LEARNERS' VIEWS ON ASYMPTOTES OF A HYPERBOLA AND EXPONENTIAL FUNCTION: A COMMOGNITIVE APPROACH

Vimolan Mudaly, Sihlobosenkosi Mpofu
University of KwaZulu-Natal, Republic of South Africa
E-mail: mudalyv@ukzn.ac.za, sihlobosenkosi@gmail.com

Abstract

Learners in South African schools often respond poorly in questions related to the asymptote. Despite the fact that there are only a few functions in the South African curriculum that actually explore the asymptote, learners still show some deficiency in their understanding of the concept. This research examined Grade 11 learners' mathematical discourses about the asymptotes of the hyperbola and exponential functions. Data were analysed using the Realisation Tree of a Function, an adaptation of the Realisation Tree Assessment tool from Weingarden, Heyd-Metzuyanim and Nachlieli. While the Realisation Tree Assessment tool focused on teacher talk, the Realisation Tree of a Function focused on learner expression and responses. A qualitative research design was essentially adopted, with exploratory, descriptive and interpretive elements complementing both its data collection and analysis. A purposive sampling strategy was implemented. Data were collected by means of a test administered to a total of 112 Grade 11 participants from four selected secondary schools. Focus group interviews were conducted with 24 of the best-performing participants by using their responses from the written mathematical tests.
The results revealed that the learners' mathematical discourse is not coherent. While learners' work on each representation was often mathematical there seemed to be a struggle when the task had an unusual orientation. Different expressions of the same mathematical object elicited different responses. The challenge is that learners exhibited a fragmented relationship between the mathematical objects of the function.
Keywords: *commognition, realization tree, ritualised learning, visual mediators.*

Introduction

Mathematical functions are one of the most important topics in the South African national curriculum as signified by the prominence given to it by the Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011) in terms of the teaching hours allocated. However, national senior certificate results (Spaull, 2013; Taylor, 2011) showed that students still struggle with questions related to functions. The research examined the learners' interpretation of the mathematical object, the asymptote, from three algebraic representations of the exponential function and a hyperbola. While the mathematical object is the same there was a varying interpretation of the asymptote by students, signifying that reification and alienation (Sfard, 2012; Nachlieli & Tabach, 2012; Nachlieli & Tabach, 2018) of the mathematical object has not yet been fully realised. The partial objectification of the mathematical object, the asymptote, is responsible for the struggles that learners have in learning of functions. Grade 11 learners' responses to the three questions on the asymptote of a hyperbola and two on the exponential function were analysed. The asymptote from the hyperbola had the equation presented in a familiar classroom type format, the two exponential functions were presented in a somewhat

unfamiliar manner although the representations are within the realms of the curriculum to which the learners were exposed.

Commognition

The theory of commognition, as proposed by Sfard (2008), posits that learning of mathematics is about how the community of mathematics communicates in a language that is understood by the members of this group (Professional community of mathematicians). Mathematical discourse is composed of the words used to describe the mathematical objects, the symbols that signify the mathematical objects, the routines used in the expression of the mathematical objects and the endorsed narratives that are agreed by the professional mathematics community (Caspi & Sfard, 2012, Nachlieli & Tabach, 2012; Sfard, 2012). Thinking is communication with oneself (Sfard, 2008). Participants of the mathematical discourse show their internal communication mostly through what they say, write, draw or sketch. Communication is seen in both talk and action. In this research we analysed how learners responded to questions on the asymptotes of the hyperbola and exponential functions expressed in algebraic form. Learners' response to these questions raised many questions about *saming*, reification and alienation of the mathematical object, in this case the asymptote (Berger, 2013; Sfard, 2008). *Saming* is when participants of the mathematical discourse fail to distinguish between two or more similar mathematical objects, for example, an undefined portion of a graph is regarded as an asymptote by some learners [consider the function $f(x) = \frac{(x-2)}{(x^2-5x+6)}$ at the point $x = 2$ and $x = 3$]. While there were no struggles with drawing a dotted line by learners to denote the asymptote it is their interpretation of the algebraic expression of the asymptote that needs scrutiny. Algebra in mathematics is a generalisation and compression of arithmetic processes. The realization tree of the function is firstly expressed in word form, where statements (written or spoken) are drawn. Secondly it is expressed in the form of ordered pairs and for neatness and aesthetic pleasure mathematicians use a table of values. Thirdly, a function is represented in the form of a diagram on a plane and finally, an algebraic form represents the realization of a function.

