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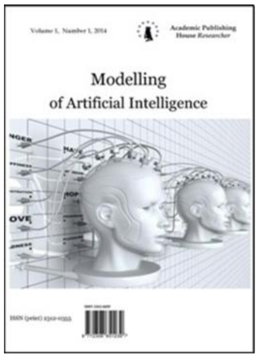

Articles and statements

Pattern Recognition by Cross Sections

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Abstract

The goal of the present paper is to investigate covariograms of convex bodies (it is equivalent to investigate the orientation dependent chord length distribution functions). The applications of these problems are known in both geometric and computer tomography. Algorithms to reconstruct convex bodies by its covariogram for finite number of directions (the same problem for orientation dependent chord length distribution function has the negative solution) is one of the main problem of stochastic geometry. In particular, find the covariograms for classes of three dimensional convex bodies. Covariogram problem for three dimensional case is an open problem, while in the planar case the problem has the positive solution and if dimensionality of space greater than or equal 4 it has negative solution. The formulation of the problems is accompanied by discussion of the existing tools and ways of their implementation.

Keywords: covariogram, kinematic measure, orientation-dependent chord length distribution, convex body.

1. Introduction

Complicated geometrical patterns occur in many areas of science. Their analysis requires creation of mathematical models and development of special mathematical tools. The corresponding area of mathematical research is called Stochastic Geometry (see Gardner, 2006 **and** Schneider, Weil, 2008). Among more popular applications are Stereology and Tomography. The objective of stereology is to draw inferences about the geometrical properties of n-dimensional structure, $n \geq 2$, when information is only available in some lower-dimensional form via linear probes, planar sections, or projections of thick slices. Its application arises in the study of geometrical structure of inclusions or pores in opaque bodies such as metals, minerals, synthetic materials, or biological tissues; in these cases the available information must come from linear probes or planar sections. The methods and formulae of stereology relate characteristics of n-dimensional structures to quantities arising from measurements of planar sections . The step from spatial structures to their sections involves a great loss of information and so stereological methods commonly yield only ``global" information of a statistical character.

At the Conference on Tomography at Oberwolfach, R. Gardner introduced the term geometric tomography. In the R. Gardner monograph (Gardner, 2006), the following definition is

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offered: ``Geometric tomography is the area of mathematics dealing with the retrieval of information about a geometric object from data about its sections, or projections, or both''. The word projection is used in the sense of a shadow, that is, the usual orthogonal projection on a line. The parallel X-ray of $\boldsymbol{D}$ in the direction φ gives the length of the chord of intersection of $\boldsymbol{D}$ with the line through x parallel to φ. The sections of bodies by random planes and lines (X-rays, cracks) are considered in many mathematical models of modern physics (computer tomography, crack tessellations etc). Two important mathematical problems are arisen: 1) for given convex body to calculate the chord distribution, 2) for given chord length distribution to reconstruct the convex body. Although there are many recent results and investigations in these directions, some problems are open, in particular, the computer programs for calculation of chord length distributions are missing. We are considered the problem of investigation of chord length distribution in n-dimensional space. Recognition of planar domains $\boldsymbol{D}$ by means of random lines intersecting $\boldsymbol{D}$ is one of the interesting problem of Stochastic Geometry.

Let $\boldsymbol{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space, $\boldsymbol{D} \subset \boldsymbol{R}^n$ be a bounded convex body with inner points, and V_n be the n-dimensional Lebesgue measure in R^n.

2. Discussion

Definition 1. (see Matheron, 1975, Schneider, Weil, 2008). The function

$$C(\boldsymbol{D}, h) = V_n\big(\boldsymbol{D} \cap (\boldsymbol{D} + h)\big), \quad h \in \boldsymbol{R}^n \tag{1}$$

is called the covariogram of the body $\boldsymbol{D}$.

Here $\boldsymbol{D} + h = \{x + h,\ x \in \boldsymbol{D}\}$. $C(\boldsymbol{D}, h)$ is called the set covariance of $\boldsymbol{D}$.

The definition of the covariogram is given by G. Matheron, who formulated it for more general sets, and even for functions. In (Matheron, 1975), G. Matheron conjectured that the covariogram of a convex body $\boldsymbol{D}$ determines $\boldsymbol{D}$ within the class of all convex bodies, up to translations and reflections. G. Averkov and G. Bianchi (Bianchi, Averkov, 2009), showed that every planar convex body is determined within all planar convex bodies by its covariogram, up to translations and reflections.

