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# New results on the stability, integrability and boundedness in Volterra integro-differential equations

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**Abstract**

The authors of this article deal with a first order non-linear Volterra integro-differential equation (NVIDE). To this end, the conditions are obtained which are sufficient for stability (S), boundedness (B), and for every solution $x$ of (NVIDE) is integrable. For properties of solutions of (NVIDE) considered three new theorems on (S), (B) and integrability properties of solutions are proved. The methods of the proofs involve constructing of a suitable Lyapunov functional (LF) which gives meaningful results for the problems to be investigated. The conditions to be given involve nonlinear improvement and extensions of those conditions found in the literature. An example is provided to illustrate the effectiveness of the proposed results. The results obtained are new and complements that found in the literature.

**Keywords:** first order; (S); (B); integrability; (LF).

## 1 Introduction

The linear and non-linear (VIDEs) models in their different aspects attracted many authors that investigated them from many sides (see, for example,

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Adivar and Raffoul [1], Becker ([2],[3],[4]), Burton ([5],[6],[7]), Burton and Haddock [8], Burton and Mahfoud ([9], [10], [11]), Chang and Wang [12], Corduneanu [13], Dung [14], Eloe et al. [15], Engler [16], Funakubo et al. [17], Furumochi and Matsuoka [18], Grace and Akin [19], Graef, and Tunç [20], Graef et al. [21], Grimmer and Seifert [22], Grimmer and Zeman [23], Gripenberg et al. [24], Grossman and Miller [25], Hara et al. ([26], [27], [28]), Hino and Murakami [29], Islam et al. [30], Jin and Luo [31], Lakshmikantham and Rama Mohan Rao ([32], [33]), Mahfoud ([34], [35], [36]), Martinez [37], Miller [38], Murakami [39], Napoles Valdes [40], Peschel and Mende [41], Raffoul ([42], [43], [44]), Rama Mohana Rao and Raghavendra [45], Rama Mohana Rao and Srinivas [46], Staffans [47], Talpalaru [48], Tunç ([49], [50], [51], [52], [53]), Tunç and Ayhan [54], Tunç and Mohammed ([55], [56]), Tunç and Tunç [57], Vanualailai [58],Vanualailai and Nakagiri [59], Wang ([60], [61]), Wang et al. [62], Wazwaz [63], Xu ([64], [65]), Zhang ([66], [67]), Da Zhang [68], the references ([69], [70], [71]), and many relative papers or books in references of these works).

When we look at the works just mentioned, in generally, the (S), (B), instability, $L^1[0,\infty)$, $L^2[0,\infty)$, etc., properties of the solutions for (LVIDE) or (NLVIDE) are investigated by employing fixed point theory, perturbation methods, integral inequalities, the Lyapunov's function(al)s, the Lyapunov-Razumikhin's function(al)s, the variations of parameters formulas, etc..

However, when we look at the related literature, it can be seen that nearly all of these results were proved by employing of the Lyapunov's function(al)s. That is, to the best of our knowledge, only in the proofs of a few results the fixed point theory, perturbation methods, the variations of parameters formulas, etc., are used to verify the problems therein. This case can be checked and seen by studying the context of the mentioned works and those found in the references of these articles and books. Indeed, this information shows the effectiveness of the (LFs) in the researches and applications raised is sciences and engineering. Here, we would not like to state the details of the applications of these methods.

Xu [65] studied the uniform asymptotic (S) of the trivial solution of the scalar (LVIDE):

$$x' = a(t)x + \int_{-\infty}^{t} D(t,s)x(s)ds. \tag{1}$$

Xu [65] has used (LFs) to give sufficient conditions for the (S) of solutions of (LVIDE) (1). However, to the best of our information, it seems that the theory of (LVIDE) (1) has not been developed further. One of the aim and novelty of this article is to develop this fact further.

In this article, we treat (NVIDE) of the type of

$$x' = -A(t)G_1(x) + \int_{-\infty}^{t} D(t,s)G_2(s,x(s))ds + P_1(t,x), \tag{2}$$

where $x \in \Re$, $A(t) : \Re \to (0,\infty)$, $G_1 : \Re \to \Re, G_1(0) = 0$ and $G_2, P_1 : \Re \times \Re \to \Re$ with $G_2(s,0) = 0$ and $D : \Re \times \Re \to \Re$ with $s \le t < \infty$ are continuous functions.