The Asymptote

Endorsed narratives are agreed upon by the community of mathematicians, and the purpose is to have the same narratives applied within the community. Functions behave in distinctive ways under particular conditions. In quadratic and cubic functions, the common characteristics are the stationary points and intercepts when the x and y variables are equated to zero. Mathematicians generally define an asymptote of a curve as a line that is tangent to a curve at infinity. As the values of the curve increase, the curve and the asymptote approach each other. According to this definition, a curve may sometimes cross the horizontal asymptote. The vertical asymptote cannot intersect with the curve. While some functions such as $f(x) = \frac{x^2-3x}{x-1}$ will have graphs not crossing the asymptote, there are functions such as $g(x) = \frac{sinx}{x}$ in which the horizontal asymptote is crossed infinitely many times as the independent variable x tends to infinity.

In terms of the school curriculum definition, the asymptote is a line whose distance from the curve tends to zero as the independent variable (either x or y) tends to infinity. Grade 11 learners are not expected to learn limits. As such, the definition with limits was not suitable for them. At the same time, the proposition that the curve should not cross the asymptote is not

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mathematically correct and should therefore be avoided in the teaching and learning situation (Mpofu & Pournara, 2018). As stated in the paragraphs above, the horizontal asymptote may, or may not cross the curve of a graph. However, it should not mean that learners should be exposed to questions that show the asymptote crossing many times. What is emphasised here is for the community of mathematicians not to have contradictions in their endorsed narratives.

There are generally three kinds of linear asymptotes, a vertical asymptote, a horizontal asymptote, and an oblique asymptote. In addition, graphs may have a removable discontinuity. In a vertical asymptote, the gradient is undefined, and the asymptote is presented in the form of $x=a$ Furthermore, the vertical asymptote is parallel to, or coincides with the y-axis. In many instances, making a distinction between the vertical asymptote and the removable discontinuity poses a challenge to the learners (Berger, 2013). A function will have a vertical asymptote at the zero point of the denominator, provided the denominator is not a factor of the numerator. If the denominator is a factor of the numerator, the zero of the denominator is then the removable movable discontinuity.

An asymptote and a removable discontinuity occur when the denominator is zero. The difference between the two is that an asymptote cannot be redefined to make the function continuous at that point. On the other hand, there is a way of defining a function in removable discontinuities such that they are continuous. If a denominator of a rational function is zero at a certain value of x and the numerator is not zero, then there is a vertical asymptote at that point. In contradistinction, when both the numerator and the denominator are zero, there is a removable discontinuity (Berger, 2013). For example, $f(x) = \frac{4}{x-2}$ at $x=2$ would be $f(2) = \frac{4}{0;}$ and $h(x) = \frac{4(x-4)^2}{x-4}$ at $x=4$ would result in $h(4) = \frac{0}{0}$. In this case, the function f has a vertical asymptote, while the function h has a removable discontinuity. At grade 11, the curriculum does not require knowledge of this difference.

Challenges Concerning Learning Asymptotes

The official secondary school mathematics curriculum in South Africa presents learners with functions whose asymptotes are either parallel to the axes or coincide with the axes. The hyperbola and the exponential functions have asymptotes at the zero points of the denominator, and there is no intersection between the horizontal asymptote and the graphs. The reviewed literature showed that learners tended to struggle with the concept of the asymptote (Flesher, 2003; Kidron, 2011; Mpofu & Pournara, 2018).

