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Some Fixed Point Results For Multivalued Operators In Vector Valued Spaces

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Abstract : The aim of this paper is to prove some fixed point theorems for multivalued operators in E-b-metric space which is a Riesz space valued b-metric space.

Keywords : b-metric space, contraction mapping theorem, dedekind complete, E-b-metric space, multivalued operator, Riesz space, vector metric space.

Introduction : F. Riesz [7] introduced the concept of Riesz space. For a more extensive treatment of the theory of Riesz space we refer C. D. Aliprantis and K. C. Border [1], W. A. J.Luxemburg and A.C. Zannen [7].

Riesz space (or vector lattice) is an ordered vector space and at the same time a lattice also. Let E be a Riesz space with the positive cone $E_+ = \{x \in E : x \geq 0\}$ for an element $x \in E$, the absolute value $|x|$, the positive part x^+, the negative part x^- are defined as $|x| = x \vee (-x)$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ respectively.

If every non–empty subset of E which is bounded above has a supremum, then E is called Dedekind complete or order complete. The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds for every $a \in E_+$.

Example 1 ([1]). Let $R^n (n \geq 1)$ be the real linear space of all real n–tuples $x = (x_1, x_2, x_3, \ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots\ldots, y_n)$ with coordinatewise addition and multiplication by real numbers. If we define that $x \leq y$ means that $x_k \leq y_k$ holds for $1 \leq k \leq n$, then R^n is a Riesz space with respect to this partial ordering.

Definition 1.1 ([1]). Let E be a Riesz space. A sequence (b_n) is said to be order convergent or o–convergent to b if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n, written as $b_n \xrightarrow{0} b$ or $o.\lim b_n = b$.

Definition 1.2 ([1]). A sequence (b_n) is said to be order Cauchy (o–Cauchy) if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$ holds for all n and p.

Definition 1.3 ([1]). A Riesz space E is said to be o–Cauchy complete if every o–Cauchy sequence is o–convergent.

If range space of a metric space is Riesz space then it becomes a vector metric space.

Definition 1.4 ([2]). Let X be a non–empty set and E be a Riesz space. Then function $d : X \times X \to E$ is said to be a vector metric (or E–metric) if it satisfies the following properties :

(a) $d(x, y) = o$ if and only if $x = y$

(b) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Also the triple (X, d, E) is said to be a vector metric space. Vector metric space is generalization of metric space. For arbitrary elements x, y, z, w of a vector metric space, the following statements are satisfied :

(i) $0 \leq d(x, y)$ (ii) $d(x, y) = d(y, x)$

(iii) $|d(x, z) - d(y, z)| \leq d(x, y)$

(iv) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$

Example 2 ([2]). A Riesz space is a vector metric space $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is said to be the absolute valued metric on E.

Definition 1.5 ([2]). A sequence (x_n) in a vector metric space (X, d, E) vectorial converges (E–converges) to some $x \in E$, written as $x_n \xrightarrow{d.E} x$ if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \le a_n$ for all n.

Definition 1.6 ([2]). A sequence (x_n) is called E–cauchy sequence whenever there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \le a_n$ holds for all n and p.

Definition 1.7 ([3]). A vector metric space X is called E–complete if each E–cauchy sequence in X, E converges to a limit in X.

For more details and results regarding vector metric spaces we refer to [3], [5].

When E = R, the concepts of vectorial convergence and metric convergence, E–cauchy sequence and Cauchy sequence in metric are same.

When also X = E and d is the absolute valued vector metric on X, then the concept of vectorial convergence and convergence in order are the same.

I.A. Bakhtin [14] defined the concept of b-metric space in 1989.

Definition 1.8 ([6]) : Let X be a non–empty set and let $s \ge 1$ be a given real number. A function $d : X \times X \to R_+$ is called a b–metric provided that, for all $x, y, z \in X$

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \le s[d(y, x) + d(y, z)]$

A pair (X, d) is called a b–metric space. It is clear from definition that b–metric space is an extension of usual metric space .

