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ON SOME NEW PROJECTION THEOREMS AND SHARP ESTIMATES IN HERZ TYPE SPACES IN BOUNDED PSEUDOCONVEX DOMAINS

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We prove new projection theorems for new Herz type spaces in various domains in C^n in the unit disk, unit ball, bounded pseudoconvex domains and based on these results we provide sharp estimates for distances in such type spaces under one condition on Bergman kernel. Similar type result in such type spaces in tubular domains over symmetric cones will be also provided.

Key words: pseudoconvex and tubular domains,the unit ball ,projection theorem, Herz spaces

Introduction

This paper is a continuation of long series of papers of first author on distances and is devoted to some new sharp results for distance function in new Herz type spaces in the unit ball and bounded pseudoconvex domains with smooth boundary. A related result in tubular domains will be also provided. These results are based on some new projection theorems in such type function spaces. Some easy details in our proofs will be omitted. Some interesting questions will be posed. We provide now some definitions and notations in pseudoconvex domains in C^n Throughout this paper $H(D)$ denotes the space of all holomorphic function on an open set $D \subset \mathbb{C}^n$. We follow notation from [13], [23]. Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, let $\delta(z) = d(z) = dist(z, \partial D)$. Then there is a neighbourhood v of $\overline{D}$ and $p \in C^\infty(U)$ such that $D = \{z \in U : \rho(z) > 0\}, |\bigtriangledown \rho(z)| \geq c > 0$ for $z \in \partial D, 0 < \rho(z) < 1$ for $z \in D$ and p is strictly plurisubharmonic in a neighbourhood U_0 of ∂D. Note that $d(z) \asymp \rho(z), z \in D$. Then there is an $r_0 > 0$ such that the domains $D_r = \{z \in U : \rho(z) > r\}$ are also smoothly bounded strictly pseudoconvex domains for all $0 \leq r \leq r_0$. Let $d\sigma_r$ be the normalized surface measure on ∂D_r and dV or dv the Lebesgue measure on D in $\mathbb{C}^n$. The following mixed norm spaces were investigated in [12]. For $0 < p < \infty, 0 < q \leq \infty, \delta > 0$ and $k = 0, 1, 2, \ldots$ set

$$||f||_{p,q,\delta;k} = \Big(\sum_{|\alpha| \leq k} \int\limits_0^{r_0} (r^\delta \int\limits_{\partial D_r} |D^\alpha f|^p d\sigma_r)^{\frac{q}{p}} \frac{dr}{r} \Big)^{\frac{1}{q}}, 0 < q < \infty \tag{1}$$

and

$$||f||_{p,\infty,\delta;k} = \sup_{0<r<r_0} \sum_{|\alpha| \leq k} (r^\delta \int\limits_{\partial D_r} |D^\alpha f|^p d\sigma_r)^{\frac{1}{p}}. \tag{2}$$

The corresponding spaces $A^{p,q}_{\delta;k} = A^{p,q}_{\delta;k}(D) = \{f \in H(D) : ||f||_{p,q,\delta;k} < \infty\}$ are complete quasi normed spaces, for $p, q \geq 1$ they are Banach spaces.

For $\delta > 0$, the space $A^\infty_\delta = A^\infty_\delta(D)$ consists of all $f \in H(D)$ such that

$$||f||_{A^\infty_\delta} = \sup_{z \in D} |f(z)| \rho(z)^\delta < \infty,$$

and for $k = 0$ the weighted Bergman space $A^p_\delta = A^p_\delta(D) = A^{p,p}_{\delta+1}(D)$ consists of all $f \in H(D)$ such that

$$||f||_{A^p_\delta} = \Big(\int\limits_D |f(z)|^p \rho(z)^\delta d(V(z)) \Big)^{\frac{1}{p}} < \infty.$$

In particular case we have known classical Bergman spaces in the unit ball B (see [10], [2], [3]).

We denote below by $B(z, r)$ the standard Kobayashi ball in D domain(see [12])

Also let further $dv_\alpha(z) = d^\alpha(z) dv(z)$ be the weighted Lebegues measure in D domain ,where $\alpha > -1$.

The next section is a continuation of our previous work [15], [14] and treats the following problem: estimate $dist_Y(f, X), f \in Y$ where (X, Y) is one of the following pairs: (X, A^∞_δ) and (Y, A^∞_δ), where X and Y are Herz type spaces in pseudo convex or tube domains. In both cases we give sharp results under some condition on kernel. Techniques used to obtain our results were previously used to study analogous problems for analytic Besov spaces in the unit ball and polydisc (see [7],[8], [11], [14-15]). The literature on the extremal problems in spaces of analytic functions is extensive, even in the case of the unit disk, a classical exposition of these problems treated by duality methods developed by S. Havinson, W. Rogosinski and H. Shapiro can be found in [16].

