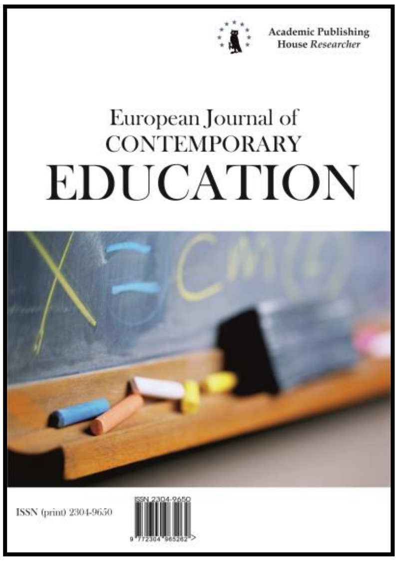
Published in the Slovak Republic
European Journal of Contemporary Education
E-ISSN 2305-6746
2019, 8(1): 4-24
DOI: 10.13187/ejced.2019.1.4
www.ejournal1.com



Visualization in Basic Science and Engineering Education

Rushan Ziatdinov (Ed.)

Advantages of Using the CAS Mathematica in a Study of Supplementary Chapters of Probability Theory

Natalia M. Mezhennaya [a, *], Oleg V. Pugachev [a]

[a] Bauman Moscow State Technical University, Moscow, Russian Federation

Abstract

Typical difficulties in learning probabilistic subjects are concerned with big data, complicated formulas and inconvenient figures in statistical analyses. The present research considers the usage of innovative teaching methods (e.g. electronic summary of lectures, presentations of lecture courses, task solution templates, electronic training materials for seminar studies) supported by the CAS (computer algebra system) Mathematica, as suggested by the authors for several topics of the course 'Supplementary Chapters of Probability Theory'. These methods help to solve tasks requiring routine calculation and simplify the ability to find analytical and graphical dependencies in the tasks under consideration. Visualisation possibilities built in the CAS contribute to students' comprehension of new theoretical material and complicated probabilistic notions. The article contains examples of CAS-performed tasks including the calculation and visualisation of the notions of the conditional probability density function, conditional expectation, order statistics and running maximum. The purpose of the suggested CAS-based materials is to solve a whole class of tasks of similar types; it is possible to obtain new results by varying the input data without spending much time elaborating the solution method.

The methods using innovative teaching materials yield advantages to both students and teachers. These methods simplify and individualise the process of education, shorten the time necessary for students' independent work and motivate students to achieve the results. Moreover,

[*] Corresponding author
E-mail addresses: natalia.mezhennaya@gmail.com (N.M. Mezhennaya),
opugachev@yandex.ru (O.V. Pugachev)

the preparation and renovation of the bank of tasks of varying levels of difficulty require much less time. Consequently, teachers and students have more time to devote toward discussing and analysing the results obtained. Another possible effect of using the novel teaching means is reducing the differentiation in students' training levels. In the present research, examples of teaching materials and scenarios of their usage are presented.

In order to clarify possible scenarios of using the suggested methods, the authors explored the necessity of their modifications, and assessed students' attitudes toward the aforementioned education methods as well as toward traditional methods, the advantages and difficulties of their usage are also explored. For this purpose, students were asked questions, and their answers have been analysed statistically. The research demonstrates that students regard interactive templates as complicated and difficult to understand; however, they do not conclude that they are unnecessary. The most remarkable aspect for students is the convenience of work with mobile devices and the possibility of a full-text search. However, the students did not make a connection between the interactive templates and the presentations prepared with their use. One can conclude that students should begin learning CAS in the first term of their university education. Additional adaptation of the teaching materials for mobile devices is necessary as well.

Keywords: CAS – computer algebra systems, correspondence analysis, Mathematica, mathematical statistics, mobile device, questioning, random process, teaching means, visualisation.

1. Introduction

Actual problems of modern science and technology require new education methods consistent with the applied direction of education (Sezonova et al., 2016). Students must learn to construct mathematical models of natural phenomena by applying the achieved skills. They must also be able to perform computer experiments with the models and draw conclusions from the results of a calculation.

The inclusion of computer technologies in the process of education and the development of on-line methods have two effects: they individualise the education process and allow for the possibility to collect data on students' education outcomes (Barba et al., 2016). Previous studies (Ivaniushina et al., 2016), (Ipek, Ziatdinov, 2017) have considered the particularities of a contemporary teaching environment that incorporates computer technologies. In regard to the visualisation of mathematical objects, e.g. functions of a complex variable (Troup et al., 2017), Taylor polynomials (Wojas, Krupa, 2017) play an important role in new education technologies and enable students to comprehend complicated mathematical notions.

Problems requiring the numerical analysis of big data are typical for modern technology (Psycharis, 2016; Rau, 2017). Often, an analytical solution of such problems turns out practically impossible. Therefore, the capability of a numerical solution and modeling becomes particularly important; thus, programming and mathematical packages ought to be included in education curriculum. Their successful mastering is necessary for a time-effective solution to problems that future engineers will encounter.

Mastering various program languages is not easy for students who have not studied them in school, and learning them requires a considerable amount of time (Broley et al., 2018). As a rule, it is sufficient for students of several specialisations to master certain program languages. Using CAS (computer algebra systems), such as MATLAB, Maple, and Mathematica, is preferable. The advantages of these program languages include their convenient interfaces and the presentation of mathematical expressions in a form familiar to students (Garner, 2004; Vlasova et al., 2017b).

Ivanov et al. (2017) discussed the application of the CAS Mathematica when teaching the course Discrete Mathematics. They specifically addressed Mathematica's main advantages and the possible ways to overcome certain difficulties. Issues related to teaching the probabilistic subjects (Vlasova et al., 2017a), especially Mathematical Statistics and Random Processes, should be investigated as well.

The probabilistic subjects (Probability Theory, Mathematical Statistics and Theory of Random Processes) hold a special place among mathematical subjects in the structure of modern education. These subjects are closely concerned with practice; their applications are significant since they deal with mathematical models of statistical conformities to natural laws. Statistical methods are widely used in technical and economical sciences due to the development of mass processes in industry, as well as experimental techniques and the necessity of a subtle analysis of experimental data. For a successful analysis of stochastic phenomena, one needs to construct, choose and verify mathematical models to generalise experience and known problems of research. Therefore, vast knowledge of the fundamental branches of mathematics (i.e. Analytic Geometry, Calculus, Differential Equations, Linear Algebra etc.) is required. Even if certain engineers do not have to apply statistical methods in their professional activities, they still need to be familiar with the basic notions and ideas of probabilistic sciences in order to understand their conclusions (Lai, Savage, 2013; Kvasz, 2018).

Typical difficulties associated with learning probabilistic subjects are concerned with big data, complicated formulas and inconvenient figures in statistical analyses of data (Holmes, 2003). In terms of another difficulty influencing the choice of an education strategy, it is possible that many students have not trained enough in mathematics in general; students' fear before certain STEM subjects should also be taken into account (Fatikhova, Sayfutdiyarova, 2017; Morán-Soto, Benson, 2018). In order to overcome these difficulties, students ought to learn computer-based methods of data analysis. One can use personal computers or tablets with special program complexes installed to solve task problems (Harrison, Lee, 2018; Davidovich, Yavich, 2018). Numerical technologies essentially contribute to one's education, allowing for a more thorough understanding of the subject studied and its laws (Kidro, Tall, 2015).

For the sake of mastering theoretical knowledge in the course 'Probability Theory, Mathematical Statistics, and Random Processes', computer lab works, including modeling and numerical experiments, are organised (Karakış et al., 2016; Mezhennaya, Pugachev, 2018). These works supplement and illustrate the mathematical theory with examples and actual calculations. Teachers shall provide students with detailed instructions for computer-solving probabilistic tasks by using program packages, including those adapted for mobile devices (MATLAB, Maple, Mathematica, etc.). The ability to construct interactive patterns of solutions to certain types of problems is another advantage of using CAS (Abell, Braselton, 2009; Fisher, 2014). When a user modifies the input data of a problem, a template automatically modifies the results of all steps of the solution in analytical or graphical form.

In the present work, new CAS-based teaching methods for probabilistic subjects are considered. The authors' suggested approach consists of supplementing the didactic complex of the subject (consisting of traditional teaching means, e.g. a manuscript summary of lectures and seminar studies, electronic training materials to seminar studies, electronic materials which help with at-home task) with innovative teaching means designed using the CAS Mathematica[*]. Presentations of the lecture course and task solution templates prepared using Mathematica are suggested. The presented material includes graphic forms; it is based on the visualisation tools built in Mathematica.

The methods that incorporate innovative teaching materials yield advantages to both students and teachers. These methods simplify and individualise the process of education, shorten the time required for students' independent work, and motivate students to achieve the results. The main advantage for teachers is that they do not have to spend much time preparing tasks with convenient numbers. On the other hand, there is the potential for the fast generation and renovation of the bank of tasks of varying difficulty levels, and the supplementary possibility of involving students in the process of education. Additional opportunities for a bilateral discussion and analysis of the obtained results exist. Another possible effect of using the innovative teaching means is a reduction in the differentiation of students' training levels. In the present research, examples of teaching materials and their usage scenarios are presented.

The materials have been prepared for fourth-year students of engineering specialisations at Bauman Moscow State Technical University (BMSTU). They are enrolled in the course 'Supplementary Chapters of Probability Theory' in the seventh or eighth half-year term.

It is well known that joint work of a student and a teacher is necessary for the achievement of positive education results (Ivaniushina et al., 2016). Therefore, it is important to take the students' opinions on new education techniques into account. For this purpose, the authors gave questionnaires to the students; statistical analysis of their responses is presented in this paper.