Example 3 ([3]) : The space $L_p (0 < p < 1)$ of all real functions x(t), $t \in [0, 1]$ such that $\int_0^1 |f(t)|^p \, dt < \infty$, is b–metric space if we take

$$d(f, g) = \left(\int_0^1 |f(t) - g(t)|^p \, dt \right)^{1/p} \quad \text{for each } f, g \in L_p$$

Several authors have investigated fixed point theorems on b–metric spaces, one can see [6], [8]

Combining the concept of vector metric space (E-metric space) and b-metric space I. R. Petre [5] defined E–b–metric space as follows:

Definition 1.9 ([5]). Let X be a non–empty set of $s \ge 1$, A functional $d : X \times X \to E_+$ is called an E–b–metric if for any $x, y, z \in X$, the following conditions are satisfied :

(a) $d(x, y) = 0$ if and only if $x = y$

(b) $d(x, y) = d(y, x)$

(c) $d(x, z) \le s[d(x, y) + d(y, z)]$

The triple (X, d, E) is called an E–b–metric space.

Example 4. Let d: $[0,1] \times [0,1] \to R^2$ defined by $d(x,y) = (\alpha |x-y|^2, \beta |x-y|^2)$ then (X,d,R^2) is an E-b-metric space where $\alpha, \beta > 0$ and $x,y \in [0,1]$.

Example 5 . The space $l_p(0 < p < 1)$, $l_p = \left\{ x = (x_i) : x_i \in R, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$ and $x = \{x_i\}$, $y = \{y_i\} \in l_p$ define $\rho(x, y) = (\alpha_1 \|x - y\|_p, \alpha_2\|x - y\|_p, \ldots \alpha_n\|x - y\|_p)$ then (l_p, ρ, R^n) is an E-b-metric space.

For more facts regarding vector metric space see [11], [12].

Let X is a non empty set and T: X → P(X) is a multivalued operator, we denote by $F_T = \{x \in X : x \in T(x)\}$, where

$p(X) = \{Y : Y \subseteq X\}$;

$P(X) = \{Y \in P(X) : Y \neq \phi\}$

And in the context of a vector metric space (X, d, E), we denote by

$P_{cl}(X) = \{Y \in P(X) : Y$ is E- closed$\}$;

$P_b(X) = \{Y \in P(X) : Y$ is E- bounded$\}$;

$Graph(T) = \{(x,y) \in X : y \in T(X)\}$.

Definition 1.10 ([4]). Let (X, d, E) be a vector metric space. The operator $T: X \to P_{cl}(X)$ is said to be a multivalued k- contraction, if and only if $k \in [0,1)$ and for any $x, y \in X$ and any $u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u,v) \leq k\, d(x, y) \qquad \text{………. (*)}$$

Definition 1.11 ([4]). Let (X, d, E) be a vector metric space. The operator T: X → P (X) be a multivalued operator. The sequence $(x_n)_{n\in N} \subset X$, recursively defined by

$$\{x_0 = x, x_1 = y;$$
$$\{x_{n+1} \in T(x_n), for all n \in N$$

is called the sequence of successive approximations of T starting from $(x,y) \in Graph(T)$.

Definition. Let (X, d, E) be an E-complete E-b-metric space. The operator $T : X \to P_{cl}(X)$ is said to be a multivalued (a,b,c,e,f)-contraction if and only if $a,b,c,e,f \in R_+$ with $a+b+c+e+f < 1$ and for any $x,y \in X$ and any $u \in T(x)$, there exists $v \in T(y)$ such that

$d(u,v) \leq ad(x,y) + bd(x,u) + cd(y,v) + e\, d(x,v) + f\, d(y,u)$

Main Results :

Theorem 1. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean and let $T : X \to P_{cl}(X)$ be a multivalued k-contraction with $sk < 1$ and $k \in (0,1]$. Then T has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations of T starting from $(x,y) \in Graph(T)$ for $n \in N$ which E-converges in (X, d, E) to the fixed point of T.

Proof : Let $x_0 \in X$ and $x_1 \in Tx_0$ then there exists $x_2 \in Tx_1$ such that

$d(x_1,x_2) \leq k\, d(x_0,x_1)$

Thus, define the sequence $(x_n) \in X$ by $x_{n+1} \in Tx_n$ and

$d(x_n,x_{n+1}) \leq k\, d(x_{n-1},x_n)$ for $n \in N$.

Inductively, we obtain,

$d(x_n,x_{n+1}) \leq k\, d(x_{n-1},x_n) \leq k^2 d(x_{n-2},x_{n-1}) \leq \ldots\ldots \leq k^n d(x_0,x_1)$ for $n \in N$.