A study by Flesher (2003) found that learners' asymptote-related difficulties were universal. In her cognitively inclined study, Flesher (2003) focused on college learners who were expected to describe the meaning of graphical representations in their own words. Some of the findings of this study indicated that learners' mathematical discourse on the asymptote was at a granular stage. Flesher's study was cognitive in nature, hence her usage of the term 'conceptual misunderstanding' because she found that the students saw the asymptote as a number and not as a line. This most probably arose from the algebraic calculations rather than from the graphs. Some of the learners in the study struggled to position the asymptote, stating that the asymptote can only coincide with the axes. This view emanates from an asymptote of parent functions, in which there are no vertical or horizontal translations. Yet, some learners could not define the horizontal and and/or vertical asymptotes after correctly calculating the position of the asymptotes on the Cartesian plane.

Furthermore, learners generalise mathematical objects according to their what they see, whether such objects are a procedure or a representation. In her cognitive study, Kidron (2011) explored the learners' concept image of the horizontal asymptote. The learner in question

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struggled to understand that the horizontal asymptote can intersect with the function. The diagrams created some cognitive conflict in which the graph and the asymptote behaved in a manner that she (Kidron, 2011) did not expect. After checking for errors in her work, she finally accepted that the horizontal asymptote could intersect with the graph infinitely. The lesson learnt from Kidron's (2011) study is that the current high school definition introduced to learners in the South African system of education will need to be changed, if learners are to continue studying rational functions. The horizontal asymptotes would intersect with the graph and in some instances, the asymptote would be a line in the form of $y = mx + c$.

In a study conducted by Yerusalemy (1997), learners defined the horizontal asymptote as a slope with a gradient of zero. Their initial thoughts on the vertical asymptotes were that they occur on every zero point of the denominator. They did not distinguish the removable discontinuity from the asymptote. Some of these learners referred to the asymptote as a point, which demonstrates that their view of the asymptote was incorrect. Similar to the study by Nair (2010), learners in the study by Yerusalemy (1997) above also stated that asymptotes were invisible lines.

Research Aim

Learners' inability to adequately understand asymptotes and the way they could be determined is a universal (common) problem (according to Flesher (2003)) and South African learners are susceptible to such difficulties. Evidence obtained from teachers at schools indicate that learners answer questions adequately in the written form but fail to explain what they have written when asked to explain. Learners tend to believe that the word "asymptote" is used only for a hyperbola. This assertion is supported by the National Senior Certificate Diagnostic report for the 2015 final examinations in South Africa, where a number of candidates listed two asymptotes for an exponential function in response to a question. This research then, attempted to explore the following research question: What is the nature of learners' mathematical discourse on the asymptotes of the hyperbola and exponential function?

Research Methodology

General Research Background

The participants (112) were selected from four rural secondary schools in Mthatha in the Eastern Cape province in South Africa. This empirical, mixed methods research was underpinned within the interpretivist paradigm. Mthatha is a small town with a dense population. Education is affected by a variety of factors, including a high illiteracy rate, and a low per household income as most residents rely on government grants and some households are child headed.

Sample

A homogeneous purposive sampling technique was used, where research participants' in possession of similar representative qualities with the study population were selected (Saunders, Lewis, & Thornhill, 2012). A dual-purpose testing mechanism was used. Furthermore, six learners were purposively selected for task-based interviews per school. The test served the purpose of examining the mathematical discourse of Grade 11 learners in the sphere of functions in general.

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Instruments

Qualitative and quantitative data were collected by means of a test and task-based interviews. The test was used only to select the participants for the task-based interviews. The validity of the test was ensured by asking five other teachers of Grade 11 classes to examine the test and by also piloting the test among 10 learners in another non-participating school. Approximately one hour was allocated for the actual test administration. The test was conducted under strict examination conditions. The test itself comprised 4 questions (40 marks) based on sections that the learners had already covered in lessons in previous and current grades.

Flexibility was used in the administration of the task-based interviews, allowing the selected participants to use multiple methods of answering questions. Some preferred to answer questions on the chalkboard, some on paper, while others preferred verbal responses only. Participants were allowed to express themselves freely in their language of choice. While most of the participants responded in English, there were some instances of some learners choosing IsiXhosa (the local language widely used in the Eastern Cape) to express themselves.