In the sequel we shall let that there is a function $G_0 : \Re \to \Re$ which is continuously differentiable and defined by

$$G_0(x) = \begin{cases} \frac{G_1(x)}{x}, x \neq 0 \\ G_1'(0), x = 0. \end{cases}$$

Hence, (NVIDE) (2) yields that

$$x' = -A(t)G_0(x)x + \int_{-\infty}^{t} D(t,s)G_2(s,x(s))ds + P_1(t,x),$$

where $x$ represents $x(t)$ and through the paper when we need it is assumed the same representation.

It is clear that (NVIDE) (2) involves (LVIDE) (1). In fact, when we take $A(t) = -a(t)$, $G_1(x) = x$, $G_2(s,x(s)) = x(s)$ and $P_1(t,x) = 0$, then (NVIDE) (2) reduces to (LVIDE) (1) discussed by Xu [65]. This information yields one of the other novelty of this paper.

We treat the (S) of trivial solution and integrabilty of solutions of (NVIDE) (2) when $P_1(t,x) = 0$ and the (B) of solutions of (NVIDE) (2) when $P_1(t,x) \neq 0$ by defining a suitable (LF), which gives meaningful new results. The use of auxiliary (LF) allows us to deduce inequalities such that all solutions must satisfy them. Hence, from which we deduce the (S), (B) of solutions and the solution $x(t)$ is integrable. The manner in which a non-positive (LF) can be used for (B) and integrabilty is one of the main and new further novelty of the paper.

We begin with the following notations, (S) and (B) definitions.

For any $t_0 \ge 0$ and $\phi \in (\alpha, t_0)$, $-\infty < \alpha \le t_0$, where $\phi$ is initial function, let $x(t) = x(t, t_0, \phi)$ denote the solution of (NVIDE) (2) on $(-\infty, \infty)$ such that $x(t) = \phi(t)$ on $(\alpha, t_0]$.

Here, the set of all continuous and real-valued functions on $(\alpha, t_0]$ and $[t_0, \infty)$ are shown by $C(\alpha, t_0]$ and $C[t_0, \infty)$, respectively.

For $\phi \in C(\alpha, t_0]$, let us assume that $|\phi|_{t_0} = \sup\{\phi(t)| : -\infty < \alpha \leq t_0\}$.

**Definition 1.** The trivial solution of (NVIDE) (2) is said stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$ there exists a $\delta = \delta(\varepsilon, t_0)$ such that $\phi \in C(\alpha, t_0]$ with $|\phi(t)|_{t_0} < \delta$ implies that $|x(t, t_0, \phi)| < \varepsilon$ for all $t \geq t_0$.

**Definition 2.** The solutions of (NVIDE) (2) are said bounded if for each $K > 0$ there exists $T > 0$ such that

$$t_0 \geq 0, \phi \in C(\alpha, t_0], |\phi(t)|_{t_0} < T \ \ \textit{and} \ \ t \geq t_0 \ \ \textit{imply} \ \ |x(t)| \leq K.$$

## 2 Main results

Let $P_1(t, x) = 0$.

**A. Assumptions**

**(A1)** There exists a positive constant $g_0$ such that

$$|G_2(t, x)| \leq g_0|x|$$

for $t, x \in \Re$, where the function $G_2(.)$ with $G_2(s, 0) = 0$ is continuous for the arguments displayed explicitly.

**(A2)**

$$A(t)G_0(x) - \int_{-\infty}^{t} g_0|D(t, s)|ds \geq 0$$

for $t, x \in \Re$, where the functions $A(.)$ and $D(.)$ are continuous and the function $G_0(.)$ with $G_1(0) = 0$ is continuously differentiable for the arguments displayed explicitly.

**(A3)**

$$\int_{-\infty}^{t} \int_{0}^{\infty} g_0|D(u, s)|duds \leq L$$

for some constant $L, L > 0$ and $t \in \Re$.

We first give a boundedness and stability result for the solutions of (NVIDE) (2).

**Theorem 1.** If hypotheses (A1)-(A3) are satisfied, then all solutions of (NVIDE) (2) are bounded and the trivial solution of (NVIDE) (2) is (S).