[*] The CAS Wolfram Mathematica with license number L3253-3375 is used.

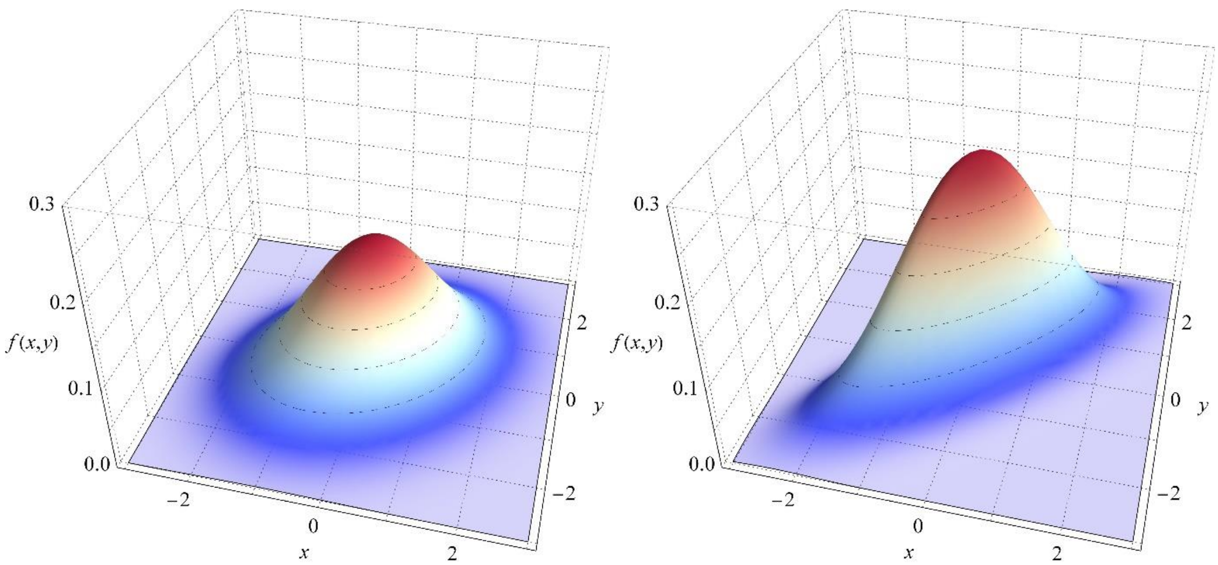
This research attempted to clarify students' attitude toward CAS-based materials and analysed their usage frequency and scenarios. Such an analysis will provide insight into changes that are needed in the education process and with the materials used. For instance, the materials can be modified according to students' needs and the stated education purposes and to improve teachers' 'client first' approach. The additional advantage of these methods is their flexibility in achieving the purposes of education, which is determined by their future use in the professional activity — e.g. a computer analysis of engineering data, the construction of neural network models etc.

2. Materials and Methods

2.1. Conditional distributions in a two-dimensional normal law

One of basic notions engineers exploit in practice is a normal distribution on a plane. Therefore, mastering this notion is highly important in the course Probability Theory, which is taken by students of engineering specialisations. It is well known that students mainly perceive the formulas concerned with a two-dimensional normal law as overly complicated and that analytical manipulations necessary for establishing the properties of this law (e.g. the calculation of its conditional expectation) are considered as an arduous routine. Therefore, it is important to use a CAS to depict and focus on graphical illustrations and to obtain answers.

The authors suggest to begin with constructing the graph and level lines of the density function of a two-dimensional normal distribution with both component, x and y, having the standard normal distribution. Their correlation coefficient is r ($-1 < r < 1$). It is necessary to consider two different values of r in order to recall and clarify its meaning. See the graphs in Figure 1.



a) b)

Fig. 1. Graphs of density functions of two-dimensional normal distributions with each component having the standard normal distribution and the correlation coefficients r: a) r = 0.3; b) r = 0.8

Figure 2 presents the scatter plot of a sample consisting of 100 values obtained from a two-dimensional normal distribution, in comparison with its probability density function. Note that the CAS Mathematica is capable of creating animated figures that show how a distribution changes depending on the value of r; one can demonstrate such animation within the course of lectures.

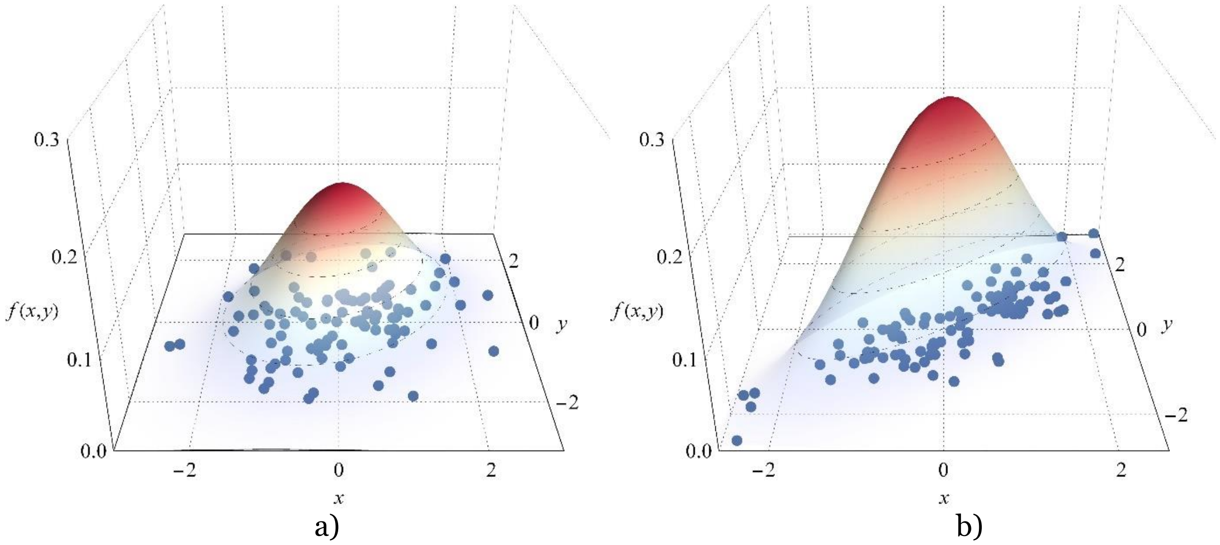


Fig. 2. Graphs of density functions of two-dimensional normal distributions combined with their scatter plots: a) correlation coefficient r = 0.3; b) r = 0.8

In Figure 1 and Figure 2, it is evident that the correlation coefficient influences the form of the graph of density function and the cloud of points.

Figure 3 illustrates the plain projections of the objects shown in Figure 2.

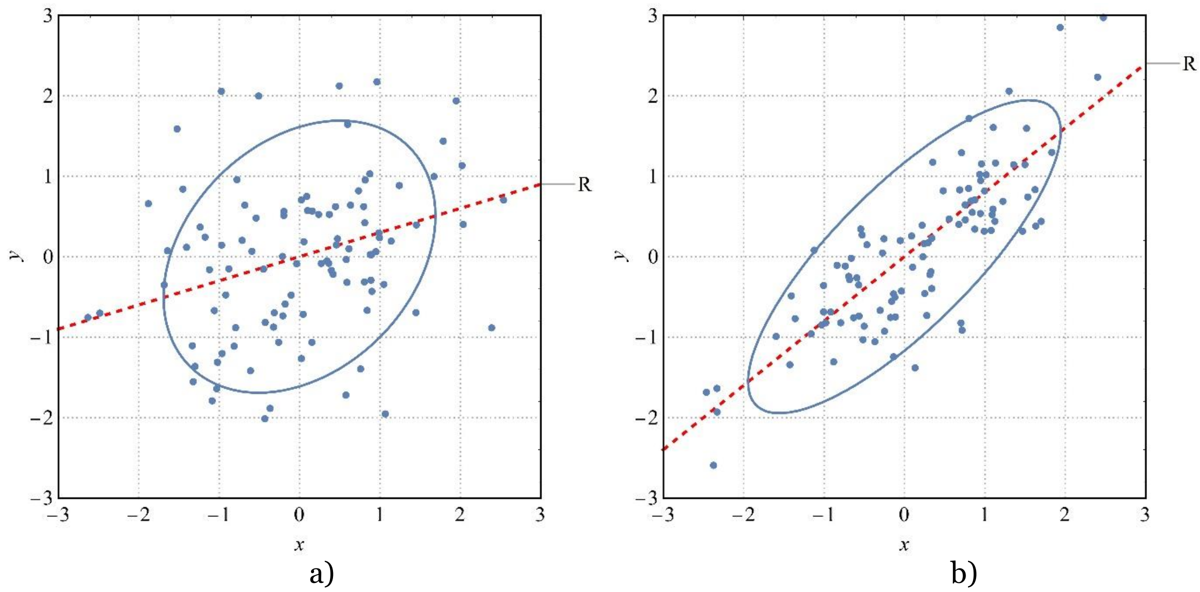


Fig. 3. Level lines of density functions of two-dimensional normal distributions combined with their scatter plots and the regression lines R of y with respect to x for different values of the correlation coefficient: a) r = 0.3; b) r = 0.8

The next illustration (Figure 4) clarifies the notion of the conditional density function of the component y in the case of the known value of x. For this purpose, the graphs of the probability density function, the regression line and the conditional and non-conditional density functions of the component y are combined for a known value of x. The graphs depict one-dimensional density functions corresponding to two different values of x.