Now, for all n and p, we have

$d(x_n,x_{n+p}) \leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \ldots\ldots\ldots + s^p d(x_{n+p-1},x_{n+p})$ for any $n \in N$

$d(x_n,x_{n+p}) \leq sk^n d(x_0,x_1) + s^2 k^{n+1} d(x_0,x_1) + \ldots\ldots\ldots + s^p k^{n+p-1} d(x_0,x_1)$ for any $n \in N$

$$= \frac{sk^n\left(1-(sk)^p\right)}{(1-sk)} \text{ d}(x_0,x_1) \le \frac{sk^n}{1-sk} \text{ d}(x_0,x_1) = a_n.a = b_n \text{ for any } n \in N,\ p \in N$$

Where $a_n = \frac{sk^n}{1-sk} \downarrow 0$ and $a = d(x_0, x_1) \in E^+$

Now, since E-Archimedean property, we get $b_n \downarrow 0$. So, the sequence $\{x_n\}$ is E-cauchy sequence in X. By the E- completeness of X, there is $z \in X$ such that $d(x_n,z) \le a_n$.

We know that $x_{n+1} \in Tx_n$ for any $n \in N$ and by the multivalued k-contraction condition it follows that there exists $u \in Tz$ such that $d(x_{n+1},u) \le k\, d(x_n,z)$ for any $n \in N$.

Then the following estimation holds:

Since $d(z,u) \le sd(z,x_{n+1}) + sd(u,x_{n+1}) = sd(x_{n+1},z) + sd(x_{n+1},u)$

$\le skd(x_n,z) + sa_{n+1}$

$\le ska_n + sa_{n+1} \le s(k+1)a_n \downarrow 0$

Thus, there exists $z = u \in Tz$ i.e. T has a fixed point in X.

Example 6. Let $E = R^2$ with componentwise ordering and let $X = [0,1]$

The mapping $d : X \to E$ is defined by

$$d(x, y) = \left(\frac{4}{3}|x-y|^2, |x-y|^2\right)$$

Then X is E-b-metric space. Let $T: X \to P_{cl}(X)$ with $T(x) = \{u(x), v(x)\}$, where $u,v : X \to X$ are defined by $u(x) = \frac{x}{2}$, $v(x) = \frac{x}{3}$

We have the following possibilities:

Case 1: for any $(x,y) \in X$ and any $\frac{x}{2} \in T(x)$, there exists $\frac{y}{2} \in T(y)$ such that

$$d\left(\frac{x}{2},\frac{y}{2}\right) \le kd(x,y)$$

$$\Rightarrow \left(\frac{4}{3}\left|\frac{x}{2}-\frac{y}{2}\right|^2, \left|\frac{x}{2}-\frac{y}{2}\right|^2\right) \le k\left(\frac{4}{3}|x-y|^2, |x-y|^2\right)$$

Case 2: for any $(x,y) \in X$ and any $\frac{x}{3} \in T(x)$, there exists $\frac{y}{3} \in T(y)$ such that

$$d\left(\frac{x}{3},\frac{y}{3}\right) \le kd(x,y)$$

$$\Rightarrow \left(\frac{4}{3}\left|\frac{x}{3}-\frac{y}{3}\right|^2, \left|\frac{x}{3}-\frac{y}{3}\right|^2\right) \le k\left(\frac{4}{3}|x-y|^2, |x-y|^2\right)$$

For all of these cases, the condition $d(u,v) \le k\, d(x, y)$ holds for $k = \frac{1}{2}$. From theorem 1, it follows that T has a fixed point in X.

Theorem 2. Let (X, d, E) be an E- complete E-b-metric space with $s \geq 1$ and E-Archimedean and let $T : X \to P_{cl}(X)$ be a multivalued (a,b,c,e,f)-contraction with $ks < 1$

Where $k = a + b + c + se + sf$

Then T has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations of T starting from (x,y) $\in$ Graph(T) which E-converges in (X, d, E) to the fixed point of T.