Very little is known regarding the covariogram problem when the space dimension is greater than 2. It is known that centrally symmetric convex bodies in any dimension, are determined by their covariogram up to translations. For $n = 3$ the problem is open. Nevertheless, for the case of bounded convex polyhedron for $n = 3$ Matheron's conjecture received a positive answer. In fact, the covariogram problem is equivalent to the problem of determining a convex domain from all orientation-dependent chord length distributions (see Bianchi, Averkov, 2009, Schneider, Weil, 2008).

The problem of finding the measure of the segments of a constant length that are contained in $\boldsymbol{D}$ has no simple solution and depends on the shape of $\boldsymbol{D}$. It is known the explicit form for the kinematic measures of some planar domains: a disk, a rectangle, if the length of the segment is less than the smaller side of the rectangle (see Santalo, 2004) and for the equilateral triangle, the rectangle (for an arbitrary length of the segment) and regular pentagon (see Gasparyan, Ohanyan, 2013).

Let S^{n-1} denote the $(n-1)$-dimensional sphere of radius 1 centered at the origin in R^n. We consider a random line which is parallel to $\mathbf{u} \in S^{n-1}$ and intersects $\boldsymbol{D}$, that is, an element from the set:

$$\Omega_1(\boldsymbol{u}) = \{\textit{lines which are parallel to } \boldsymbol{u} \textit{ and intersect } \boldsymbol{D}\}.$$

Let $\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}$ be the orthogonal projection of $\boldsymbol{D}$ onto the hyperplane $\boldsymbol{u}^{\perp}$ (here $\boldsymbol{u}^{\perp}$stands for the hyperplane with normal $\mathbf{u}$, passing through the origin).

A random line which is parallel to $\boldsymbol{u}$ and intersects $\boldsymbol{D}$ has an intersection point (denoted by x) with $\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}$. We can identify the points of $\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}$ and the lines which intersect $\boldsymbol{D}$ and are parallel to $\mathbf{u}$, meaning that we can identify the sets $\Omega_1(\boldsymbol{u})$ and $\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}$. Assuming that the intersection point x is uniformly distributed over the convex body $\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}$, we can define the following distribution function.

Definition 2. The function

$$F(\boldsymbol{u},t)=\frac{V_{n-1}\{x\in\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}:V\_1(g(\boldsymbol{u},x)\cap\boldsymbol{D})<t)\}}{b_{\boldsymbol{D}}(\boldsymbol{u})} \tag{2}$$

is called orientation-dependent chord length distribution function of $\boldsymbol{D}$ in direction $\boldsymbol{u}$ at point $t\in R^1$, where $g(\boldsymbol{u},x)$ is the line which is parallel to $\boldsymbol{u}$ and intersects $\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D}$ at point x and $b_{\boldsymbol{D}}(\boldsymbol{u})=\mathrm{V}_{\mathrm{n-1}}(\Pi\, r_{\boldsymbol{u}^{\perp}}\boldsymbol{D})$.

Observe that each vector $h\in\boldsymbol{R}^n$ can be represented in the form $h=(\boldsymbol{u},t)$, where $\boldsymbol{u}$ is the direction of h, and t is the length of h.

Lemma 1. (see Matheron, 1975) *Let $\boldsymbol{u}\in S^{n-1}$ and $t>0$ be such that $\boldsymbol{D}\cap(\boldsymbol{D}+t\boldsymbol{u})$ contains inner points. Then $C(\boldsymbol{D},\boldsymbol{u},t)$ is differentiable with respect to t and the following equality holds:*

$$-\frac{\partial\, C(\boldsymbol{D},\boldsymbol{u},t)}{\partial\, t}=\big(1-F(\boldsymbol{u},t)\big)\cdot\, b_{\boldsymbol{D}}(\boldsymbol{u}). \tag{3}$$

At $t=0$ the right-hand derivative exists, and the same equality holds.

Let $L(\omega)$ be a random segment of length $l>0$, which is parallel to a given fixed direction $\boldsymbol{u}\in S^{n-1}$ and intersects $\boldsymbol{D}$. Consider the random variable $|L|(\omega):=V_1(L(\omega)\cap\,\boldsymbol{D})$\$, where $L(\omega)\in\Omega_2(\boldsymbol{u})$, and the set $\Omega_2(\boldsymbol{u})$ is defined as follows:

$$\Omega_2(\boldsymbol{u})=\{segments\ of\ lengths\ l, which\ are\ parallel\ to\ \boldsymbol{u}\ and\ intersect\ \boldsymbol{D}\}.$$