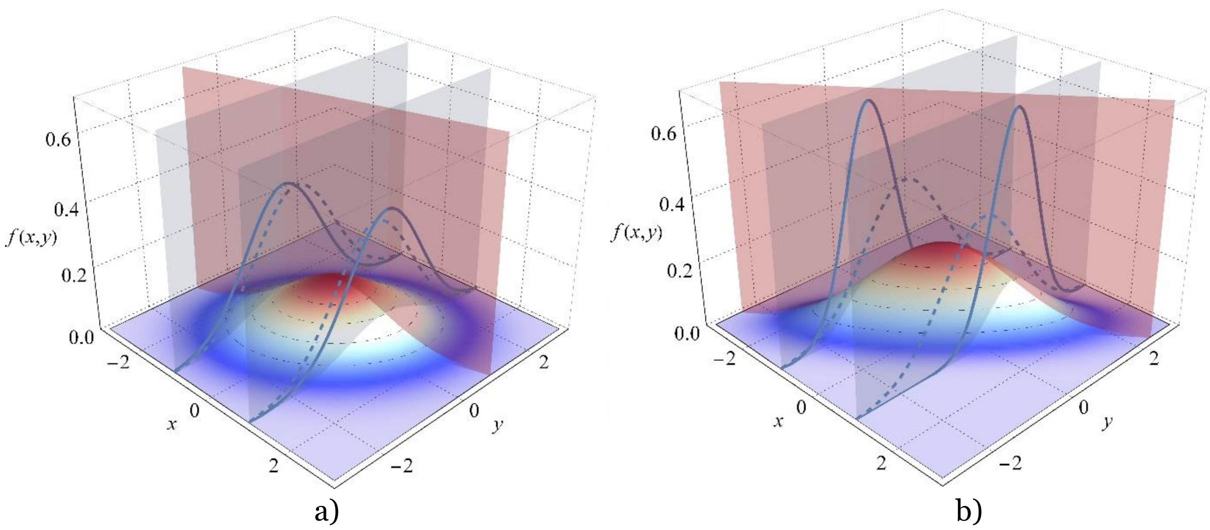


Fig. 4. Graphs of density functions of two-dimensional normal distributions, the planes of regression of y with respect to x (red), conditional density functions (solid lines) of y under the conditions $x = -1$ and $x = 1$ (in the gray vertical planes) and non-conditional density functions (dash lines) of y: a) correlation coefficient $r = 0.3$; b) $r = 0.8$

The graphs present a detailed explanation of the meaning of the correlation coefficient and its influence on the form of the cloud of points and the conditional probability density function. Moreover, the teacher can provide the students with CAS Mathematica code files for an independent study or for use in lab work, in order to show how its usage enables one to quickly and effectively perform routine calculations, whereas one obtains the answer in the traditional mathematical form.

2.2. Functions of normal random vectors

It is reasonable to explore the distributions of a normal random vector's functions. Consider the following task statement: The random values X and Y have a two-dimensional normal distribution; hereby, each component has the standard normal distribution, and their correlation coefficient is r. Consider the problem of finding the probability density function of the length R of the radius vector of a point with coordinates (X, Y), i.e.

$$R = \sqrt{X^2 + Y^2}.$$

Figure 5 illustrates this problem in the cases of two different values of r.

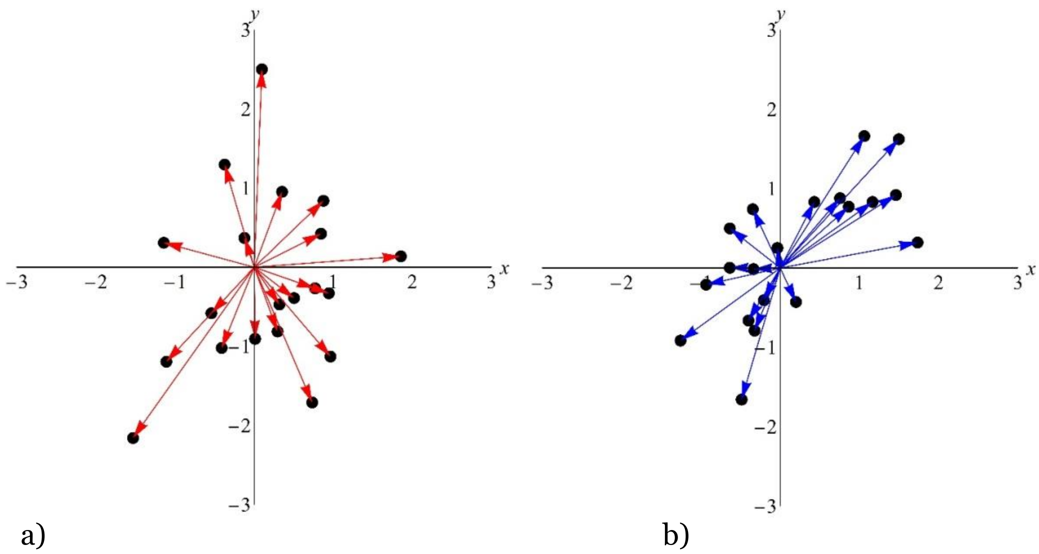


Fig. 5. Clouds of points and the corresponding values of radius vectors for a sample volume of 20 from two-dimensional normal distributions: a) for r = 0; b) for r = 0.8

It is well known that in the case of $r = 0$ (Figure 5, a), the value R has a Rayleigh distribution; if the correlation coefficient is nonzero, then the probability density function of R should be expressed via special functions, and it is difficult to calculate analytically. Nevertheless, the corresponding expression is easy to find in the CAS Mathematica. Below, Figure 6 shows the scatter plots of random samples of R and the graph of its probability density function for two different values of correlation coefficients.

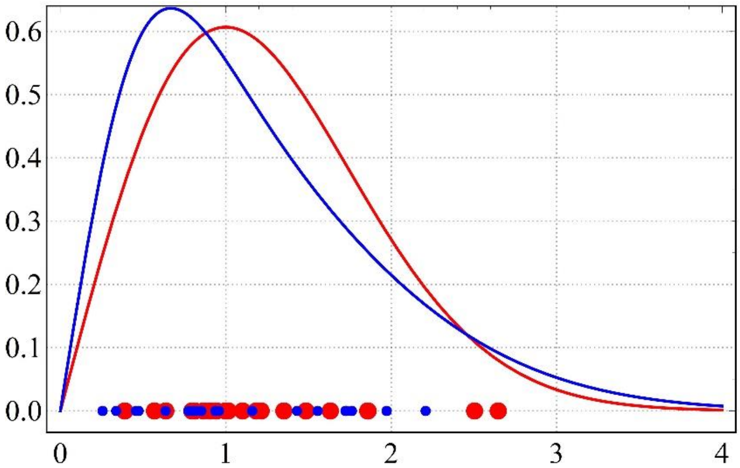


Fig. 6. Scatter plots (sample volume = 20) of length R of the radius vectors of normally distributed random points and the graphs of probability density functions of R: red colour corresponds to $r = 0$; dark blue corresponds to $r = 0.8$

Figure 6 enables one to compare the probability density functions corresponding to different correlation coefficients. Although the difference in the shape of the graphs of probability density functions seems insignificant, it appears that the distribution corresponding to $r = 0.8$ possesses a significantly heavier tail, as one can see in the diagram.

2.3. Conditional expectation of minimum under a known value of maximum

Let X_1 and X_2 be independent random values with the same Rayleigh distribution. Consider the new random values

$$Y_1 = \min\{X_1, X_2\},\ Y_2 = \max\{X_1, X_2\}.$$

The authors made a graphical illustration of the notion of conditional expectation $\mathbf{E}(Y_1|Y_2)$. First, they generated a sample of 100 values of the random vector (X_1, X_2); next, they transformed it into the sample of 100 values of the random vector (Y_1, Y_2). In Figure 7, the first set of points appears in a dark blue colour; the transformed set of points is shown in red.

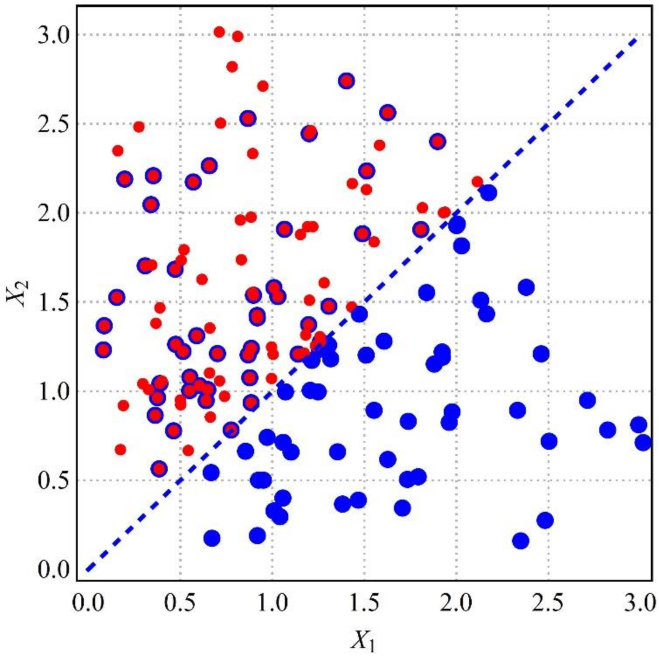


Fig. 7. A cloud of 100 sample points of the random vector (X_1, X_2) (dark blue points) and the corresponding 100 points of the random vector (Y_1, Y_2) (smaller red ones). The dark blue dash line is the reflecting screen (the bisector of the first quadrant)

The next figure (Figure 8) illustrates the notion of the regression line. The scatter plot of the random vector (Y_1, Y_2) is combined with the regression line and the reflecting screen.