Proof : Let $x_0 \in X$ and $x_1 \in Tx_0$ then there exists $x_2 \in Tx_1$ such that

$d(x_1,x_2) \leq a\, d(x_0,x_1) + b\, d(x_0,x_1) + c\, d(x_1,x_2) + e\, d(x_0,x_2) + f\, d(x_1,x_1)$

$d(x_1,x_2) \leq ad(x_0,x_1) + bd(x_0,x_1) + cd(x_1,x_2) + e\, d(x_0,x_2)$,

Inductively, we define the sequence $(x_n) \in X$, $x_{n+1} \in Tx_n$ for $n \in N$.

$d(x_n,x_{n+1}) \leq a\, d(x_{n-1},x_n) + b\, d(x_{n-1},x_n) + c\, d(x_n,x_{n+1}) + e\, d(x_{n-1},x_{n+1}) + f\, d(x_n,x_n)$

$d(x_n,x_{n+1}) \leq a\, d(x_{n-1},x_n) + b\, d(x_{n-1},x_n) + c\, d(x_n,x_{n+1}) + e\, d(x_{n-1},x_{n+1})$

$d(x_n,x_{n+1}) \leq a\, d(x_{n-1},x_n) + b\, d(x_{n-1},x_n) + c\, d(x_n,x_{n+1}) + se\, d(x_{n-1},x_n) + se\, d(x_n, x_{n+1})$

$(1-c-se)\, d(x_n,x_{n+1}) \leq (a+b+se)\, d(x_{n-1},x_n)$ for any $n \in N$ ……..(1)

Further,

$d(x_{n+1},x_n) \leq a\, d(x_{n-1},x_n) + b\, d(x_{n+1},x_n) + c\, d(x_{n-1},x_n) + e\, d(x_n,x_n) + f\, d(x_{n-1},x_{n+1})$

$d(x_{n+1},x_n) \leq a\, d(x_{n-1},x_n) + b\, d(x_{n+1},x_n) + c\, d(x_{n-1},x_n) + sf\, d(x_{n-1},x_n) + sf\, d(x_n,x_{n+1})$

$(1-b-sf)\, d(x_{n+1},x_n) \leq (a+c+sf)\, d(x_{n-1},x_n)$ for any $n \in N$ ……..(2)

From (1) and (2) ,

$(1-b-c-se-sf)\, d(x_n, x_{n+1}) \leq (2a+b+c+se+sf)\, d(x_{n-1}, x_n)$

$$d(x_n, x_{n+1}) \leq \frac{2a+b+c+se+se}{1-(b-c-se-sf)}\, d(x_{n-1}, x_n)$$

$d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n)$

where $\lambda = \dfrac{2a+b+c+sc+sf}{1-(b+c+se+sf)} < 1$

Now, $d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n) + \lambda^2 d(x_{n-2}, x_{n-1}) + \ldots\ldots\ldots\ldots\ldots + \lambda^n d(x_0, x_1)$ for any $n \in N$

We have

$d(x_n,x_{n+p}) \leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \ldots\ldots\ldots\ldots + s^p\, d(x_{n+p-1},x_{n+p})$ for any $n \in N$

$d(x_n,x_{n+p}) \leq s\lambda^n\, d(x_0,x_1) + s^2 \lambda^{n+1}\, d(x_0,x_1) + \ldots\ldots\ldots\ldots + s^p \lambda^{n+p-1}\, d(x_0,x_1)$ for any $n \in N$

$$= \frac{s\lambda^n\left(1-(s\lambda)^p\right)}{(1-s\lambda)}\, d(x_0,x_1) \leq \frac{s\lambda^n}{1-s\lambda}\, d(x_0,x_1) = a_n.a = b_n \text{ for any } n \in N,\ p \in N$$

Where $a_n = \dfrac{s\lambda^n}{1-s\lambda} \downarrow 0$ and $a = d(x_0, x_1) \in E^+$. Note that $s\lambda < 1$, since $sk < 1$.

On the other hand, by E-Archimedean property, we get $b_n \downarrow 0$. So, the sequence $\{x_n\}$ is E-cauchy sequence in X. By the E-completeness of X, there is $z \in X$ such that $d(x_n,z) \leq a_n$.

We know that $x_{n+1} \in Tx_n$ for any $n \in N$ and by the multivalued (a,b,c,e,f)-contraction condition it follows that there exists $u \in Tz$ such that

$d(x_{n+1},u) \leq a\, d(x_n,z) + b\, d(x_n, x_{n+1}) + c\, d(z,u) + e\, d(x_n,u) + f\, d(z, x_{n+1})$ for any $n \in N$.

