\eta) = \Phi_3(\eta) \ge 0, I_2(\eta) = \Phi_4(\eta)$ $Q(\eta) = \Phi_5(\eta) = \Phi_3(0) \ge 0.$

Let C denote the Banach space $C([-\tau, 0], \mathbb{R}^5)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^5$. With a biological meaning, we further assume that $\Phi_i(\eta) = \Phi_i(0) \ge 0$ for $i = 1,2,3,4,5.$

$$\begin{cases} \dot{S}(t) = \rho + \mu - \nu - (\mu_1 + d)S - (\beta + k)SQ, \\ \dot{I}(t) = \beta SQ - (\mu_2 + d)I - \gamma e^{-\mu_2^1 \tau} S(t-\tau)Q(t-\tau), \\ \dot{I}_1(t) = \alpha_1 I - (\mu_2^1 + d)I_1 - \gamma e^{-\mu_2^1 \tau} S(t-\tau)Q(t-\tau), \\ \dot{I}_2(t) = \alpha_2 I - (\mu_2^2 + d)I_2 - \gamma e^{-\mu_2^2 \tau} S(t-\tau)Q(t-\tau), \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d)Q, \end{cases} \quad (2.3)$$

With the initial conditions.

$$S(\eta) = \Phi_1(\eta), I(\eta) = \Phi_2(\eta), I_1(\eta) = \Phi_3(\eta), I_2(\eta) = \Phi_4(\eta), Q(\eta) = \Phi_5(\eta), \quad -\tau \le \eta \le 0, \quad (2.4)$$

Where $\Phi_1(0) \ge 0, \Phi_2(0) \ge 0, \Phi_3(0) \ge 0, \Phi_4(0) \ge 0, \Phi_5(0) \ge 0 \quad -\tau \le \eta \prec 0.$

The region $\Omega_1 = \{(S, I, I_1, I_2, Q) \in \mathsf{R}_+^5, \quad S + I + I_1 + I_2 Q \le N < \frac{\rho + \mu - \nu}{\mu_1 + d}\}$ is positively invariant set of (2.3).

4.1. Equilibrium and stability

An equilibrium point of system (2.3) satisfies

$$\begin{cases} \rho + \mu - \nu - (\mu_1 + d)S - (\beta + k)SQ = 0, \\ \beta SQ - (\mu_2 + d)I - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) = 0, \\ \alpha_1 I - (\mu_2^1 + d)I_1 - \gamma e^{-\mu_2^1 \tau} S(t-\tau)Q(t-\tau) = 0, \\ \alpha_2 I - (\mu_2^2 + d)I_2 - \gamma e^{-\mu_2^2 \tau} S(t-\tau)Q(t-\tau) = 0, \\ \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d)Q = 0, \end{cases} \quad (2.5)$$

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection $I = 0$, the system (2.5) has a disease-free equilibrium $(E_0)_1$:

$$(E_0)_1 = (\hat{S}, \hat{I}, \hat{I}_1, \hat{I}_2, \hat{Q})^T = \left(\frac{\rho + \mu - \nu}{\mu_1 + d}, 0, 0, 0, 0\right)^T. \quad (2.6)$$

The eigenvalues can be determined by solving the characteristic equation of the linearization of (2.3) near E_0. So the eigenvalues are

$$A_1 = -(\mu_1 + d), A_2 = -(\mu_2 + d + \alpha_1 + \alpha_2),$$
$$A_3 = -(\mu_2^1 + d), A_4 = -(\mu_2^2 + d)$$

In order for A_1, A_2, A_3, A_4 to be negative, then the basic reproduction number of the infection $(R_0)_1$ as follows.

$$(R_0)_1 = \frac{(\rho + \mu - \nu)}{(\mu_1 + d)(\mu_3 + d)}\left(\gamma e^{-\mu_2 \tau} + \gamma_1 e^{-\mu_2^1 \tau} + \gamma_2 e^{-\mu_2^2 \tau}\right) \quad (2.7)$$