Observe that each random segment $L(\omega)$ lying on a line $g(\boldsymbol{u},x)$ can be specified by the coordinates $(g(\boldsymbol{u},x),y)$, where y is the one-dimensional coordinate of the center of $L(\omega)$ on the line $g(\boldsymbol{u},x)$. As the origin on the line $g(\boldsymbol{u},x)$ we take one of the intersection points of the line $g(\boldsymbol{u},x)$ with the boundary of domain $\boldsymbol{D}$. Using the above notation, we can identify $\Omega_2(\boldsymbol{u})$ with the following set:

$$\Omega_2(\boldsymbol{u})=\left\{(x,y):\ x\in\Pi\, r_{u^{\perp}}\boldsymbol{D},\ y\in\left[-\frac{l}{2},\ \chi(\boldsymbol{u},x)+\frac{l}{2}\right]\right\},$$

where $\chi(\boldsymbol{u},x)=V_1(g(\boldsymbol{u},x)\cap\,\boldsymbol{D})$. Note that the set $\Omega_2(\boldsymbol{u})$ does not depend on the choice of the origin of the line $g(\boldsymbol{u},x)$, and the choice of the positive direction follows from the explicit form of the range of variation of y. Further, we set

$$B_{\boldsymbol{D}}^{\boldsymbol{u},t}=\{(x,y)\in\Omega_2(\boldsymbol{u}):\ |L|(x,y)<\ t\},\quad t\in\ \boldsymbol{R}^1,$$

and observe that the sets $\Omega_2(\boldsymbol{u})$ and $B_{\boldsymbol{D}}^{\boldsymbol{u},t}$ are measurable subsets of $\boldsymbol{R}^n$.

Definition 3. *The function*

$$F_{|L|}(\boldsymbol{u},t)=\frac{V_n(B_{\boldsymbol{D}}^{\boldsymbol{u},t})}{V_n(\Omega_2(\boldsymbol{u}))}=\frac{1}{V_n(\Omega_2(\boldsymbol{u}))}\int_{B_{\boldsymbol{D}}^{\boldsymbol{u},t}}dxdy \tag{4}$$

is called orientation-dependent distribution function of the length of a random segment L in direction $\boldsymbol{u}\in S^{n-1}$.

Let G_n be the space of all lines g in $\boldsymbol{R}^n$. A line $g\in G_n$ can be specified by its direction $\boldsymbol{u}\in S^{n-1}$ and its intersection point x in the hyperplane $\boldsymbol{u}^{\perp}$. The density $d\boldsymbol{u}^{\perp}$ is the volume element $d\,\boldsymbol{u}$ of the unit sphere S^{n-1} and dx is the volume element on $\boldsymbol{u}^{\perp}$ at x. Let $\mu(\cdot)$ be a locally finite measure on G_n, invariant under the group of Euclidian motions. It is well known that the element of $\mu(\cdot)$ up to a constant factor has the following form (see Santalo, 2004):

$$\mu(dg)=dg=du\,dx.$$

Denote by $O_{n-1}=\sigma_{n-1}(S^{n-1})$ the surface area of the unit sphere in $R\hat{}n$. For each bounded convex body D, we denote the set of lines that intersect D by

$$[D]=\{g\in G_n, g\cap\ D\neq\emptyset\}.$$

We have (see Santalo, 2004)

$$\mu([D])=\frac{O_{n-2}V_{n-1}(\partial\, D)}{2(n-1)}.$$

A random line in $[\boldsymbol{D}]$ is the one with distribution proportional to the restriction of μ to $[\boldsymbol{D}]$. Therefore, for any $t\in\ \boldsymbol{R}^1$, we have

$$F(t)=\frac{\mu(\{g\in\ [\boldsymbol{D}],\backslash,\backslash,V_1(g\cap\boldsymbol{D})<t\})}{\mu([\boldsymbol{D}])},$$

which is called the chord length distribution function of $\boldsymbol{D}$.

Let L be a random segment of length l in $\boldsymbol{R}^n$ and let $K(\cdot)$ be the kinematic measure of L (Santalo, 2004).

If $g \in G_n$ is the line containing L and y is the one-dimensional coordinate of the center of L on the line g, then the element of the kinematic measure up to a constant factor is given by

$$dK = dg\, dy\, dK_{[1]},$$

where dy is the one-dimensional Lebesgue measure on g and $dK_{[1]}$ is a motion element in $\boldsymbol{R}^n$, that leaves g unchanged (see Santalo, 2004 and Schneider, Weil, 2008).