Since $d(z,u) \leq sd(x_{n+1},u) + sd(x_{n+1},z)$

$\leq sa\, d(x_n,z) + sb\, d(x_n, x_{n+1}) + sc\, d(z,u) + se\, d(x_n,u) + sf\, d(z, x_{n+1}) + sd(x_{n+1},z)$

$\leq sa\, a_n + sb\, d(x_n, x_{n+1}) + sc\, d(z,u) + se\, [sd(x_n,z) + sd(u,z)] + sf\, a_{n+1} + sa_{n+1}$

$\leq s(a+f+1)\, a_n + sb\, d(x_n, x_{n+1}) + sc\, d(z,u) + s^2e\, d(x_n,z) + s^2e\, d(z,u)$

$(1-sc-s^2e)\, d(z,u) \leq s(a+f+1)a_n + sb\, d(x_n, x_{n+1}) + s^2e\, a_n$

$$d(z,u) \leq \frac{s(a+f+se+1)}{(1-sc-s^2e)} a_n + \frac{sbd(x_n, x_{n+1})}{(1-sc-s^2e)} \downarrow 0, \quad \text{note that } 1-sc-s^2e > 0.$$

Thus, we have there exists $z = u \in Tz$ i.e. T has a fixed point in X.

Theorem 3. Let (X, d, E) be a complete E–b–metric space with E-Archimedean and let $T : X \rightarrow P_{cl}(X)$ be a multivalued mapping and satisfies the following conditions :

(i) for any $x \in X$, $d(u,v) \leq kL(x,y)$ where $u \in Tx$, $v \in Ty$, $ks<1$

and

$$L(x, y) \in \{d(x,y), d(x,u), d(y,v), \frac{1}{2}[d(x,v) + d(y,u)], \frac{1}{2}[d(x,u) + d(y,v)]\}$$

Then T has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations of T starting from $(x,y) \in Graph(T)$ which E-converges in (X, d, E) to the fixed point of T.

Proof : Let $x_0 \in X$ and $x_1 \in Tx_0$.

Inductively, we define the sequence $\{x_n\} \in X$, $x_{n+1} \in Tx_n$ for $n \in N$.

We first show that

$$d(x_n, x_{n+1}) \leq kL(x_{n-1}, x_n) \text{ for all n.}$$

Now we have to consider the following cases :

Case 1 : $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$ for all n.

Case 2 : $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$ for all n.

Case 3 : $d(x_n,x_{n+1}) \leq kd(x_n, x_{n+1})$

$\Rightarrow$ $d(x_n, x_{n+1}) = 0$ for all n.

Case 4 : $d(x_n,x_{n+1}) \leq k\frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]$

$$d(x_n,x_{n+1}) \leq \frac{k}{2}[d(x_{n-1},x_{n+1})]$$

$$\leq \frac{k}{2}s\,[d(x_{n-1},x_n) + d(x_n,x_{n+1})]$$

$$\left(1-\frac{k}{2}s\right)d(x_n,x_{n+1}) \leq \frac{k}{2}s\, d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1}) \leq \left(\frac{\frac{k}{2}s}{1-\frac{ks}{2}}\right)d(x_{n-1},x_n) \qquad \left\{\frac{ks}{2} < \frac{1}{2} \text{ i.e. } ks < 1\right\}$$

Thus $d(x_n, x_{n+1}) \le \lambda_1 d(x_{n-1},x_n)$ where $\lambda_1 = \left(\dfrac{\frac{k}{2}s}{1-\frac{ks}{2}}\right) < 1$

Case 5 : $d(x_n,x_{n+1}) \le k\frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$

$$\le \frac{k}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$\left(1-\frac{k}{2}\right)d(x_n,x_{n+1}) \le \frac{k}{2}d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1}) \le \left(\frac{\frac{k}{2}}{1-\frac{k}{2}}\right)d(x_{n-1},x_n) \quad \left\{\because \frac{k}{2} < \frac{1}{2}\right\}$$

$d(x_n,x_{n+1}) \le \lambda_2 d(x_{n-1},x_n)$ where $\lambda_2 = \dfrac{\frac{k}{2}}{1-\frac{k}{2}} < 1$

