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Connecting Noncommutative Geometry to $f(R)$ Modified Gravity

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Abstract It is shown in this note that a noncommutative-geometry background determines the modified-gravity function $f(R)$ for modeling dark matter.

Keywords: Noncommutative geometry; $f(R)$ modified gravity.

1 Introduction

It is well known that $f(R)$ modified gravitational theories can account for dark matter in the sense that the galactic dynamics of massive test particles can be explained in the framework of $f(R)$ gravity without the need for dark matter [1,2,3]. For a general discussion of dark matter as a geometric effect of $f(R)$ gravity, see Ref. [4]. Less well known is that noncommutative geometry can play a similar role [5,6]. The purpose of this note is to explain the reason for the connection between the two theories. Given that noncommutative geometry is an offshoot of string theory, the results of this note may be viewed as indirect evidence for the latter.

2 Noncommutative Geometry and $f(R)$ Modified Gravity

Noncommutative geometry is based on the following outcome of string theory: coordinates may become noncommuting operators on a D-brane [7,8]. This statement refers to the commutator $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix. Noncommutativity replaces point-like structures by smeared objects. As discussed in Refs. [9,10], the aim is to eliminate the divergences that normally occur in general relativity. A good way to accomplish the smearing effect is to assume that the energy density of the static and spherically symmetric and particle-like gravitational source has the form [11,12]

$$\rho(r) = \frac{M\sqrt{\beta}}{\pi^2(r^2+\beta)^2} \tag{1}$$

in spherical coordinates and is interpreted to mean that the mass M of the particle is diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty.

The study of dark matter and dark energy has led to a renewed interest in modified theories of gravity. One of the most important of these, $f(R)$ modified gravity, replaces the Ricci scalar R in the Einstein-Hilbert action

$$S_{\text{EH}} = \int \sqrt{-g}\, R\, d^4x$$

by a nonlinear function $f(R)$:

$$S_{f(R)} = \int \sqrt{-g}\, f(R)\, d^4x.$$

(For a review, see Refs. [13,14,15].)

Since both theories can account for dark matter, it is important to determine a possible connection.

3 Noncommutative Geometry and Dark Matter

We start with the general metric of a static and spherically symmetric line element, using units in which $c = G = 1$ [16]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2}$$

Here $m(r)$ is the effective mass inside a sphere of radius r with $m(0) = 0$. We also require that $\lim_{r\to\infty}\Phi(r) = 0$ and $\lim_{r\to\infty}m(r)/r = 0$, usually referred to as asymptotic flatness.

Because of the spherical symmetry, the only nonzero components of the stress-energy tensor are $T^t{}_t = -\rho(r)$, the energy density, $T^r{}_r = p_r(r)$, the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t(r)$, the lateral pressure. Assuming the conservation law $T^\alpha{}_{\beta;\,\alpha} = 0$, there are only two independent Einstein field equations,

$$\rho(r) = \frac{1}{8\pi}\frac{2m'(r)}{r^2} \tag{3}$$

and

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{2m(r)}{r^3} + \frac{2\Phi'(r)}{r}\left(1 - \frac{2m(r)}{r}\right)\right]. \tag{4}$$

Next, we need to recall that galaxies exhibit flat rotation curves (constant tangential velocities) sufficiently far from the galactic center, due to the existence of dark matter [17]. This behavior indicates that the matter in the galaxy increases linearly in the outward radial direction. More precisely, the total mass $M_T(r)$ enclosed in a sphere of radius r has the form

$$M_T(r) = v^2 r, \tag{5}$$

where v is the constant tangential velocity; here $v^2 = 0.000001$ in geometrized units [18].

To connect these ideas to noncommutative geometry, we start with a thin spherical shell of radius $r = r_0$. So instead of a smeared object, we now have a smeared spherical surface. Let us consider the smearing in the outward radial direction only, that being the analogue of a smeared particle at the origin. So Eq. (1) is replaced by

$$\rho(r) = \frac{M_{r_0}\sqrt{\beta}}{\pi^2[(r - r_0)^2 + \beta]^2}, \tag{6}$$

where M_{r_0} is the mass of the shell. According to Ref. [5], the smeared mass $m_\beta(r)$ is given by

$$m_\beta(r) = \frac{2M_{r_0}}{\pi}\left[\tan^{-1}\frac{r - r_0}{\sqrt{\beta}} - \frac{(r - r_0)\sqrt{\beta}}{(r - r_0)^2 + \beta}\right]. \tag{7}$$

Observe that

$$\lim_{\beta\to 0} m_\beta(r) = M_{r_0}.$$

So the mass of the shell is zero at $r = r_0$ and rapidly rises to M_{r_0}. (We will see later that $r - r_0$ has to exceed $\sqrt{\beta}$.)