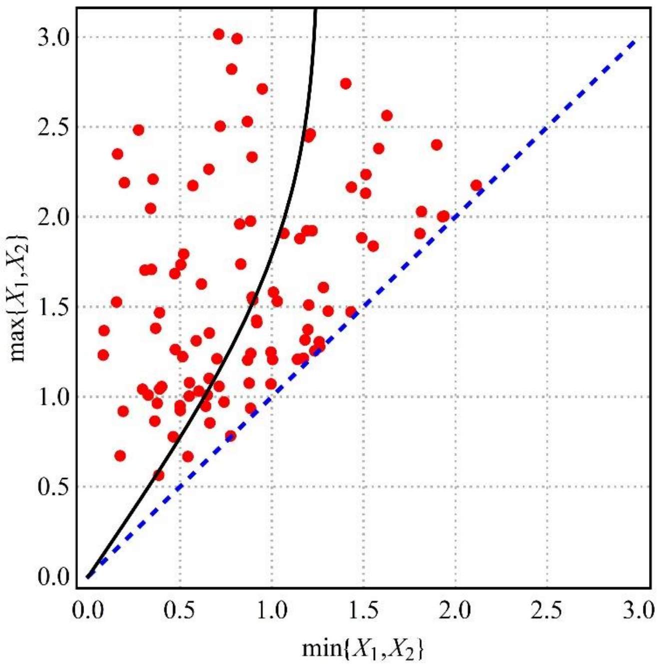


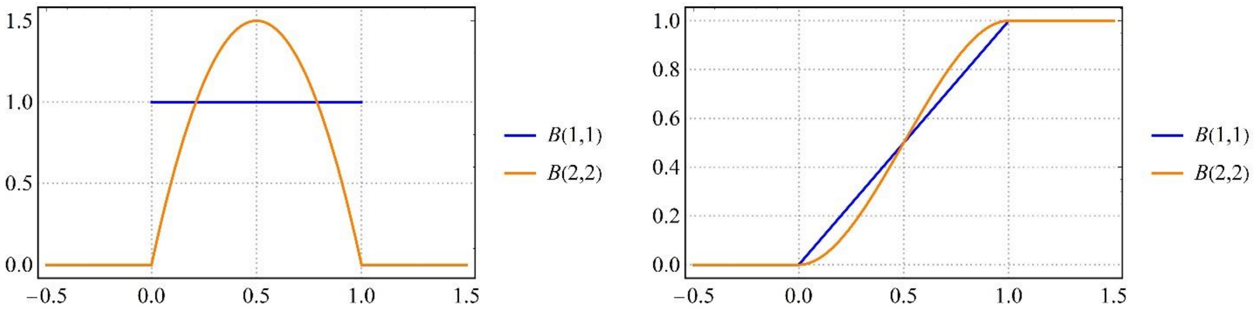
Fig. 8. A cloud of 100 sample points of the random vector (Y_1, Y_2) (red points) and the regression line $y_1 = \mathrm{E}(Y_1|Y_2 = y_2)$ (black solid line). The dark blue dash line is the bisector of the first quadrant

This way, one can use the graphical possibilities of CAS Mathematica for a simple explanation of the often difficult to understand notion of conditional expectation.

2.4. Comparison of samples of Beta distributions

Consider the random values X and Y distributed in the segment [0;1] as having different laws of distribution: X is uniformly distributed, and Y has B(2,2) distribution. Note that the uniform distribution is the same as B(1,1). Both distributions are symmetric with respect to the middle of the segment, but they have different density functions. The task is to observe resemblances and differences in the characters of the distribution laws of X and Y to demonstrate non-evident differences in their distributions and to clarify the basic notions and laws of sampling theory.

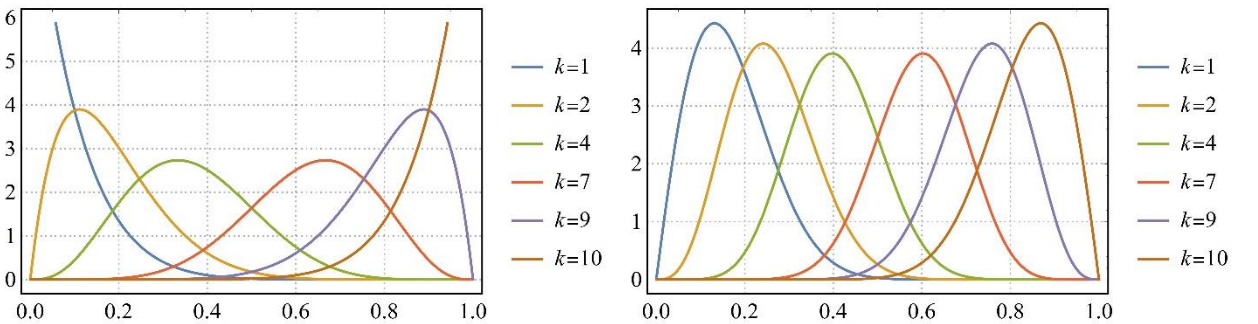
The researchers constructed the graphs of probability density functions and distribution functions of X and Y. For this task, they used functions built in the CAS Wolfram Mathematica. Figure 9 shows the results of running the commands.



a) b)

Fig. 9. Graphs of probability density functions (a) and distribution functions (b) of the laws B(1,1) and B(2,2)

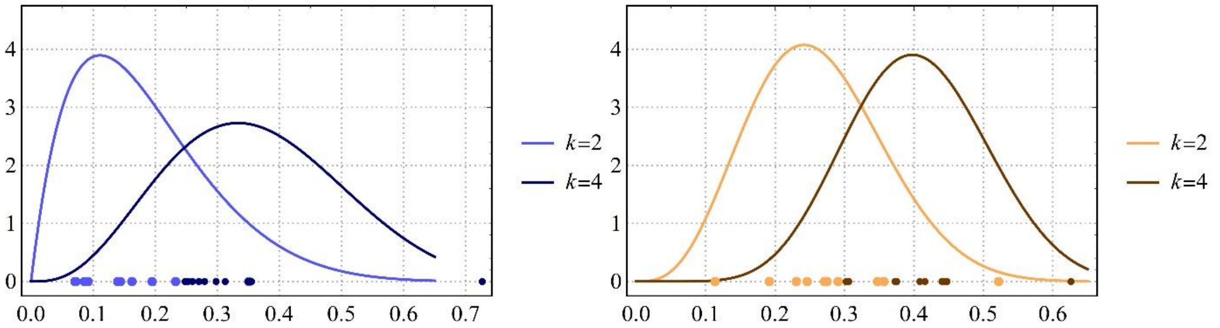
The authors then explored the distributions of the order statistics. They used the well-known formulas of their probability density functions (Van der Waerden, 1969) and constructed their graphs (see Figure 10).



a) b)

Fig. 10. Graphs of probability density functions of the k-th order statistics, k = 1, 2, 4, 7, 9, 10, for samples of volume 10 from the laws B(1,1) (a) and B(2,2) (b)

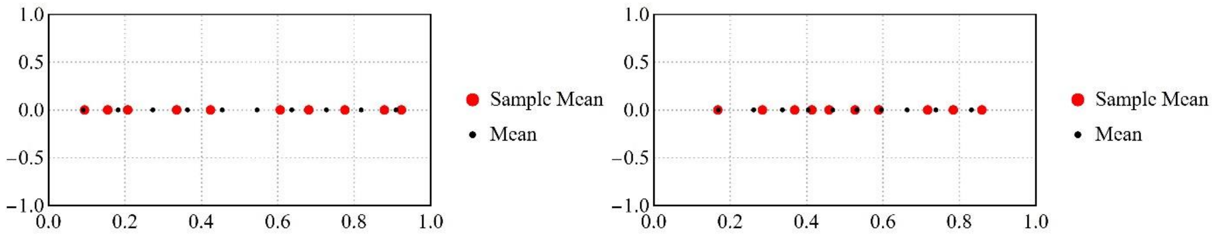
In addition, one can illustrate these graphs with samples of the order statistics obtained. To do so, generate 10 samples, with each having volume 10, from the laws B(1,1) (a) and B(2,2) (b). Figure 11 shows the graphs of probability density functions of the second and the fourth order statistics and the samples of their obtained values.



a) b)

Fig. 11. Graphs of probability density functions of the second and the fourth order statistics of samples of volume n = 10 from the laws B(1,1) (a) and B(2,2) (b) and samples of volume 10 of their values (points below the graphs)

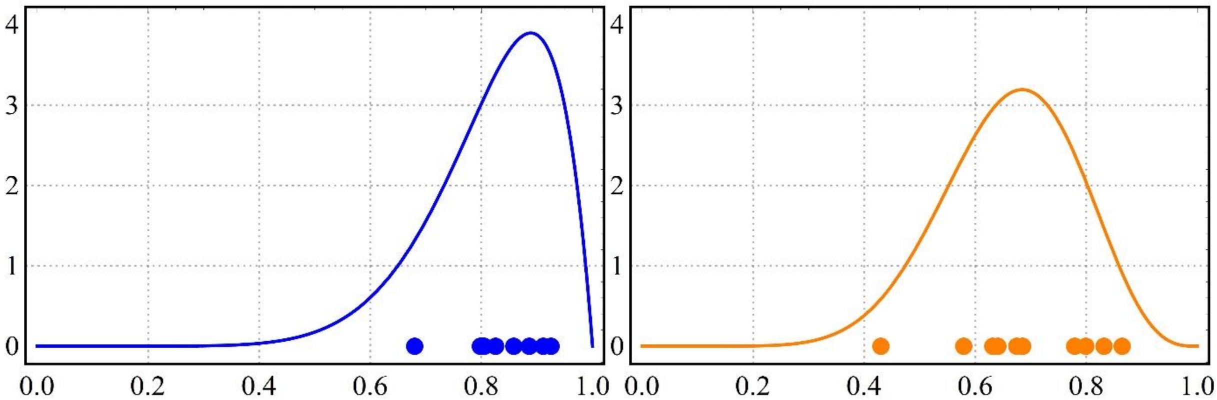
The researchers then compared the sample mean values of the order statistics of 10 samples with their theoretical values obtained via numerical methods. Figure 12 shows the results of running Mathematica commands.



a) b)

Fig. 12. Sample mean values of the k-th order statistics (k = 1,…, 10) compared with their theoretical values for 10 samples of volume n = 10 from the laws B(1,1) (a) and B(2,2) (b)

The researchers completed the example by exploring the problem of distribution of the sample range. For this purpose, they found analytical expressions of its probability density functions. In the case of the B(2,2) law, the expression is too bulky; thus, it is difficult to calculate it without using a CAS. But if one uses Mathematica, the density functions can be found quickly and effectively. Next, plot the probability density functions of the sample range of the two laws under consideration and compare them with the empirical values obtained earlier (Figure 13).



a) b)

Fig. 13. Scatter plots of sample ranges (sample volume = 10) for the laws B(1,1) (a) and B(2,2) (b) and their probability density functions

The graphs clarify differences between the distributions of the samples.