The test was administered first because it was used to select the participants for the task-based interviews. For purposes of enhancing "the authorial voice" of the participants, the questions selected for both the test and the task-based interviews were as close as possible to the participants' everyday classroom experiences.

Four focus groups, comprising six participants each (a group per school) were interviewed. This aspect of the research was not included in the study because the intention was to look specifically at the individual participant's discourses related to the asymptote and therefore the focus group discussions had no tangible influence on these findings.

The Realization Tree of a Function

A realization is nothing more than a representation of a mathematical object (Weingarden, Heyd-Metzuyanim and Nachlieli, 2018). Sfard (2008) coined the term realization because she believes that mathematical objects do not exist in themselves but are a result of human communication. For example, quadratic equations are solved using four basic methods. One can factorise, complete the square, use a formula or use a graph. The four ways of solving quadratic equations are the realizations of the quadratic equation. Learners need to realise the connectivity between methods and that the result from each of the methods is the same. What enhances learners' realisation that they are actually doing the same process using different means is allowing them (learners) to explore different ways of solving *factorizable* quadratic equations rather than strictly using the formula for those equations that have irrational roots. Mudaly (2014) states that visualisation can be used to make connections between concepts. There are four basic realizations of a function and these are namely graphical, words, algebraic and ordered pairs. Ordered pairs are usually written in the form of a table of values. What learners see are discrete values that are not connected to each other. Sfard (2012) refers to this kind of communication as impoverished realisations.

Realisation Tree of a Function Analysis tool

The Realisation Tree of a Function analysis tool (RTF) was adopted from the Realization Tree Assessment tool (Weingarden, Heyd-Metzuyanim & Nachlieli, 2018). While Weingarden et al. (2018) used a four staged shade coded tool to analyse the realisations by the teacher in class, this research reduced the Realization Tree of an asymptote to only three steps to analyse learners' response to questions on the asymptote of a hyperbola and exponential function. The RTF has three stages and for convenience, renamed them levels. Level one is where the learners' realization became completely unacceptable to the community of mathematicians, for example when an asymptote is written as a number (the asymptote is two) or when an asymptote is

written in the form of coordinates (2;1). The second level is when learners' response is partially acceptable. For example, instead of writing two asymptotes of a hyperbola, the learner just writes one asymptote. The last and the highest level of realisations is when the response is acceptable to the community of mathematicians and there is no ambiguity. Whether spoken or written when learners say the asymptote is y is equal to 2 and x is equal to 1. The highest stage was named level three.

Classification	Description	Examples
Level 3	Objectified communication. Fluidity in communication on the four representation of a function	Able to relate all the realizations of a function. e.g. an asymptote is a line; expressed in the form of $x = a$ and $y = b$.
Level 2	Partial objectification of the mathematical object. Expressing related mathematical objects as if they are not related	Drawing a line for an asymptote but writing a number for the same mathematical object
Level 1	No objectification at all.	Failure to respond to questions or unmathematical statements e.g. an asymptote is a point.

The findings were that students readily recognised the asymptotes of functions in equations or formula that are generally used in classrooms or tests but when there was a slight change the number of correct responses declined.

Research Results

Participants' Responses to the Equations of the Asymptotes

The participants were asked to *Write the equations of the asymptotes of f in* $f(x) = \frac{3}{x-1} - 2$. Most of the participants performed well in this question. Over 75% expressed the asymptotes correctly, that is, $x = 1$ and $y = -2$. However, 25 participants failed to express the asymptote in any mathematically acceptable manner. Four participants did not respond to the question at all, and simply left blank spaces. Twelve participants had all their responses incorrect and could not name either of the asymptotes. Thirteen participants' responses were partially correct. Most of the participants (75%) responded in a manner that is acceptable to the community of mathematicians by stating two asymptotes that were perpendicular to each other. Participant A's responses in relation to the equation of the asymptotes of f was simply $x = 1$ and $y = -2$.