Thus for all n and p, we have

$$d(x_n,x_{n+p}) \le sd(x_n, x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \dots + s^p d(x_{n+p-1},x_{n+p})$$

$$\le s\lambda^n d(x_0, x_1) + s^2 \lambda^{n+1} d(x_0,x_1) + \dots + s^p\lambda^{n+p-1} (x_0,x_1)$$

$$= \frac{s\lambda^n(1-(s\lambda)^p)}{(1-s\lambda)} d(x_0,x_1) \le \left(\frac{s\lambda^n}{1-s\lambda}\right) d(x_0,x_1)$$

$= a_n.a = b_n$ for any $n \in N$ and $p \in N$

Now, since E is Archimedean, we have $b_n \downarrow 0$. So the sequence $\{x_n\}$ is E–Cauchy in X. By the E–completeness of X, there is $z \in X$ such that $d(x_n, z) \le a_n$.

We know that $x_{n+1} \in Tx_n$ and $T : X \to P_{cl}(X)$ be a multivalued mapping so it follows that there exists $w \in Tz$ such that

$d(x_{n+1},w) \le kL(x_n,z)$ for any $n \in N$

Then the following estimation holds:

$d(z,w) \le sd(x_{n+1},z) + sd(x_{n+1},w)$

$\le ska_n + skL(x_n,z)$

Where $L(x_n,z) \in \{d(x_n,z), d(x_n,x_{n+1}), d(z,w), \frac{1}{2}[d(x_n,w) + d(z,x_{n+1})], \frac{1}{2}[d(x_n,x_{n+1}) + d(z,w)]\}$

Case 1 : $d(z,w) \le ska_n + skL(x_n,z) \le ska_n + ska_{n-1} \le 2ska_{n-1}\downarrow 0$

Case 2 : $d(z,w) \le ska_n + skd(x_n,x_{n+1}) \le ska_n + sk[sd(x_n,z) + sd(z,x_{n+1})]$

$\le ska_n + s^2ka_{n-1} + s^2ka_n \le ska_n + 2s^2ka_{n-1} \le sk(1 + 2s)a_{n-1}$ $(\because a_n \le a_{n-1})$

Case 3 : $d(z,w) \le ska_n + skd(z,w)$

$(1-sk)\, d(z,w) \le ska_n$

$$d(z,w) \leq \left(\frac{sk}{1-sk}\right) a_n \downarrow 0$$

$d(z,w) = 0$

Case 4 : $d(z,w) \leq ska_n + \frac{1}{2} sk[d(x_n,w) + d(z,x_{n+1})] \leq ska_n + \frac{sk}{2} [\{sd(x_n, z) + sd(z,w)\} + d(x_{n+1}, z)]$

$$\leq ska_n + \frac{s^2k}{2} d(x_n, z) + \frac{s^2k}{2} d(z,w) + \frac{sk}{2} d(x_{n+1}, z)$$

$$\leq ska_n + \frac{s^2k}{2} a_{n-1} + \frac{s^2k}{2} d(z,w) + \frac{sk}{2} a_n$$

$$\left(1-\frac{s^2k}{2}\right) d(z,w) \leq \left(\frac{s^2k}{2} + \frac{3sk}{2}\right) a_{n-1}$$

$$d(z,w) \leq \frac{\left(\frac{s^2k}{2} + \frac{3sk}{2}\right)}{\left(1-\frac{s^2k}{2}\right)} a_{n-1}$$

$\Rightarrow d(z,w) = 0$

Case 5 : $d(z,w) \leq ska_n + \frac{1}{2} sk[d(x_n, x_{n+1}) + d(z,w)] \leq ska_n + \frac{sk}{2} [\{sd(x_n, z) + sd(x_{n+1}, z)\} + d(z,w)]$

$$\left(1-\frac{sk}{2}\right) d(z,w) \leq ska_n + \frac{s^2k}{2} a_{n-1} + \frac{s^2k}{2} a_n$$

$$d(z,w) \leq \frac{sk(s+1)}{1-\frac{sk}{2}} a_{n-1} \downarrow 0$$

$\Rightarrow d(z,w) = 0$

Therefore T has a common fixed point in X.

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