2.5. The Wiener process distribution and running maximum

The topic of normal laws of distribution, as discussed in section 2.1, closely relates to exploring the properties of one of the most well-known random processes: the Wiener process.

First, it is important to review the properties of one-dimensional distributions of the standard Wiener process w_t (Figure 14).

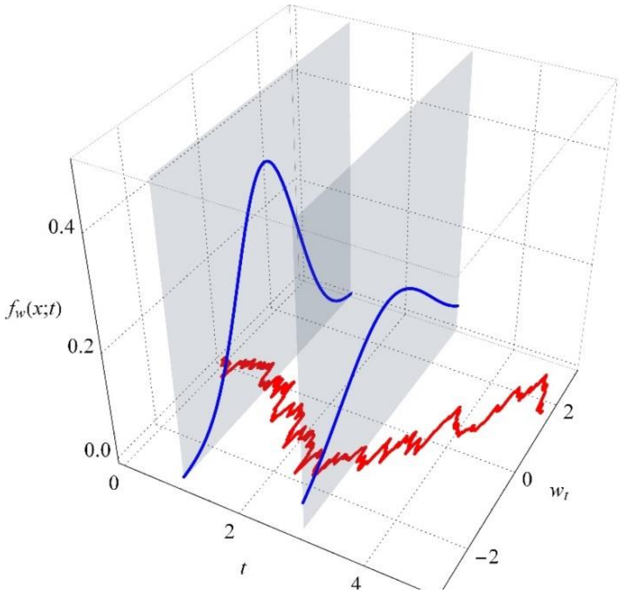


Fig. 14. A path of the standard Wiener process w_t (red line) and its probability density functions at the moments t = 1 and t = 3 (dark blue lines)

As Figure 14 demonstrates, the probability density function changes as t increases: whereas the mode is constant, the volatility increases.

In the same way, one can explore the conditional laws of distribution under a known value of the process during a fixed moment in time (Figure 15).

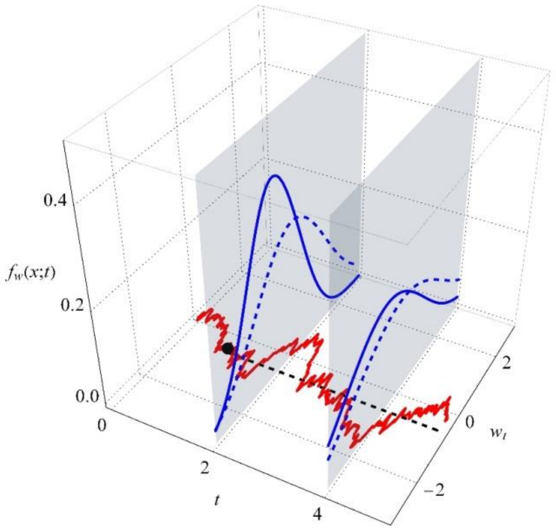


Fig. 15. A path of the standard Wiener process w_t (red line), its conditional expectation $E(w_t|w_1=x_0)$ (black dash line) and conditional probability density functions at the moments t = 2, 4 (dark blue solid lines). The dark blue dash lines show the non-conditional probability density functions at the moments t = 2, 4 for comparison

Figure 15 enables one to compare the conditional and non-conditional probability density functions of a Wiener process during the same moment of time (the solid dark blue lines and the dark blue dashes, respectively). Moreover, Figure 15 demonstrates the fact that knowledge of the value of the process at a given moment of time decreases its volatility.

The running maximum is expressed as m_t, i.e. the maximum of a standard Wiener process w_t until the moment t. Figure 16 clarifies the notion of m_t.

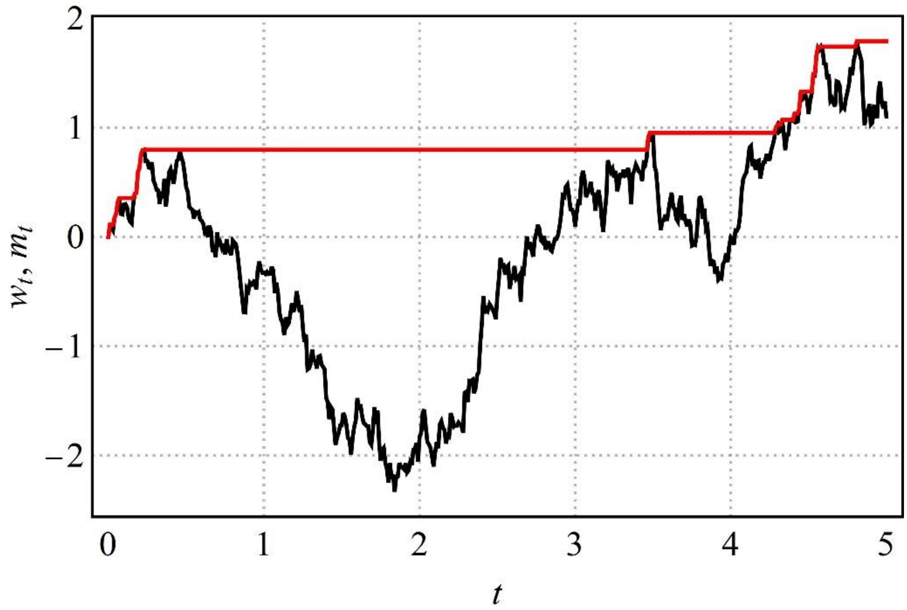


Fig. 16. A path of the standard Wiener process w_t (black line) and the corresponding path of running maximum m_t (red line)

The next figure (Figure 17) shows the empirical and theoretical distribution functions of m_t at a given moment in time, i.e. t. Figure 18 presents its histogram and probability density function.

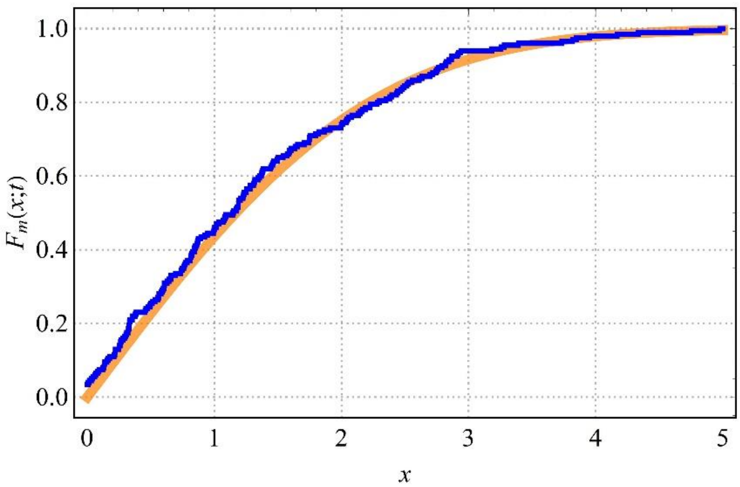


Fig. 17. Graphs of theoretical (orange line) and empirical (dark blue line) distribution functions of the running maximum m_t

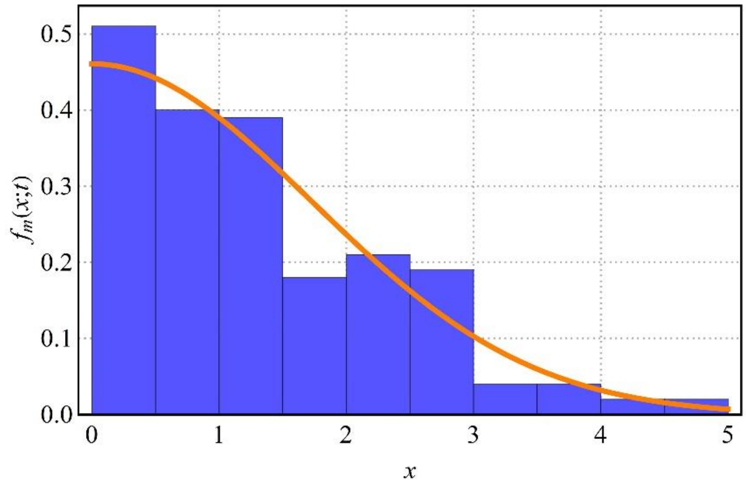


Fig. 18. Graph of theoretical probability density function (orange line) and a sample histogram (dark blue) of the running maximum m_t

A comparison of the empirical and theoretical laws of distribution is especially important for understanding the foundation of the sampling method. Moreover, the examples under consideration clearly demonstrate how one can apply the simplest statistical laws to study much more complicated notions such as the running maximum.