This response corresponds with the discourse of the community of mathematicians. Participant A wrote the two asymptotes of $f(x) = \frac{3}{x-1} - 2$ mathematically because she expressed them as evidence of interpreting the symbolic mediator; that is, the function f. Her explanation of why she expressed her response in the form of an equation was that "…*it is because an asymptote is a function*". We classified her communication on the asymptote of the hyperbola as being at level three because every member of the professional community of mathematicians would have interpreted it in a similar way.

Participant B produced a non-mathematical response, which could be classified as a ritualised routine and an example of a level two communication. Ritualised routines are a result of ritualised learning where procedures are committed to memory and are recalled and applied mechanistically when a particular application calls for its enactment. His initial response was correct ($x = 1$ and $y = -2$), and then went on to write the coordinate (1; -2). The asymptote is

regarded as a coordinate in the same manner that one would do for a turning point, or intercept where the *x* and *y-values* are written as a coordinate. When asked why he wrote the coordinate, Participant B pointed to the intersection of the two asymptotes as the reason for doing so. However, it was clear that participants knew how to identify the asymptote from an equation, but probably did not know what the asymptote actually was. Participant B could interpret the symbolic visual mediator $f(x) = \frac{3}{x-1} - 2$, which was classified according to the RTF analysis tool as level two because his communication exhibits a partial development in the learning of asymptotes, although he could write the equations for the asymptotes.

The work of the participants reflected the second type of mathematically unacceptable responses when they presented the equation in terms of the *p* and *q* parameters. Participant C presented his answer for the asymptote as $p = x = 1$ and $y = q = -2$. When a function is generalised, *p* and *q* are used to represent the vertical and horizontal shifts respectively. These parameters are used in almost all the functions in the Further Education and Training (FET) phase in South Africa. The participants' responses were intentional and incorrect because they tended to equate the vertical and horizontal shifts to the asymptotes. Adler and Ronda (2015) supported this argument when they stated that learners misconstrue these mnemonics as part of the Cartesian plane. From the researchers' experience, mathematics teachers have inculcated the notion of simplifying mathematics by emphasising the parameters to such an extent that learners know the parameters to be the real asymptotes. Participant C wrote his equations and included the letters *p* and *q* together with the equations, which showed that he is able to interpret the symbolic visual mediator, as he is able to name asymptotes from the equation.

Participant C's interpretation of the asymptote was classified as partially mathematical because he could identify the equations of the asymptote. Participant C's routines are "ritualised mathematical" because he included *p* and *q* in his equation. Participant C's communication remained at the ritualised stage. His communication was classified as being in level two because his communication has both mathematically accepted symbolic visual mediators and mnemonics. The example of a third level classification according to the RTF analytical tool is shown in the discussions that follow. The participants had some knowledge of the asymptote of a hyperbola, which was demonstrated by the three respondents excluding the *y* and writing the equation as if it was *only* a number.

Participant D used inductive reasoning to arrive at her answer. Naming the asymptote as *-2* is mathematically incorrect, since *-2* cannot be located on the Cartesian plane. There is a tendency by some learners to think that *p* or *q* could replace *x* and *y* respectively, or that they were synonymous such that one could replace the other interchangeably. Participant D first wrote a general equation $f(x) = \frac{a}{x-p} - q$, and had to be reminded of the asymptotes of the function. She then substituted the number three for *a*. She further equated *q* to the asymptote. Finally, she wrote that *asymptote = -2*. The object, which Participant D wrote, did not exist on the Cartesian plane because the word asymptote did not show a clear representation on the coordinate system. She did not do anything about the vertical asymptote. She also did not mention the vertical asymptote. The participant's routines were ritualised because she followed-up the routines of others as her own routines, which did not lead to mathematically endorsed narratives. Participant D's equations were not mathematical as they could not be located on the Cartesian plane. Her level of communication is at level one because she failed to identify the asymptote from an algebraic representation showing some disjointed relationships.

In a question that required learners to write the equation of the asymptote of *f(x) − 5* given $f(x) = \left(\frac{1}{3}\right)^{-5x}$, gave credence to the notion that participants' routines were ritualised. Participant R simply wrote down the answer $y = -5$, without showing any working, procedure or method. The answer was acceptable, as it was a horizontal asymptote. There was no

ambiguity with her response, and the community of mathematicians would have deduced the same meaning. Her interpretation was endorsed in the mathematical discourse. Participant W also provided a similar solution, like many of the other participants.