The length $|L|$ of a random segment L, provided that it hits the body D}, has the following distribution function:

$$F_{|L|}(t) = \frac{K(L: L \cap D \neq \emptyset, V_1(L \cap D) < t)}{K(L: L \cap D \neq \emptyset)}\ , \quad t \in R^1.$$

Theorem 1. *(see Gasparyan, Ohanyan, 2014). We establish a relationship between the distribution function of the random variable $|L|(\omega)$ and the orientation-dependent chord length distribution function in R^n, given by the following formula:*

$$F_{|L|}(u,t) = \begin{cases} 0, & for \quad t \leq 0 \\ \frac{b_D(u)[2t + F(u,t)(l-t) - \int_0^t F(u,z)\, dz]}{V_n(D) + l\, b_D(u)}, & for \ \ 0 \leq t \leq l, \\ 1, & for \quad t > l, \end{cases} \tag{5}$$

Note that explicit forms for orientation-dependent chord length distribution function $F(u,t)$ for triangles, ellipses, regular polygons and parallelograms were obtained in the papers (Gasparyan, Ohanyan, 2013; Gasparyan, Ohanyan, 2014). Hence substituting in (10) $n = 2$ and the corresponding formulas for $F(u,t)$, we get explicit expressions for $F_{|L|}(u,t)$ for the mentioned planar convex domains.

Theorem 2. *(see Gasparyan, Ohanyan, 2014). The distribution function of the random variable $|L|(\omega)$ and the covariogram over the interval $[0,l]$ are related by the following formula:*

$$F_{|L|}(u,t) = 1 + \frac{1}{V_n(D) + l\, b_D(u)} \left[\frac{\partial\, C(D,u,t)}{\partial\, t}(l-t) - C(D,u,t)\right]. \tag{6}$$

Theorem 3. *(see Gasparyan, Ohanyan, 2014). The following relationship between the distribution function of the length of a random segment intersecting D and the chord length distribution function of D in R^n:*

$$F_{|L|}(t) = \begin{cases} 0, & for \quad t \leq 0 \\ \frac{O_{n-2} V_{n-1}(\partial D)(2t + F(t)(l-t) - \int_0^t F(z)\, dz)}{(n-1) O_{n-1} V_n(D) + l O_{n-2} V_{n-1}(\partial D)}, & for \ \ 0 \leq t \leq l, \\ 1, & for \quad t > l, \end{cases} \tag{7}$$

Denote by $P(L(u,\omega) \subset D)$ probability, that random segment $L(u,\omega)$ (of fixed length l and direction u) entirely lying in body D.

Proposition 1. *(see Aharonyan, Ohanyan, 2018) Probability $P(L(u,\omega) \subset D)$ in terms of distribution function $F(u,z)$ has the following form:*

$$P(L(u,\omega) \subset D) = \frac{V_n(D) - l\, b_D(u) + b_D(u) \int_0^l F(u,z)\, dz}{V_n(D) + l\, b_D(u)}\ , \tag{8}$$

while in the terms of the covarigramm of body D has the form:

$$P(L(u,\omega) \subset D) = \frac{C(D,u,l)}{V_n(D) + l\, , b_D(u)}. \tag{9}$$

Denote by $P(L(\omega) \subset D)$ probability, that random segment of length l in R^n having a common point with body D entirely lying in body D (in this case the direction of the segment $L(\omega)$ is arbitrary). Note, that probability $P(L(\omega) \subset D)$ can be obtain from probability $P(L(u,\ \omega) \subset D)$ by integration over all directions $u \in S^{n-1}$.

Proposition 2. Probability $P(L(\omega) \subset D)$ in terms of chord length distribution function $F(t)$ has the following form:

$$P(L(\omega) \subset D) = \frac{O_{n-2} V_{n-1}(\partial D)\left(\int_0^l F(z) dz - l\right) + (n-1) O_{n-1} \{n-1\} V_{n(D)}}{(n-1) O_{n-1} V_n(D) + l O_{n-2} V_{n-1}(\partial D)}. \tag{10}$$

Since the ball $B_n(R)$ is an isotropic body, then $C(B_n(R), u, l) = C(B_n(R), l)$ does not depend on direction $u \in S^{n-1}$. Therefore, we get

$$P(L(u,\omega) \subset B_n(R)) = P(L(\omega) \subset B_n(R)) = \frac{C(B_n(R), l)}{V_n(B_n(R)) + l \cdot b_{B_n(R)}(u)}. \tag{11}$$

It is known that the volume of n-dimensional ball of radius R equals to

$$V_n\left(\boldsymbol{B}_{\boldsymbol{n}(\boldsymbol{R})}\right) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \cdot R^n, \tag{12}$$

while $b_{\boldsymbol{B}_{\boldsymbol{n}(\boldsymbol{R})}(u)}$ is the projection of n-dimensional ball of radius R on hyperplane $\boldsymbol{u}^{\perp}$ equals

$$b_{\boldsymbol{B}_{\boldsymbol{n}(\boldsymbol{R})}(u)} = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot R^{n-1},$$

Where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ is the Gamma function.