3. Methodology of the Research

3.1. General background of the research

Research on students' attitudes toward various teaching methods (e.g. a manuscript summary of lectures, an electronic summary of lectures, presentations of the lecture course prepared using CAS Mathematica, manuscript materials for seminar studies, electronic training materials for seminar studies, electronic methodic materials that help with at-home tasks, task solution templates prepared using CAS Mathematica) has been conducted. The research includes data derived from the responses of fourth-year students (165 students, 98 male and 67 female, aged 20 to 27 years) who took the course 'Supplementary Chapters of Probability Theory' in the years 2016-17 and 2017-18. A questionnaire consisting of four questions that focused on gender, age, the applied teaching methods and students' attitudes toward such methods was used as a data collection tool for the research. The purpose of the study was to determine the main advantages and disadvantages of the applied methods in order to improve them in the future. The sample data were collected from December 2016 until June 2018 and were analysed in June–July 2018.

3.2. Sample for the research

The research included a questionnaire that was given to fourth-year students who had learned 'Supplementary Chapters of Probability Theory' and agreed to take part in the survey. All of the students were in their fourth year at Bauman Moscow State Technical University during the years 2016-17 or 2017-18. The research presents the data of those students who accurately completed the questionnaires. In other words, they provided answers to all of the questions, without any contradictions in their answers to questions 3 and 4. (The questionnaires that did not include a teaching means in response to question 3 but included its evaluation in question 4 were not taken into account). As a result, a total 165 questionnaires were chosen for statistical analysis. Eleven questionnaires were excluded due to their lack of demographic data or mistakes related to questions 3 and 4. The interview procedure eliminated the possibility for discussions and cribbing.

The research includes the sample data collected from the questionnaires completed by 165 students (98 male and 67 female) aged 20 to 27 years. Their mean age was 21.72 with a standard deviation of 0.84. Such a distribution in gender and age is typical for students of technical specialisations.

All the data about the students and their education outcomes were presented in impersonal form. In other words, in the research, the authors did not include the students' names, specialisations and groups, their education marks or their answers. Only the obtained quantitative data on education outcomes and the questioning results were analysed.

3.3. Instrument and procedures

The questionnaire, as presented in Table 1, included four questions and was the tool of research. The main purpose of distributing a questionnaire to students was to compare and analyse their attitudes toward traditional teaching methods (e.g. manuscript and electronic summaries of lectures, manuscript materials for seminar studies, electronic training materials for seminar studies, electronic materials to help with at-home tasks) and innovative methods prepared using CAS Mathematica (e.g. presentations of the lecture course, task solution templates). The questionnaire contained typical demographic questions (gender and age) and asked the students about the teaching means within the half-year term. In order to determine the students' attitudes toward the applied teaching means, they were asked to assess convenience of each means based on several statements, with each statement having values of suitability that ranged from '5' (fully corresponds) to '1' (fully contradicts). Typical statements were as follows: a) this teaching means is easy to understand; b) this teaching means is easy to use at home; c) this teaching means is easy to use in city transport etc. Table 1 provides a complete list of the statements.

It is commonly known that modern education programmes involve many components; as such, many students view the time assigned to independent work as insufficient. Consequently, students often do not have enough time and/or motivation for an independent study of a subject and to perform related tasks. Electronic presentations can clarify complicated physical phenomena; hence, they support text searches and other tools that save students' time and enable the teacher to apply the 'client first' approach. This advantage becomes particularly important during the reforming and optimisation of the education process, the aggregation of lecture audiences etc. Another reason is that penetration of interactive task templates, in addition to saving time, stimulates the development of the skills of independent solution of engineering tasks in the future.

It is interesting to clarify the students' attitudes toward these matters. The statements given in the fourth question of the questionnaire aimed to explore the following important aspects related to teaching means:

1) do they regard a certain teaching means as a) easy/difficult, b) necessary/unnecessary;

2) which scenarios of independent using the teaching means do they choose: a) using at home/fluent using (e.g. in transport);

3) does the teaching means save time (e.g. by supporting the search of text information).

Table 1. The list of statements evaluated by students

1. Your gender	□ female □ male
2. Your age (full years)	
3. Mark the teaching means you have used in this term: □ A1 Manuscript summary of lectures; □ A2 Electronic summary of lectures; □ A3 Presentations of the lecture course prepared using CAS Mathematica; □ B1 Manuscript materials for seminar studies; □ B2 Electronic training materials for seminar studies; □ C1 Electronic methodic materials that help with at-home tasks; □ C2 Task solution templates prepared using CAS Mathematica.	
4. For each of the teaching means used, mark each statement based on your impression (5 – fully corresponds, 1 – fully contradicts). If you have not used the teaching means, leave the cell empty.	

Statement / Teaching means	A1	A2	A3	B1	B2	C1	C2
Q1 This teaching means is easy to understand.							
Q2 This teaching means is easy to use at home.							

Q3 This teaching means is easy to use in city transport.							
Q4 This teaching means is suitable for quickly finding information.							
Q5 This teaching means is not convenient to use.							
Q6 This teaching means is difficult to understand.							
Q7 This teaching means saves time.							
Q8 I like this teaching means.							
Q9 I don't like this teaching means.							
Q10 This teaching means is necessary.							
Q11 This teaching means is unnecessary.							

The questioning was carried out during the 14th-16th week of each term. (It was the seventh or the eighth half-year term, depending on the students' specialisations.) Groups of students who were interested in responding to the questionnaire (no more than 20 students in each group) were invited after the studies into a lecture hall suitable for 180 students. They sat spaced out from each other so that they were unable to see each other's answers. They were given 20 minutes to complete the questionnaire. The questionnaire was anonymous and contained no personal information.

3.4. Data analysis

The collected sample data were processed with Excel, Statistica, and Mathematica. The correctness of the answers to questions 3 and 4 was checked. (As previously mentioned, the questionnaires that did not include a teaching means in response to question 3 but included its evaluation in question 4 were not taken into account). As a result, 165 correct questionnaires were chosen for statistical analysis.

3.5. Results

The overwhelming majority of students had more or less used all of the teaching means. The number of students not using each of the teaching means (question 3) and the associated percentages are provided in Table 2. Note that relatively more students ignored the teaching means C2 'Task solution templates prepared with using CAS Mathematica'. Possible reasons for this finding are students' fear prior to using program languages and the absence of equipment.

Table 2. The number of students not using each of the teaching means and the percentages

Teaching means	A1	A2	A3	B1	B2	C1	C2
Number of students not using	7	3	7	9	4	5	16
Percentage	4.2	1.8	4.2	5.5	2.4	3.0	9.7

Table 3 (below) presents the frequency with which students selected '4' and '5' for items in the fourth question.

Table 3. Frequency of '4' and '5' answers to each statement and the associated teaching means

Statement \ Teaching means	A1	A2	A3	B1	B2	C1	C2
Q1	74	89	121	92	97	102	59
Q2	99	112	132	111	117	121	109
Q3	67	122	135	69	124	120	24
Q4	89	129	127	96	125	122	92
Q5	41	26	24	37	27	21	48

Q6	37	33	27	34	19	23	59
Q7	70	112	131	90	129	134	80
Q8	89	94	121	87	98	115	64
Q9	32	23	21	24	19	14	42
Q10	134	114	102	128	109	105	98
Q11	5	4	14	12	5	8	23

Correspondence analysis (Clausen, 1998; Hoffman, Franke, 1986) was used to determine the linkage between the statements and the teaching means. Figure 19 presents the graph of eigenvalues for the data from Table 3, wherefrom one sees that two dimensions are necessary. The first dimension explains 81 % of the inertia; the second dimension explains 12 % of the inertia. Figure 20 depicts the correspondence analysis map.

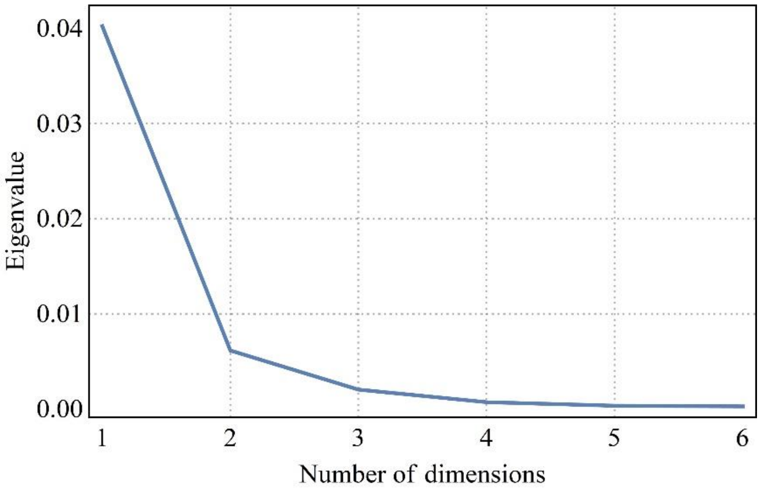


Fig. 19. Graph of eigenvalues for Table 3

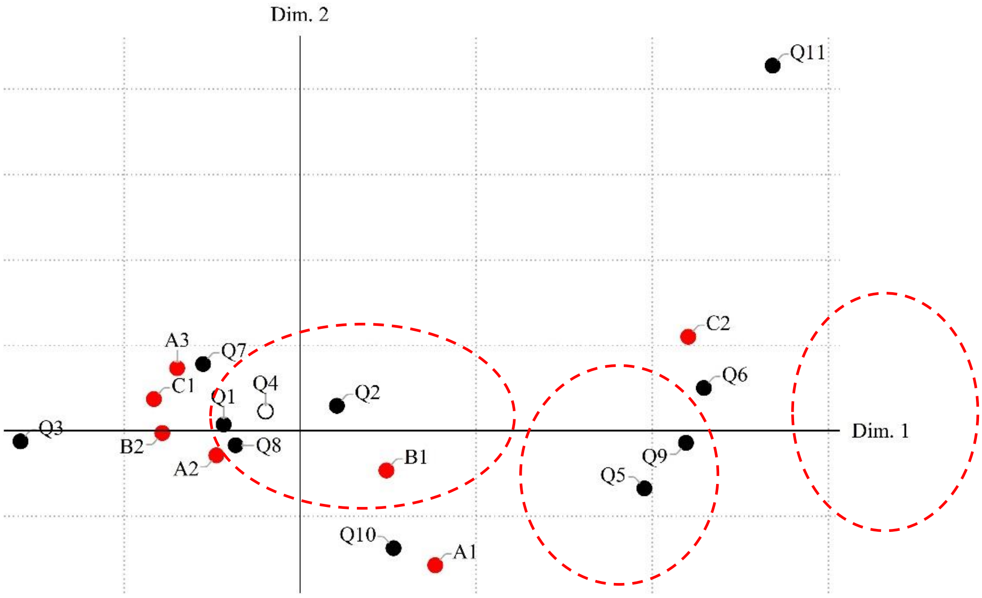


Fig. 20. Correspondence analysis map of the data from Table 3 for two dimensions. The first dimension explains 81 % of the inertia; the second dimension explains 12 % of the inertia