Participant TT produced a different dimension to the asymptote of an exponential function by writing $f(x) = (\frac{1}{3})^{-5x}$. In responding to the question requiring the participants to write the equation of the asymptote of $f(x) - 5$, Participant TT multiplied the constant *-5* with the exponent *x*. An asymptote is a linear function, but what Participant TT wrote was not a linear equation. Participant TT did not interpret the symbolic visual mediator $f(x) - 5$ in a mathematically acceptable manner. He confused $f(x) - 5$ with. In fact $f(5x)$, Participant TT only responded to the former $(f(x) - 5)$ by making substitutions to the equation and did not respond to the question directly. There is nothing in the response that relates to an asymptote. He did not know what an asymptote was and did not know how it is represented as an equation, which is an example of level one communication on the Realization Tree of a Function.

In this section, the four categories of the participants' test responses were discussed. The first category is that of participants providing a correct answer without showing any process or method. The second category was that of participants who provided the desired responses and also showed the processes leading to the answer. The third category of participants wrote a vertical line, instead of a horizontal asymptote. As a result, they misrepresented the asymptote of an exponential function by writing an equation showing a vertical line instead of a horizontal one. The last category consisted of a discordant depiction of responses, which had nothing to do with the equation of an asymptote.

In the third question that the participants were expected to respond to, the phrase *exponential function* did not appear in the question, but the equation was exponential $[\theta = 60\left[2^{\frac{-t}{15}}\right] + 20]$. The number of participants who successfully named the asymptote decreased to 29 (when compared to the 84 who successfully answered question one). This decline can be attributed to the general ritualization of learning of functions. The difference in these two questions was that familiar mathematical language was used in the first question, whereas changed, unfamiliar symbols were used in the third question.

While the variables in $\theta = 60\left[2^{\frac{-t}{15}}\right] + 20$ are θ *and* t learners used the variable y for the asymptote. None of them used the variables from the equation. A few learners like Participant H were able to identify the asymptote from the equation $\theta = 60\left[2^{\frac{-t}{15}}\right] + 20$ but they still expressed their answer as $y = 20$ *instead of* $\theta = 20$. Although the variables were incorrect, we classified Participant H's communication as level two because she could identify the asymptote from the equation indicating that she recognized the exponential function, but her routines are ritualized because the variable is θ and not y.

Discussion

Mathematics is not all about writing or stating mathematically acceptable responses to questions. Learning mathematics necessitates that learners be able to show how mathematical objects relate to each other, by displaying an understanding of the concept. Learning of mathematics does not mean that learners should simply reproduce through rote learning what they have done in class. Often, learners, through the authority of the teacher or textbook, adhere strictly to the procedures demonstrated in class or in the textbook. Engelbrecht, Bergsten and Kagesten (2016: 570) state that "a*s is the case in many countries in the world, mathematics teaching at the upper secondary level often has an emphasis on procedural skills rather than conceptual understanding"*. This explains why some participants could successfully complete certain questions whilst they struggled with those that were slightly different.

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When learners in this research displayed a low level of communication about the asymptote, it showed that their understanding of the concept was still very poor. The learners could replicate the simple solutions that they had already seen but had no idea about how they should respond to questions that had symbols that were unfamiliar. The learners adhered to the symbolic artefacts that they used previously but could not equate those symbols to new ones. This implied that their narratives depended entirely on replicating procedure rather than engaging with the mathematics required. Roberts and Le Roux (2018) agreed by stating that "relational thinking could produce more than one way to solve a problem and each method would produce a different branch on the realisation tree". These learners should have discursively related their previous ideas about the asymptote to the new equation and produced a solution that was appropriate. The learners' inability to extrapolate their solution from the ritualised version of the problem to a changed version implies that there was a deficit of real understanding of the concept and the process. Understanding can generally be viewed as the ability to use a concept flexibly. This research showed that the learners were rigid in their application of the concept and any variation in the symbols confused the learners.