We recall in our next section some of the definitions and results from [1,5,6], [17-22] on tube and pseudoconvex domains. We alert the reader some proofs are omitted since from our point of view these proofs can be easily recovered based on remarks we make in the text of paper and interested readers can recover these details easily. We use common convention regarding constants: letter C denotes a constant which can change its value from one occurrence to the next one.

Distance estimates in Herz spaces in tube and pseudoconvex domains

In this section we provide first basic facts on pseudconvex domains and tubular domains over symmetric coes then formulate and prove our main results.

Since $|f(z)|^p$ is subharmonic (even plurisubharmonic) for a holomorphic f, we have $A^p_s(D) \subset A^\infty_t(D)$ for $0 < p < \infty, sp > n$ and $t = s$. Also, $A^p_s(D) \subset A^1_s(D)$ for $0 < p \leq 1$ and $A^p_s(D) \subset A^1_s(D)$ for $p > 1$ and t sufficiently large. Therefore we have an integral representation

$$f(z) = \int_D f(\xi)K(z,\xi)\rho^t(\xi)dV(\xi), f \in A^1_t(D), z \in D, \tag{3}$$

where $K(z,\xi)$ is a kernel of type t, that is a smooth function on $D \times D$ such that $|K(z,\xi| \leq C|\widetilde{\Phi}(z,\xi)|^{-(n+1+t)}$, where $\widetilde{\Phi}(z,\xi)$ is so called Henkin-Ramirez function for D. Note that (3) holds for functions in any space X that embeds into A^1_t. We review some facts on $\widetilde{\Phi}$ and refer [19]-[22] reader to for details. This function is C^∞ in $U \times U$, where U is a neighbourhood of $\overline{D}$, it is holomorphic in z, and $\widetilde{\Phi}(\zeta,\zeta) = \rho(\zeta)$ for $\zeta \in U$. Moreover, on $\overline{D} \times \overline{D}$ it vanishes only on the diagonal $(\zeta,\zeta), \zeta \in \partial D$. Locally, it is up to a non vanishing smooth multiplicative factor equal to the Levi polynomial of ρ. From now on we work with a fixed Henkin-Ramirez function $\widetilde{\Phi}$.

We refer the reader to [12] and [22] for standard properties of r-lattices in bounded pseudoconvex domains with smooth boundary and for same type properties in the unit ball to [9] and for r-lattices in tube to [1],[5],[6]. In the unit disk these results are classical (see for example [4]). We are going to use the following results from [12] and [17].

Lemma 1. *Assume $K(z,\xi)$ is a kernel of type $t, t > -1$.*

(a) For $0 < r < r_0$ we have

$$\int_{\partial D_r} |K(z,\xi)|d\sigma(z) \leq C(\rho(\zeta) + r)^{-t-1}, \zeta \in D.$$

(b) (Forelly-Rudin estimate) Assume $\sigma > 0$ satisfies $\sigma - t - 1 < 0$. Then we have

$$\int_D |K(z,\xi)|\rho^{\sigma-1}(z)dV(z) \leq C\rho^{\sigma-t-1}(\zeta), \zeta \in D.$$

For part a) of the above lemma see Corollary 3.9. of [19], [12], [17], for part b) see [21], [12], [17]. We note that the same estimates are valid if K is replaced by $\widetilde{K}(z,\zeta) = K(z,\zeta)$.

The estimates of such type for strictly pseudoconvex domains have a long history, the basis for such results were constructive methods in several complex variables, namely integral representation formulas developed by Henkin and Ramires around 1970. E. Ligocka obtained an important factorization theorem for the weighted Bergman kernel, see [20], building on the previous work by Kerzman and Stein, see [19]. For further results in this directions see [21], [22] and recent papers [23].

Now we add some basic facts for analytic spaces in tube domains over symmetric cones.

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space V. $H(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. Following the notation of [1] and [5],[6] we denote the rank of the cone Ω by r and by Δ the determinant function on V.

Letting $V = \mathbb{R}^n$, we have as an example of a symmetric cone on $\mathbb{R}^n$ the Lorentz cone Λ_n which is a rank 2 cone defined for $n \geq 3$ by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, y_1 > 0\}.$$

The determinant function in this case is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2$$

(see for example [1],[5],[6]).

Let us introduce some convenient notations regarding multi-indices.