**Remark 1.** It is well-known from the stability theory of the ordinary or functional differential and integro-differential equations that if we find a Lyapunov functional, which is positive definite and its time derivative along the solutions of the considered ordinary or functional differential and integro-differential equation(s) is negative semidefinite or negative definite, then we can guarantee the stability and uniformly asymptotically stability of the zero solution of that equation(s), respectively. In addition, by applying the Gronwall's inequality to the results of the time derivative of possible Lyapunov functional(s), we can conclude the boundedness and integrability of the solutions for the considered equation(s). In this paper, when we do calculation through the proofs of our main results, we will have these ideas in our mind.
**Proof.** We construct an auxiliary (LF) $v = v(t) = v(t, x(t))$ by

$$v = |x| + \int_{-\infty}^{t} \int_{0}^{\infty} g_0 |D(u, s)||x(s)| du ds.$$

Hence, we get

$$v(t, 0) = 0$$

and

$$v(t) = v(t, x) \geq |x|. \tag{3}$$

Thus, the auxiliary (LF) is clearly positive definite. Differentiating the auxiliary functional along the solutions of (NVIDE) (2), we get

$$\begin{aligned} v' &= -\frac{x}{|x|}[A(t)G_1(x(t)) - \int_{-\infty}^{t} D(t, s)G_2(s, x(s))ds] \\ &+ \int_{t}^{\infty} g_0|D(u, t)|du|x(t)| - \int_{-\infty}^{t} g_0|D(t, s)||x(s)|ds \\ &\leq -A(t)G_0(x)|x| + \int_{-\infty}^{t} |D(t, s)||G_2(s, x(s)|ds \\ &+ \int_{t}^{\infty} g_0|D(u, t)|du|x(t)| - \int_{\infty}^{t} g_0|D(t, s)||x(s)|ds \\ &\leq -A(t)G_0(x)|x| + \int_{-\infty}^{t} g_0|D(t, s)||x(s)|ds \\ &+ \int_{t}^{\infty} g_0|D(u, t)|du|x(t)| - \int_{\infty}^{t} g_0|D(t, s)||x(s)|ds \\ &= -A(t)G_0(x)|x| + \int_{t}^{\infty} g_0|D(u, s)|du|x(t)|. \end{aligned}$$

Using assumptions (A1) and (A2), that is,

$$|G_2(t,x)| \le g_0|x|$$

and

$$A(t)G_0(x) - \int_{-\infty}^{t} g_0|D(u,t)|du \ge 0,$$

we have

$$v'(t) \le -[A(t)G_0(x) - \int_{-\infty}^{t} g_0|D(t,s)ds]|x| \le 0. \tag{4}$$

Integrating inequality (4) from $t_0$ to $t$, we get

$$v(t) \le v(t_0) \quad for \quad all \quad t \ge t_0. \tag{5}$$

Then, in view of (3) and (5), it follows that

$$|x(t)| \le v(t) \le v(t_0) \tag{6}$$

for all $t \ge t_0$. From (6), we can get that all solutions of (NVIDE) (2) are bounded.

Now, from the above estimate, assumption (A3) and the fact that

$$v(t_0) = |\phi(t_0)| + \int_{-\infty}^{t_0}\int_{0}^{\infty} g_0|D(u,s)||\phi(s)|duds \le |\phi|_{t_0}L_0,$$

where

$$L_0 = 1 + \int_{-\infty}^{t_0}\int_{0}^{\infty} g_0|D(u,s)|duds,$$

we get

$$|x(t)| \le |\phi|_{t_0}L_0$$

for all $t \in \Re$. It immediately follows that the trivial solution of (NVIDE) (2) is stable, that is, for any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{L_0}$, and so for $\phi \in (\alpha, t_0]$, $-\infty < \alpha \le t_0$, with $|\phi|_{t_0} \le \delta$, we have

$$|x(t)| \le \delta L_0 = \varepsilon.$$

Hence, we can conclude that the trivial solution of (NVIDE) (2) is stable. Hence, we can reach the desired result of Theorem 1.

In our coming theorem, Theorem 2, we show that all solutions of (NVIDE) (2) are integrable.

**B. Assumptions**

**(H1)** There exists a positive constant $\delta_0$ such that

$$A(t)G_0(x) - \int_{-\infty}^{t} g_0|D(t,s)||x(s)|ds \geq \delta_0$$

for $t \geq t_1$ and $x \in \Re$, where the functions $A(.)$ and $D(.)$ are continuous and the function $G_0(.)$ with $G_1(0) = 0$ is continuously differentiable for the arguments displayed explicitly.

**Theorem 2.** In addition to assumptions (A1) and (A3), if assume assumption (H1) holds, then every solution of (NVIDE) (2) is integrable.

**Proof.** From Theorem 1, any solution of (NVIDE) (2) is bounded and satisfies (4) and (6). If assumption (H1) holds, then from (4) we get

$$v'(t) \leq -\delta_0|x| \quad for \quad t \geq t_1.$$

Integrating the last estimate from $t_1$ to $t$, we find

$$v(t) - v(t_1) \leq -\delta_0 \int_{t_1}^{t} |x(s)|ds$$

so that

$$\delta_0 \int_{t_1}^{t} |x(s)|ds \leq v(t_1) - v(t) \leq v(t_1),$$

i.e.,

$$\delta_0 \int_{t_1}^{t} |x(s)|ds \leq v(t_1).$$

Hence, we see that the solution $x(t)$ of (NVIDE) (2) is integrable. The former inequality implies the desired idea of Theorem 2.