This lack of understanding is explained by Rittle-Johnson and Schneider's (2015) argument that conceptual knowledge can be described as knowledge rich in relationships. The learners had not created sufficiently detailed relationships between the concept of the asymptote and the symbols associated with them. They recalled that which the teacher had done but they could not complete the exercises that involved symbols that were different.

The learners' failure to respond to a variation in questions reinforced the view that participants' mathematical discourses were mostly ritualised routines. As alluded to by Nachlieli and Tabach, (2018), there is no guarantee that ritualised learning will lead to flexibility in the learning and application of mathematics. The teaching and learning process should be more about learners exploring mathematical objects and discovering endorsed narratives by themselves rather than a learning situation where rules and procedures of doing mathematics are often given to the learner.

Lavie, Steiner and Sfard (2019) claim that rituals are a necessary part of learning and thus cannot be excluded from the classroom altogether. They claim that seminal routines are essential at the stage where new discourses are likely to develop. One of the ways of beginning the process of concept development is by creating it as a ritual. However, further exploration and activity must be used to develop that routine into deeper understanding so that the learner would become aware of some practical application of the concept.

Sfard (2016) also posited the argument that it was the intangibility of mathematical objects, which constituted the main challenge in the learning of mathematics. This is true for most mathematics but in order to influence the discursive narratives of the learners, they need to engage in more explorative activities. The learners in this research were able to only work with what their teacher did. It was argued that they simply followed routines learned. But Robert and Le Roux (2018) theorised that "explorations are the most sophisticated form of routine. *Explorative* discourse is characterised by narratives about mathematical objects that are endorsable in terms of mathematical axioms, definitions and theorems".

This is exactly why Anderson et al. (2001) state that "in instructional settings, learners are assumed to construct their own meaning based on their prior knowledge, their current cognitive and metacognitive activity, and the opportunities and constraints they are afforded in the setting, including the information that is available to them" (p 38). Through an explorative discourse, learners are able to negotiate meaning and accrue a deeper understanding that enables them to answer questions that are changed from the ones that they have already interacted with.

Conclusions

There was a lack of coherence in learners' mathematical discourse, as what learners wrote in one solution was different from the next one. Presenting the same question in a different way, resulted in learners struggling with the same concept, signifying a ritualised nature of learning of functions. The learners had memorised the procedure and the concept and applied it to questions that they recognised. Their understanding of the process of identifying the asymptote was shallow, isolated and disconnected. The procedure was accomplished by rote execution of the procedure. There was no indication that the learners had attained a well-connected understanding of the concepts.

Learning of mathematics is enhanced by means of explorations, rather than merely expecting learners to follow ritualistically the solutions and procedures presented by their teachers, other learners and textbooks. Opportunities for learning and deepening of mathematical discourses have a chance of occurring when learning and teaching move from rituals to the exploratory mode. Ritualised learning will often produce uncoordinated and fragmented communication from learners. While learners were able to use mathematical words and symbols, they did so without making connections with the Realisation Tree of that particular mathematical object. For learners to solve unseen, non-routine problems their exploratory routines should be strong.

Use of different *visual mediators* was a hindrance to learners' success mainly because learners had a different interpretation of each visual mediator due to their inexperience. These learners should have been exposed to different *visual mediators* during their exploratory work as they engaged with the operational and symbolic relationships between the equation and the derivation of the asymptote. Learners were not able to locate the asymptote because of the symbols used. Each realisation morphs into a tree of understanding that enables the learner to successfully produce correct solutions for the asymptote.

A further study focusing on interpretation of algebraic visual mediators of functions will help in understanding some of the challenges the learners had with the interpretation of the asymptotes from algebraic statements.

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Vimolan Mudaly *(Corresponding author)*	PhD, Associate Professor, University of KwaZulu-Natal, Private Bag X03, Ashwood, 3605 Durban, Republic of South Africa. E-mail: mudalyv@ukzn.ac.za
Sihlobosenkosi Mpofu	PhD Student, University of KwaZulu-Natal, Republic of South Africa. E-mail: sihlobosenkosi@gmail.com