If $t = (t_1, \ldots, t_r)$, then $t^* = (t_r, \ldots, t_1)$ and, for $a \in \mathbb{R}^n, t + a = (t_1 + a, \ldots, t_r + a)$. Also, if $t, k \in \mathbb{R}^r$, then $t < k$ means $t_j < k_j$ for all $1 \leq j \leq r$.

We are going to use the following multi-index

$$g_0 = \left((j-1)\tfrac{d}{2}\right)_{1 \leq j \leq r}, \text{ where } (r-1)\tfrac{d}{2} = \tfrac{n}{r} - 1.$$

For $\tau \in \mathbb{R}_+$ and the associated determinant function $\Delta(x)$ we set

$$A_\tau^\infty(T_\Omega) = \left\{F \in H(T_\Omega) : ||F||_{A_\tau^\infty} = \sup_{x+iy \in T_\Omega} |F(x+iy)\Delta^\tau(y) < \infty\right\}. \quad (4)$$

It can be checked that this is a Banach space. Below we denote by Δ_s the generalized power function (see [1], [5],[6].)

For $1 \leq p, q < +\infty$ and $\nu \in \mathbb{R}$, and $\nu > \frac{n}{r} - 1$ we denote by $A_\nu^{p,q}(T_\Omega)$ the mixed-norm weighted Bergman space consisting of analytic functions f in T_Ω such that

$$||F||_{A_\nu^{p,q}} = \left(\int_\Omega \left(\int_V |F(x+iy)|^p dx\right)^{q/p} \Delta^\nu(y) \frac{dy}{\Delta(y)^{\frac{n}{r}}}\right)^{1/q} < \infty.$$

This is a Banach space.

It is known that the $A_\nu^{p,q}(T_\Omega)$ space is nontrivial if and only if $\nu > \frac{n}{r} - 1$, (see [1], [5],[6]).

When $p = q$ we write (see [4])

$$A_\nu^{p,q}(T_\Omega) = A_\nu^p(T_\Omega)$$

This is the classical weighted Bergman space with usual modification when $p = \infty$.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ and it is given by the following integral formula (see [1],[5],[6])

$$P_\nu f(z) = C_\nu \int_{T_\Omega} B_\nu(z,\omega) f(\omega) dV_\nu(\omega),$$

where

$$B_\nu(z,\omega) = C_\nu \Delta^{-(\nu+\frac{n}{r})}((z-\overline{\omega}/i))$$

is the Bergman reproducing kernel for $A^2_\nu(T_\Omega)$ (see [1], [5],[6]).

Here we used the notation $dV_\nu(\omega) = \Delta^{(\nu-\frac{n}{r})}(v)dudv$. We denote by $dV(\omega)$ or $dv(\omega)$ the Lebegues measure on tubular domain over symmetric cone. Below and here we use constantly the following notations $\omega = u+iv \in T_\Omega$ and also $z = x+iy \in T_\Omega$ .For any f function from A^∞_τ for large enough ν we have

$$f(z) = C_\nu \int_{T_\Omega} B_\nu(z,\omega) f(\omega) dV_\nu(\omega),$$

(see[1],[5],[6]) In this case sometimes below we say simply that the f function allows Bergman representation via Bergman kernel with ν index.

This fact together with Forelli-Rudin estimate (see below)is needed for our proof.

Let us first recall the following known basic integrability properties for the determinant function, which appeared already above in definitions.

Lemma 2. . *Let $\alpha \in \mathbb{C}^r$ and $y \in \Omega$.*

1) *The integral*

$$J_\alpha(y) = \int_{\mathbb{R}^n} \left|\Delta_{-\alpha}(\frac{x+iy}{i})\right| dx$$

converges if and only if $Re\alpha > g_0^* + \frac{n}{r}$. In this case $J_\alpha(y) = C_\alpha |\Delta_{-\alpha+n/r}(y)|$.

2) *For any multi-indices s and β from C^r and $t \in \Omega$ the function*

$$y \mapsto \Delta_\beta(y+t)\Delta_s(y)$$

belongs to $L^1(\Omega, \frac{dy}{\Delta^{n/r}(y)})$ if and only if $Re(s) > g_0$ and $Re(s+\beta) < -g_0^*$. In this case we have

$$\int_\Omega \Delta_\beta(y+t)\Delta_s(y)\frac{dy}{\Delta^{n/r}(y)} = C_{\beta,s}\Delta_{s+\beta}(t).$$

.

We refer to [5],[6] for the proof of the above lemma or [1].