Finally, we give a boundedness theorem for solutions of (NVIDE) (2).

Let $P_1(t,x) \neq 0$.

**C. Assumptions**

**(C1)** There exists a positive constant $M$ such that

$|P_1(t,x)| \leq (M+|x|)|Q(t)|$ and $|Q(t)|$ is an integrable function for $t \geq t_1$, i.e, $\int_{t_1}^{\infty} |Q(s)|ds < \infty$.

**Theorem 3.** In addition to assumptions (A1) and (A2) if we assume that assumption (C1) holds, then all solutions of (NVIDE) (2) are bounded.

**Proof.** From Theorem 1, any solution of (NVIDE) (2) satisfies the estimate (4). To complete the proof of this theorem, we benefit from the functional $v = v(t) = v(t, x(t))$ just used in the proof of Theorem 1.

Obviously, we have

$$v(t) \geq |x|.$$

Next, in the light of the assumptions (A1), (A2) and (C1), the time derivative of the auxiliary functional $v = v(t) = v(t, x(t))$ can be re-revised as

$$\begin{aligned} v' &\leq |P_1(t,x)| \\ &\leq (M + |x|)|Q(t)| \\ &\leq M|Q(t)| + v(t)|Q(t)|. \end{aligned}$$

Integrating the last estimate from $t_1$ to $t$, we have

$$v(t) \leq v(t_0) + M\int_{t_1}^{t} |Q(s)|ds + \int_{t_1}^{t} v(s)|Q(s)|ds.$$

Hence, applying the Gronwall's inequality, we can obtain

$$|x(t)| \leq v(t) \leq K\exp[\int_{t_1}^{\infty} |Q(s)|ds],$$

where

$$K = v(t_0) + M\int_{t_1}^{\infty} |Q(s)|ds.$$

Consequently, one can arrive at the desirable result that every solution of (NVIDE) (2) is bounded.

**Example 1.** We consider the following scalar (NVIDE) of first order

$$x' = -x^3 + \int_{-\infty}^{t} D(t,s)f(x(s))ds \tag{7}$$

with

$$\int_0^\infty |D(t,s)|ds < 1, \int_t^\infty |D(t,s)| \in L^1[0,\infty),$$

$$|f(x)| \leq \alpha|x|^3, 0 \leq \alpha \leq 1,$$

(see, also, Burton [6]).

Define the auxiliary functional by

$$v = |x| + \int_{-\infty}^t \int_t^\infty |D(u,s)||x(s)|^3 duds.$$

Hence, the time derivative of this functional along the solutions of (NVIDE) (7) gives

$$\begin{aligned} v' &= -\frac{x}{|x|}[x^3 - \int_{-\infty}^t D(t,s)f(x(s))ds] \\ &+ \int_t^\infty |D(u,t)|du|x(t)|^3 - \int_{-\infty}^t |D(t,s)||x(s)|^3 ds \\ &\leq -|x|^3 + \int_{-\infty}^t |D(t,s)||f(x(s))|ds \\ &+ \int_t^\infty |D(u,t)|du|x|^3 - \int_{-\infty}^t |D(t,s)||x(s)|^3 ds \\ &\leq -|x|^3 + \alpha \int_{-\infty}^t |D(t,s)||x(s)|^3 ds \\ &+ \int_t^\infty |D(u,t)|du|x|^3 - \int_{-\infty}^t |D(t,s)||x(s)|^3 ds \\ &\leq -[1 - \int_t^\infty |D(u,t)|du]|x|^3 \\ &\leq -\beta|x|^3 \end{aligned}$$

for some $\beta > 0$.

In view of the discussion made, we can conclude that the zero solution of (NVIDE) (7) is stable. In addition, we can say that the zero solution of (NVIDE) (7) is also uniformly asymptotically stable.

## 3 Conclusion

We consider a functional (NVIDE) of first order. The (S), (B) and integrability features of solutions of the functional (NVIDE) considered are investigated by constructing a suitable (LF). We aim to fulfill the (S) problems obtained for (LVIDEs) to (NVIDES) for (S), in addition, (B) and integrabilty of the solutions. The results obtained have a contribution to the literature, and they improve and generalize the results of Xu [65], and that can be found in the related literature.

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