It is also shown in Ref. [5] that

$$M_T(r) = M_{r_0}(r - r_0), \tag{8}$$

in agreement with Eq. (5). As noted in Ref. [5], M_{r_0} must now be viewed as a dimensionless constant of proportionality which can be interpreted as the change in the smeared mass per unit length and is therefore constant throughout. This also follows from Eq. (8) since $dM_T(r)/dr = M_{r_0}$. We also have from Eqs. (5) and (8) that $v^2 = M_{r_0}\left(1 - \frac{r_0}{r}\right)$. So for reasonably large r,

$$v^2 \approx M_{r_0}. \tag{9}$$

To reiterate, v^2 is approximately equal to the change in the smeared mass per unit length. Since Eq. (9) holds for every shell, we could simply replace $r - r_0$ in Eq. (6) by a new variable. However, from a calculational standpoint, it would be simpler to let $r_0 = 0$. Then Eq. (6) becomes

$$\rho(r) = \frac{M_{r_0}\sqrt{\beta}}{\pi^2(r^2 + \beta)^2}, \tag{10}$$

where M_{r_0} now assumes its original meaning as the mass of the shell.

As a final comment, it was noted after Eq. (7) that $r - r_0$ must exceed $\sqrt{\beta}$. To make use of Eq. (10) in Sec. 4, we will need the more precise condition (with $r_0 = 0$)

$$r \geq a > \sqrt{\beta}, \quad a > 0. \tag{11}$$

4 The Connection to $f(R)$ Gravity

Returning now to $f(R)$ gravity, it is convenient, in view of line element (2), to denote $M_T(r)$ by $m(r)$. According to Ref. [19], the Ricci scalar R is given by

$$R = \frac{4m'(r)}{r^2}. \tag{12}$$

From Eq. (5) we then obtain

$$\frac{dM_T(r)}{dr} = m'(r) = v^2 \tag{13}$$

and

$$R(r) = \frac{4v^2}{r^2}. \tag{14}$$

This equation yields

$$r(R) = \sqrt{\frac{4v^2}{R}}. \tag{15}$$

In $f(R)$ gravity, Eq. (3) is replaced by [19]

$$\rho(r) = F(r)\frac{2m'(r)}{r^2}, \tag{16}$$

where $F = \frac{df}{dR}$. Eq. (10) now yields

$$F(r) = \frac{r^2}{2m'(r)}\rho(r) = \frac{r^2}{2m'(r)}\frac{M_{r_0}\sqrt{\beta}}{\pi^2(r^2+\beta)^2} \tag{17}$$

and from Eqs. (12) and (15),

$$F(R) = \frac{2M_{r_0}\sqrt{\beta}}{\pi^2}\frac{1}{R\left(\frac{4v^2}{R}+\beta\right)^2}. \tag{18}$$

Integrating, we get

$$\begin{aligned} f(R) &= \frac{2M_{r_0}\sqrt{\beta}}{\pi^2\beta^2}\left[\ln\left(\beta R + 4v^2\right) - \frac{\beta R}{\beta R + 4v^2} + \ln C\right] \\ &= \frac{2M_{r_0}}{\pi^2\beta^{3/2}}\left[\ln\left(4v^2\right) + \ln\left(1 + \frac{\beta R}{4v^2}\right) - \frac{\beta R}{\beta R + 4v^2} + \ln C\right], \end{aligned} \tag{19}$$

where C is an arbitrary constant.

To simplify the analysis, consider the third term inside the brackets on the right-hand side of Eq. (19). From Condition (11), $r \geq a > \sqrt{\beta}, a > 0$, r is bounded away from 0. As a result,

$$\left|-\frac{\beta}{\beta + 4v^2/R}\right| = \left|-\frac{\beta}{\beta + r^2}\right| \leq \left|-\frac{\beta}{\beta + a^2}\right| \approx 0$$

since a^2 is fixed and β is close to zero.

In Eq. (19), letting $\ln C = -\ln 4v^2$ results in

$$f(R) \approx \frac{2M_{r_0}}{\pi^2\beta^{3/2}}\ln\left(1 + \frac{\beta R}{4v^2}\right). \tag{20}$$

Now recalling that $R = 4v^2/r^2$, we have

$$\frac{\beta R}{4v^2} = \frac{\beta}{r^2} < 1$$

since $r > \sqrt{\beta}$. Thus

$$\ln\left(1 + \frac{\beta R}{4v^2}\right) = \frac{\beta R}{4v^2} - \frac{1}{2}\left(\frac{\beta R}{4v^2}\right)^2 + \frac{1}{3}\left(\frac{\beta R}{4v^2}\right)^3 - \cdots \approx \frac{\beta R}{4v^2}$$

and

$$f(R) \approx \frac{M_{r_0}}{2v\,\pi^2\sqrt{\beta}} R. \tag{21}$$

Observe that the coefficient of R is a dimensionless constant. Eq. (21) also implies that

$$f(R) \approx \frac{M_{r_0}}{2v\,\pi^2\sqrt{\beta}} R^{1+\epsilon}, \quad \epsilon \ll 1. \tag{22}$$

According to Ref. [4], $f(R)$ modified gravity can account for flat galactic rotation curves if

$$f(R) = kR^{1+\epsilon}, \quad \epsilon \ll 1, \quad \text{with constant } k, \tag{23}$$

which has the same form as Eq. (22).

5 Conclusion

It is shown in this note that a noncommutative-geometry background yields the form

$$f(R) \approx kR^{1+\epsilon}, \quad \epsilon \ll 1, \quad \text{with constant } k, \tag{24}$$

which is known to account for galactic rotation curves. Noncommutative geometry can therefore serve as a model for dark matter.

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