In Figure 20, the points corresponding to the teaching means are coloured red; the points corresponding to the statements are black. The projection of point Q4 onto the two-dimensional plane is unsatisfactory; hence, it is marked separately. For all other points, the correlation coefficients with the axes are greater than 0.5, and the quality of the projection onto the plane is good or excellent. The authors then proceeded to a qualitative analysis of the results obtained.

It is important to notice that the statement Q11 'This teaching means is unnecessary' stands separately and is irrelevant to the teaching means. The first dimension can be interpreted as 'Simplicity vs. difficulty of fluent usage' (the statements Q3 and Q6 load this dimension), the second dimension as 'Usefulness vs. uselessness of a teaching means' (the statements Q10 and Q11 load this dimension). The points make several clusters surrounded by red dash lines. The teaching means C2 stands opposite to other electronic means. One can also interpret the first dimension as 'Electronic teaching means without vs. with CAS compilation'.

The researchers then performed an analysis of the clusters and found that the statements Q5, Q6 and Q9 were associated with the teaching means C2. Therefore, for many students, the interactive templates seem complicated and difficult to understand. However, the students did not conclude that this means is unnecessary. There are several reasons for this finding. First, many students are unfamiliar with CAS, and the syntax of CAS strongly differs from traditional program languages. Hence, students do not understand how to modify the input data to obtain the answer. Second, the CAS Mathematica is not effectively adaptable on mobile devices, especially if calculations are complicated. Students remarked that they would like to be able to calculate fluently using a smart phone or a tablet.

The traditional teaching means A1 and B1 were expectedly grouped together. Mostly, the statements Q2 and Q10 corresponded to them. This finding is related to students' habit of using the materials from lectures and seminars in extra-curricular work.

The other teaching means, i.e. A2, A3, B2, and C1, are electronic, and the students' attitudes toward them were approximately equal. The statements Q1, Q3, Q7, and Q8 were relevant to them. The students regarded these teaching means as simpler compared to the traditional means of A1 and B1 and more convenient to use, especially in terms of the ability to fluently apply them. The most remarkable aspects for the students are the convenience of work with mobile devices and the possibility of full-text searches. An interesting fact is that the students did not detect a strong linkage between interactive templates (C2) and the presentations (A3) prepared using these templates. Therefore, one can conclude that CAS learning should begin in the first term of students' university education.

4. Discussion

The article describes teaching methods using the visualisation capabilities of the CAS Mathematica within the course 'Supplementary Chapters of Probability Theory' for students of technical specialisations. The new teaching means—i.e. an electronic summary of lectures, presentations of lecture courses and task solution templates prepared with CAS Mathematica, and electronic training materials for seminar studies—were used alongside the usual summaries of lectures and seminar studies. The examples given in the article are concerned with the topics 'Mathematical statistics' and 'Theory of random processes'.

The teaching methods under consideration present many new possibilities to students and teachers. First, they save time during the calculation, preparation and demonstration of graphic materials. Analogous conclusions were drawn in Vlasova et al. (2017b). Moreover, it is unnecessary to choose tasks that require an elegant analytical solution. On the other hand (Psycharis, 2016), it is possible to pay more attention to the basic principles of sampling theory, which are important in future professional activities. In addition, one advantage of using CAS as handheld calculators is their capability to perform algebraic, graphic and numeric calculations (Garner, 2004). In previous research (Emmioğlu, Capa-Aydin, 2012; Ivaniushina et al., 2016; Ivanov et al., 2017), the positive impact of computer teaching methods on education outcomes was established. Moreover, the high individualisation of teaching processes and the variability of teaching purposes achieved were identified. Each of these factors has a positive influence on students' motivation.

Another advantage of used supplementary teaching means is their variability, i.e. the capability to adapt to any tasks or purposes related to education. Since stochastic conformities to natural laws must be taken into account in the analysis of real data, probabilistic subjects form a base for the further study of the professional circle of subjects (e.g. the analysis of big data, system analysis, experiment planning etc.), and for application of the obtained knowledge in practice. Interactive presentations can explain and visualise the basic stochastic laws. In addition, they provide training in skills for independent work with CAS, independent searches for the solution algorithm and the choice of the method of visualisation of dependencies. Moreover, they teach students to create certain templates that save time and can be useful for a wide range of tasks. This way, students learn to solve a certain task and to construct a solution tool so that it is maximally general and flexible. It is especially important in consideration of the expansion of new methods, such as neural net programming, machine teaching and the analysis of big data.

Note that contact and an understanding between students and a teacher in the process of education is necessary. Hence, the teaching means and their usage scenarios should be aimed not only to the certain purposes of education but also to each student's needs.

In order to clarify the students' attitudes toward applied teaching means, a questionnaire containing several significant questions about traditional and CAS-based teaching methods was developed. A total of 165 students, 59 % men and 4 1% women, completed the questionnaire. The produced samples are representative of engineering students since men more often choose scientific-quantitative occupations (Del Pero, Bychkova, 2013).

First, it is important to note that the majority of students used all of the teaching means given. Although the degree of their usage varied, a common interest in and increasing motivation to computer teaching are detectable. There are similar conclusions on computer usage in other studies (Ivaniushina et al., 2016; Ivanov et al., 2017; Korres, 2018; Psycharis, 2016).

Attention was directed toward the causes of the observed low level of usage of task solution templates prepared with CAS Mathematica. The following causes were discovered:

1. Using the desktop version of Mathematica is not always convenient; on the other hand, functions of the mobile application are quite restricted, especially if one works with big data and complicated analytical calculations;
2. Students' possible fear of STEM subjects;
3. Motivation to achieve results by using new means (e.g. program languages) instead of traditional means is insufficient;
4. Possible gender differences related to the perception and usage of various teaching means (not considered in the present article).

The first statement, i.e. the necessity to adapt the teaching materials to mobile devices, is well known (Davidovich, Yavich, 2018; Geiger et al., 2018; Ipek, Ziatdinov, 2017). The CAS developing companies have taken steps to make programs suitable on such devices. Moreover, the possible difficulties that arise from the necessity to adapt the whole course to one computing platform have been noted. It is well known that different programs are often convenient for solving different mathematical problems. Harrison and Lee (2018) discussed the criteria for choosing applications, possible difficulties and ways to partially overcoming them. In addition, research has shown that the majority of students regard program languages as the most difficult subjects (Broley et al., 2018). Hence, it is not reasonable to prepare templates using several different systems.

Other possible causes of low education outcomes exist. For instance, students may be fearful prior to difficult courses such as probability theory, statistics and programming (Griffith et al., 2012; Peng et al., 2014). Its possible consequence is that students reject the use of interactive patterns either a priori or in the first stage of a solution. In order to get over this psychological obstacle, it is necessary to perform explanatory work and to use supplementary stimulating measures, e.g. additional rating points for students using the interactive templates (Morán-Soto, Benson, 2018). Again arises the problem of motivation playing the key role in education outcomes.

In addition, it is necessary to explore possible gender differences in the perception and usage of the given materials. The hypothesis that gender differences influence students' perception of using tablets in education was posed and partially proved (Davidovitch, Yavich, 2018). Thus, while females would attribute more weight to the affective dimension, males would attribute more weight to the cognitive dimension. Further research on the influence of gender differences on perception and usage of the given teaching means is necessary for planning measures of their penetration and popularisation.

The students' attitudes toward various electronic teaching means are approximately equal. They regard these teaching means as simpler compared to the traditional means and as more convenient to use, especially in terms of being able to fluently use them. The most remarkable aspects for students are the convenience of working on mobile devices and the possibility of full-text searches. The general positive impact of electronic teaching means has been noted (Ivaniushina et al., 2016; Ivanov et al., 2017; Korres, 2018; Psycharis, 2016). An interesting fact is that the students did not detect a strong connection between interactive templates and presentations that were prepared using the templates.

5. Conclusion

Computer technologies play an essential role in almost all stages of the modern education process. Therefore, the teaching means used ought to help students and teachers achieve education purposes. The visualisation of complicated mathematical notions is especially important, and the means used must be the most simple and variable. One of the best means of this kind involves using the inbuilt graphical methods of various CAS (MATLAB, MathCad, Mathematica etc.). The article presents miscellaneous examples of using such means within learning certain topics of the course 'Supplementary Chapters of Probability Theory' and describes their advantages. In order to assess the suitability of such teaching means, the students were asked to reflect on their attitude and level of usage of each of the teaching means considered. Based on the findings, the immediate usage of CAS is regarded rather as difficult, mostly because of students' insufficient training in programming. Therefore, it is necessary to incorporate CAS into the education process beginning in the first term, especially in students' independent work.