As a corollary of one dimensional versions of these estimates (see, for example, [1],[5],[6]) we obtain the following vital estimate (A).

$$\int_\Omega \Delta^\beta(y)|B_{\alpha+\beta+\frac{n}{r}}(z,\omega)| dV(z) \leq C\Delta^{-\alpha}(v), \tag{A}$$

$\beta > -1, \alpha > \frac{n}{r} - 1, z = x+iy, \omega = u+iv$ (see [1],[5],[6]).

Let τ be the set of all triples (p,q,ν) such that $1 \leq p,q < \infty, \nu > \frac{n}{r} - 1$.

Let $B_D(z,r)$ or $B_{T_\Omega}(z,r)$ be a Bergman or Kobayashi ball in bounded strongly pseudoconvex D or tube T_Λ domain we study in this paper.

Let $T_\Lambda^m = T_\Lambda \times \cdots \times T_\Lambda$; $D^m = D \times \cdots \times D$. Let $H(D^m)$ be the space of all analytic function in D^m and $H(T_\Lambda^m)$ be the space of all analytic functions in T_Λ^m. We define new analytic Herz type spaces in these domains as follows.

In pseudoconvex domains let

$$N_{\vec{\alpha}}^{\vec{p},q} = \{f \in H(D^m) : \sum_{k_1 \geq 0} \cdots \sum_{k_m \geq 0} \Big(\int\limits_{B_{D(a_{k_1},r)}} \cdots \Big(\int\limits_{B_{D(a_{k_m},r)}} |f(\vec{\omega})^{p_1} dv_{\alpha_1}(\omega)\Big)^{\frac{p_r}{p_1}} dv_{\alpha_m}(\omega)\Big)^{\frac{q}{p_m}} < \infty\}$$

$$M_{\vec{\alpha}}^{\vec{p},q} = \{f \in H(D^m) : \int\limits_D \Big(\int\limits_{B_{D(z,r)}} \cdots \Big(\int\limits_{B_{D(z,r)}} |f(\vec{\omega})^{p_1} dv_{\alpha_1}(\omega_1)\Big)^{\frac{p_2}{p_1}} dv_{\alpha_m}(\omega_m)\Big)^{\frac{q}{p_m}} dv(z) < \infty\},$$

$$0 < p_j < \infty, \alpha_j > -1, j = 1, \ldots, m, q \in (0, \infty)$$

similarly we can define such spaces in tube domains via Bergman balls $B_{T_\Omega}(z,r)$ We consider in this paper only particular cases of these spaces(so called X type spaces) though our results may be valid also in general case .

We now provide a simple assertion a new projection theorem for $q \leq p \leq 1$ in these new Herz type spaces in the unit disk. Then this assertion leads easily to a sharp distance theorem.It moreover at the same time also allows various far reaching extensions to function spaces of several complex variables in various domains under certain condition on Bergman kernel.

Let $f \in H(D_1)$, where D_1 is a unit disk then let $q \leq p \leq 1$, then for $\beta > \beta_0$, where β_0 is large enough we have that $||P_\beta^+(|f|)||_Y \leq C||f||_Y$ for all $\alpha > (-1), \alpha_1 > -1, \widetilde{\alpha} \geq 0$ if

$$Y = X_{p,q,\widetilde{\alpha}} = X(D_1) = \{f \in H(D_1) : \sum_{k \geq 0} \Big(\int\limits_{D(a_k,r)} |f(\omega)|^p dV_{\widetilde{\alpha}}(\omega)\Big)^{\frac{q}{p}} < \infty\};$$

or

$$Y = \widetilde{X}_{p,q,\alpha,\alpha_1} = \widetilde{X}(D_1) = \{f \in H(D_1) : \int\limits_{D_1} \Big(\int\limits_{D(z,r)} |f(\omega)|^p dV_\alpha(\omega)\Big)^{\frac{q}{p}} dV_{\alpha_1}(z) < \infty\},$$

where $H(D_1)$ is a space of all analytic functions in D_1, $dV(\omega)$ is a Lebegues measure on D_1; $dV_\alpha(z) = (1-|z|)^\alpha dV(z), \alpha > -1$, and $D(z,r)$ is the Bergman ball in the unit disk (see ,for example, [9]). We for simplicity further omit indexes in definitions of these spaces and where in addition

$$P_\beta^+(|f|)(z) = \int\limits_{D_1} \frac{|f(\omega)|(1-|\omega|)^\beta}{|(1-\overline{\omega}z)^{\beta+2}|} dV(\omega), \beta > \beta_0, z \in D_1$$

be a Bergman projection with positive Bergman kernel.