In the presence of infection $I \neq 0$, substituting in the system, Ω_1 also contains a unique positive, endemic equilibrium $(E_\tau^*)_1 = (S_\tau^*, I_\tau^*, (I_1^*)_\tau, (I_2^*)_\tau, Q_\tau^*)^T$ where

$$\begin{cases} S_\tau^* = \dfrac{\mu_3 + d}{(\mu_1 + d)(R_0)_1}, \\ I_\tau^* = \left(1 - \dfrac{\gamma e^{-\mu_2 \tau}}{\beta + k}\right) \times \left(\dfrac{\rho + \mu - \nu}{\mu_2 + d + \alpha_1 + \alpha_2}\right) \times \left(\dfrac{(R_0)_1 - 1}{(R_0)_1}\right), \\ \left(I_1^*\right)_\tau = \dfrac{\alpha_{1I_\tau^*}}{\mu_2^1 + d} + \left(\dfrac{\gamma_1 e^{-\mu_2^1 \tau}}{\mu_2^1 + d}\right) \times \left(\dfrac{\rho + \mu - \nu}{\beta + k}\right) \times \left(\dfrac{(R_0)_1 - 1}{(R_0)_1}\right) \\ \left(I_2^*\right)_\tau = \dfrac{\alpha_{2I_\tau^*}}{\mu_2^2 + d} + \left(\dfrac{\gamma_2 e^{-\mu_2^2 \tau}}{\mu_2^2 + d}\right) \times \left(\dfrac{\rho + \mu - \nu}{\beta + k}\right) \times \left(\dfrac{(R_0)_1 - 1}{(R_0)_1}\right) \\ Q_\tau^* = \dfrac{\mu_1 + d}{\beta + k}\left((R_0)_1 - 1\right) \end{cases} \tag{2.8}$$

So $\left(E_\tau^*\right)_1 = \left(S_\tau^*, I_\tau^*, \left(I_1^*\right)_\tau, \left(I_2^*\right)_\tau, Q_\tau^*\right)^T$ is the unique positive endemic equilibrium point which exists if $R_0 > 1.$

Theorem 3 *R_0 is the basic reproduction number of system (2.3), and $(R_0)_1$ is the basic reproduction number of system (2.3), then*

$$R_0 = (R_0)_1 \times \left[\frac{\gamma e^{-\mu_2 \tau}}{\gamma e^{-\mu_2 \tau} + \gamma_1 e^{-\mu_2^1 \tau} + \gamma_2 e^{-\mu_2^2 \tau}}\right] \tag{2.9}$$

Proof

We have (1.8), then

$$\frac{(\rho + \mu - \nu)}{(\mu_1 + d)(\mu_3 + d)} = \frac{R_0}{\gamma e^{-\mu_2 \tau}} \tag{2.10}$$

We remplace (2.10) in (2.7) ,

$$(R_0)_1 = \frac{R_0}{\gamma e^{-\mu_2 \tau}}\left(\gamma e^{-\mu_2 \tau} + \gamma_1 e^{-\mu_2^1 \tau} + \gamma_2 e^{-\mu_2^2 \tau}\right)$$

$$\frac{(R_0)_1}{R_0} = \frac{\gamma e^{-\mu_2 \tau} + \gamma_1 e^{-\mu_2^1 \tau} + \gamma_2 e^{-\mu_2^2 \tau}}{\gamma e^{-\mu_2 \tau}}$$

$$\frac{R_0}{(R_0)_1} = \frac{\gamma e^{-\mu_2 \tau}}{\gamma e^{-\mu_2 \tau} + \gamma_1 e^{-\mu_2^1 \tau} + \gamma_2 e^{-\mu_2^2 \tau}}$$

Then

$$R_0 = (R_0)_1 \left(\frac{1}{1 + \frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1 - \mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2 - \mu_2\right)\tau}}\right) \tag{2.11}$$

Theorem 4 *The disease-free equilibrium* $(E_0)_1$ *is locally asymptotically stable if* $R_0 < \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right)$, *and if* $R_0 > \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right)$, *system* (2.3) *has a unique non-trivial equilibrium* $\left(E_\tau^*\right)_1$ *is locally asymptotically stable.*

Proof

1. We have $R_0 < \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right)$. With (2.11),

$$(R_0)_1 \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right) < \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right);$$

$$\left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right) > 0$$

Then

$$(R_0)_1 < 1$$

2. We have $R_0 > \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right)$. With (2.11),

$$(R_0)_1 \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right) > \left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right);$$

$$\left(\frac{1}{1+\frac{\gamma_1}{\gamma} e^{-\left(\mu_2^1-\mu_2\right)\tau} + \frac{\gamma_2}{\gamma} e^{-\left(\mu_2^2-\mu_2\right)\tau}} \right) > 0$$

Then

$$(R_0)_1 > 1$$

5. Conclusion

This paper addresses a the equilibrium and stability of the first epidemic model with temporary immunity, in the absence of infection, the system has a disease-free equilibrium, in the presence of infection the system, has a unique positive, endemic equilibrium. Then we study equilibrium of the first model with âge.

Both systems have the unique positive equilibrium point locally asymptotically stable if $R_0 < 1,\ (R_0)_1 < 1$ and has a unique non-trivial equilibrium is locally asymptotically stable, if $R_0 > 1, (R_0)_1 > 1.$

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