References

Abell, Braselton, 2009 – *Abell, M.L., Braselton, J.P.* (2009). Mathematica by example. Amsterdam: Elsevier Academic Press.

Barba et al., 2016 – *Barba, P.G., Kennedy, G.E., Ainley, M.D.* (2016). The role of students' motivation and participation in predicting performance in a MOOC. *Journal of Computer Assisted Learning*, 32(3), 218-231. DOI: 10.1111/ jcal.12130

Broley et al., 2018 – *Broley, L., Caron, F., Saint-Aubin, Y.* (2018). Levels of programming in mathematical research and university mathematics education. *International Journal of Research in Undergraduate Mathematics Education*, 4(1), 38-55. DOI: 10.1007/s40753-017-0066-1

Clausen, 1998 – *Clausen, S.* (1998). Quantitative Applications in the Social Sciences: Applied correspondence analysis. Thousand Oaks, CA: SAGE Publications Ltd. DOI: 10.4135/9781412983426

Davidovich, Yavich, 2018 – *Davidovich, N., Yavich, R.* (2018). The impact of mobile tablet use on students' perception of learning processes. *Problems of Education in the 21st Century,* 76(1), 29-42.

Del Pero, Bychkova, 2013 – *Del Pero, A.S., Bychkova, A.* (2013). A bird's eye view of gender differences in education in OECD countries. *OECD Social, Employment and Migration Working Papers*, No. 149, OECD Publishing. DOI: 10.1787/5k40k706tmtb-en

Emmioğlu, Capa-Aydin, 2012 – *Emmioğlu, E., Capa-Aydin, Y.* (2012). Attitudes and achievement in Statistics: A meta-analysis study. *Statistics Education Research Journal*, 11(2), 95-102.

Fatikhova, Sayfutdiyarova, 2017 – *Fatikhova, L.F., Sayfutdiyarova, E.F.* (2017). Improvement of Methodology of Teaching Natural Science for Students with Intellectual Disabilities by Means of 3D-Graphics. *European Journal of Contemporary Education*, 6(2), 229-239. DOI: 10.13187/ejced.2017.2.229

Fisher, 2014 – *Fisher, C.R.* (2014). A pedagogic demonstration of attenuation of correlation due to measurement error. *Spreadsheets in Education (eJSiE),* 7 (1), article 4 [Electronic resource]. URL: http://epublications.bond.edu.au/ejsie/vol7/iss1/4

Garner, 2004 – *Garner, S.* (2004). The CAS classroom. *Australian Senior Mathematics Journal,* 18(2), 28-42.

Geiger, 2018 – *Geiger, V., Mulligan, J., Date-Huxtable, L. et al.* (2018). An interdisciplinary approach to designing online learning: fostering pre-service mathematics teachers' capabilities in mathematical modelling. *ZDM Mathematics Education,* 50(1-2), 217-232. DOI: 10.1007/s11858-018-0920-x

Griffith et al., 2012 – *Griffith, J.D., Adams, L.T., Gu, L.L., Hart, C.L., Nichols-Whitehead, P.* (2012). Students' attitudes toward Statistics across the disciplines: A mixed-methods approach. *Statistics Education Research Journal,* 11(2), 45-56.

Harrison, Lee, 2018 – *Harrison, Lee* (2018). iPads in the mathematics classroom: Developing criteria for selecting appropriate learning apps. *International Journal of Education in Mathematics, Science and Technology (IJEMST),* 6(2), 155-172. DOI: 10.18404/ijemst.408939

Hoffman, Franke, 1986 – *Hoffman, D.L., Franke, G.R.* (1986). Correspondence analysis: graphical representation of categorical data in marketing research. *Journal of Marketing Research,* 25, 215-227.

Holmes, 2003 – *Holmes, P.* (2003). 50 years of statistics teaching in English schools: some milestones. *Journal of the Royal Statistical Society: Series D (The Statistician),* 52(4), 439-474. DOI: 10.1046/j.1467-9884.2003.372_1.x

Ipek, Ziatdinov, 2017 – *Ipek, I., Ziatdinov, R.* (2017). New approaches and trends in the philosophy of educational technology for learning and teaching environments. *European Journal of Contemporary Education*, 6(3), 381-389. DOI: 10.13187/ejced.2017.3.381

Ivaniushina, 2016 – *Ivaniushina, V.A., Alexandrov, D.A., Musabirov, I.L.* (2016). The structure of students' motivation: Expectancies and values in taking data science course. *Educational Studies Moscow, 4*, 229-250. DOI: 10.17323/1814-9545-2016-4-229-250.

Ivanov et al., 2017 – *Ivanov, O.A., Ivanova, V.V., Saltan, A.A.* (2017). Discrete mathematics course supported by CAS Mathematica. *International Journal of Mathematical Education in Science and Technology,* 48(6), 953-963. DOI: 10.1080/0020739X.2017.1319979

Karakış et al., 2016 – *Karakış, H., Karamete, A., Okçu, A.* (2016). The effects of a computer-assisted teaching material, designed according to the ASSURE instructional design and the ARCS model of motivation, on students' achievement levels in a Mathematics lesson and their resulting attitudes. *European Journal of Contemporary Education*, 15(1), 105-113. DOI: 10.13187/ejced.2016.15.105

Kidron, Tall, 2015 – *Kidron, I., Tall, D.* (2015). The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics, 88* (2), 183-189. DOI: 10.1007/s10649-014-9567-x

Korres, 2018 – *Korres, K.* (2018). Students' attitudes towards discovery learning/Constructivistic approach using computers as cognitive tools in higher mathematics education. *European Journal Of Engineering Research And Science*, *CIE 2017*, 44-49. DOI: 10.24018/ejers.2018.0.cie.643

Kvasz, 2018 – *Kvasz, L.* (2018). On the roles of language in mathematics education. In P. Ernest. *The Philosophy of Mathematics Education Today*, 229-240. Springer: Cham. DOI: 10.1007/978-3-319-77760-3_14.

Lai, Savage, 2013 – *Lai, A., Savage, P.* (2013). Learning management systems and principles of good teaching: instructor and student perspectives. *Canadian Journal of Learning and Technology,* 39(3), 1-21.

Mezhennaya, Pugachev, 2018 – *Mezhennaya, N.M., Pugachev, O.V.* (2018). On the results of using interactive education methods in teaching Probability Theory. *Problems of Education in the 21 century, 76* (5), 678-692.

Morán-Soto, Benson, 2018 – *Morán-Soto, G., Benson, L.* (2018). Relationship of mathematics self-efficacy and competence with behaviors and attitudes of engineering students with poor mathematics preparation. *International Journal of Education in Mathematics, Science and Technology (IJEMST),* 6(3), 200-220. DOI: 10.18404/ijemst.428165

Peng et al., 2014 – *Peng, Y., Hong, E., Mason, E.* (2014). Motivational and cognitive test-taking strategies and their influence on test performance in mathematics. *Educational Research and Evaluation*, 20(5), 366-385. DOI: 10.1080/13803611.2014.966115

Psycharis, 2016 – *Psycharis, S.* (2016). The impact of computational experiment and formative assessment in inquiry-based teaching and learning approach in STEM education. *Journal of Science Education and Technology,* 25(2), 316-326. DOI: 10.1007/s10956-015-9595-z

Rau, 2017 – *Rau, M.A.* (2017). Conditions for the effectiveness of multiple visual representations in enhancing STEM learning. *Educational Psychology Review,* 29(4), 717-761. DOI: 10.1007/s10648-016-9365-3

Sezonova et al., 2016 – *Sezonova, O.N., Galchenko, S.A., Khodirevskaya, V.N.* (2016). The efficiency of higher education institutions as a basis for forming competent personnel for region economy. *European Journal of Contemporary Education,* 18(4), 464-471. DOI: 0.13187/ejced.2016.18.464

Troup et al., 2017 – *Troup, J., Soto-Johnson, H., Karakok, G., Diaz, R*. (2017). Developing Students' Geometric Reasoning about the Derivative of Complex Valued Functions. *Digital Experiences in Mathematics Education,* 3(3), 173-205. DOI: 10.1007/s40751-017-0032-1

Van der Waerden, 1969 – *Van der Waerden, B.L.* (1969). Mathematical Statistics. George Allen & Unwin Ltd., London, Springer-Verlag Berlin Heidelberg GmbH.

Vlasova et al., 2017a – *Vlasova, E.A., Mezhennaya, N.M., Popov, V.S., Pugachev, O.V.* (2017a). Methodological aspects of the discipline "Probability theory" in a technical university. *Modern High Technologies*, 17(11), 96-103. DOI: 10.17513/snt.36852

Vlasova et al., 2017b – *Vlasova, E.A., Mezhennaya, N.M., Popov, V.S., Pugachev, O.V.* (2017b). The use of mathematical packages in the framework of methodological support of probabilistic disciplines in a technical university. *Bulletin of Moscow Region State University. Series: Physics and Mathematics*, 4, 114-128. DOI: 10.18384/2310-7251-2017-4-114-128

Wojas, Krupa, 2017 – *Wojas, W., Krupa, J.* (2017). Some remarks on Taylor's polynomials visualization using Mathematica in context of function approximation. In: Kotsireas I., Martínez-Moro E. (eds). *Applications of Computer Algebra*. ACA 2015. Springer Proceedings in Mathematics & Statistics, vol. 198. Springer, Cham.