It is easy to see, that the covariogram of n-dimensional ball of radius R equals to twice the volume of n-dimensional spherical cap of high $R - l/2$. Using the formula for n-dimensional spherical cap (see Gasparyan, Ohanyan, 2015) we get

$$C\left(\boldsymbol{B}_{\boldsymbol{n}(\boldsymbol{R})}, l\right) = 2 \cdot \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot R^n \cdot \int_0^{\phi} \sin^n \theta \, d\theta, \tag{14}$$

where $\phi = arccos\, \frac{l}{2R}$.

Therefore, putting $C(\boldsymbol{B_n}(\boldsymbol{R}), l)$ from (14) we get that the probability that the segment of the length $l \leq 2R$ entire lies in n-dimensional ball of radius R equals to (see Aharonyan, Ohanyan, 2018)

$$\boldsymbol{P}\left(L(\omega) \subset \boldsymbol{B}_{\boldsymbol{n}(\boldsymbol{R})}\right) = \frac{2R}{R\,\frac{\sqrt{\pi}\Gamma\left(\frac{(n+1)}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} + l} \int_0^{\phi} \sin^n \theta \, d\theta.$$

Obviously, for any dimension n we have $\boldsymbol{P}(L(\omega) \subset \boldsymbol{B_n}(\boldsymbol{R})) = 1$ for $l = 0$ and $\boldsymbol{P}(L(\omega) \subset \boldsymbol{B_n}(\boldsymbol{R})) = 0$ for $l \geq 2R$.

We have a computer program for calculating the chord length distribution for an arbitrary convex polygon.

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References

Aharonyan et al., 2010 – *Aharonyan N. G., Harutyunyan H. S., Ohanyan V. K.* (2010). Random copy of a segment within a convex domain, *Journal of Contemporary Mathematical Analysis* (Armenian Academy of sciences), 45(5), 348-356.

Aharonyan, Ohanyan, 2005 –Aharonyan N. G. and Ohanyan V. K. (2005). Chord lenght distribution functions for polygons, 40(4), 43-56.

Aharonyan, Ohanyan, 2011 –*Aharonyan N. G., Ohanyan V. K.* (2011).Kinematic measure of intervals lying in domains, *Journal of Contemporary Mathematical Analysis* (Armenian Academy of sciences), 46(5), 280-288.

Aharonyan, Ohanyan, 2018 – *Aharonyan N. G., Ohanyan V. K.* (2018). Calculation of geometric probabilities using Covariogram of convex bodies, J*ournal of Contemporary Mathematical Analysis* (Armenian Academy of sciences), 53(3).

Bianchi, Averkov, 2009 – *Bianchi G., Averkov G.* (2009). Confirmation of Matheron's Conjecture on the covariogram of a planar convex body, J*ournal of the European Mathematical Society*, 11, 1187-1202.

Gardner, 2006 – *Gardner R.* (2006). Geometric Tomography, Cambridge University Press, Cambridge, UK, 2nd ed.

Gasparyan, Ohanyan, 2013 – *Gasparyan A., Ohanyan V.K.* (2013). Recognition of triangles by covariogram, Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), 48 (3), 110-122.

Gasparyan, Ohanyan, 2014 – *Gasparyan A., Ohanyan V.K.* (2014). Covariogram of a parallelogram, *Journal of Contemporary Mathematical Analysis* (Armenian Academy of sciences), 49(4), 194-206.

Gasparyan, Ohanyan, 2015 – *Gasparyan A., Ohanyan V. K.* (2015). Orientation-dependent distribution of the length of a random segment and covariogram, *Journal of Contemporary Mathematical Analysis* (Armenian Academy of sciences), 50(2), 90-97.

Harutyunyan, Ohanyan, 2014 – *Harutyunyan H. S., Ohanyan V.K.* (2014). Orientation-dependent section distributions for convex bodies, Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), 49 (3), 139-156.

Matheron, 1975 – *Matheron G.* (1975). Random Sets and Integral Geometry, Wiley, New York.

Ohanyan, Aharonyan, 2009 – *Ohanyan V.K., Aharonyan N.G.* (2009). Tomography of bounded convex domains, SUTRA: *International Journal of Mathematical Sciences*, 2(1), 1-12.

Santalo, 2004 – *Santalo L.A.* (2004). Integral Geometry and Geometric Probability, Addision-Wesley, Reading, MA.

Schneider, Weil, 2008 – *Schneider R., Weil W.* (2008). Stochastic and Integral Geometry, Springer.