The proof in the unit disk follows directly from following well-known simple estimates (see, for example ,[4]). Based on lemma 1 and properties of r-lattices in the pseudoconvex domains (see for example [22]) as it will be indicated below ,almost all of them are valid in the unit ball and even in bounded strongly pseudoconvex domain with smooth boundary.

We have in the unit disk (and similarly in the ball based on properties r-lattices in the ball see [9])

$$\Big(\int\limits_{D_1}\frac{|f(\omega)|(1-|\omega|)^{\beta}}{|(1-\overline{\omega}z)^{\beta+2}|}dV(\omega)\Big)^{p}\leq C\int\limits_{D_1}\frac{|f(\omega)|^{p}(1-|\omega|)^{\beta p+2p-2}dV(\omega)}{|(1-\overline{\omega}z)^{(\beta+2)p}|};p\leq 1;\beta>(-1);$$

$$\int\limits_{D(a_k,r)}\frac{(1-|\omega|)^{\tau}dV(\omega)}{|(1-\overline{\omega}z)|^{\nu}}\leq\frac{\widetilde{c}}{|1-\bar{z}a_k|^{\nu-(r+2)}};\tau\geq 0;\nu>\tau+2;a_k,z\in D_1$$

Using subharmonicity of $\frac{1}{|1-\overline{\omega}a_k|^s}$ and Forelly-Rudin estimate and obvious elementar estimates based on properties of r lattices we have (the same chain of estimates is valid for Bergman kernel in tube and pseudoconvex domains but for other values of parameters.)

$$\sum_{k\geq 0}\Big(\frac{(1-|a_k|)^{\beta}}{|(1-\omega\overline{a_k})^{s}|}\Big)\leq C_2\sum_{k\geq 0}\Big(\int\limits_{D(a_k,r)}\frac{dV(z)}{|1-\overline{\omega}z|^{s}}\Big)(1-|a_k|)^{-2+\beta}\leq\frac{\widetilde{C_1}}{(1-|\omega|)^{s-\beta}};\beta>1,s>\beta.$$

$$||f||^{q}_{A^q_\alpha}\leq C_6\int\limits_{D_1}\Big(\int\limits_{D(z,r)}|f(\omega)|^{p}dV_{\tau-2}(\omega)\Big)^{\frac{q}{p}}dV_{\tau_1}(z)$$

where $(\tau-2)\frac{q}{p}+\tau_1=\alpha, 0<p,q<\infty$

Then in the unit disk and in the unit ball we have also.

$$I=\int\limits_{D(z,r)}\frac{(1-|\omega|)^{\tau}dV(\omega)}{|1-\overline{\nu\omega}|^{\alpha}}\leq\frac{\widetilde{c}(1-|z|)^{\tau}}{|1-\bar{\nu}z|^{\alpha-2}};\tau>-1,\alpha>2,z,\nu\in D_1$$

$$I\leq\frac{C_3}{|1-\bar{\nu}z|^{\alpha-2-\tau}};\alpha>\tau+2;\tau\geq 0;z,\nu\in D_1.$$

Now combining these simple estimates we have, for example, for $X_{p,r}$ spaces, for all $\beta>\beta_0,\beta_0$ is large enough.

$$\sum_{k\geq 0}\Big(\int\limits_{D(a_k,r)}|(P^{+}_{\beta})(f)(z)|^{p}dV_{\widetilde{\alpha}}(z)\Big)^{\frac{q}{p}}\leq C_4\sum_{k\geq 0}\Big(\int\limits_{D(a_k,r)}\Big(\int\limits_{D_1}\frac{|f(\omega)|^{p}(1-|\omega|)^{\beta p+2p-2}}{|(1-\overline{\omega}z|)^{(\beta+2)p}}dV(\omega)\Big)dV_{\widetilde{\alpha}}(z)\Big)^{\frac{q}{p}}\leq$$

$$\leq C_5\sum_{k\geq 0}\Big(\int\limits_{D_1}\frac{|f(\omega)|^{p}(1-|\omega|)^{\beta p+2p-2}}{|(1-\overline{\omega}a_k)|^{(\beta+2)p-2}}dV(\omega)\Big)^{\frac{q}{p}}(1-|a_k|)^{\frac{\widetilde{\alpha}q}{p}};$$

It remains to use one more time first and second estimates for $\frac{q}{p}\leq 1$ and $\beta=\frac{\widetilde{\alpha}q}{p},\widetilde{\alpha}>1$ and finally the simple estimate

$$||f||^{q}_{A^q_\alpha}\leq C_6\sum_{k\geq 0}\Big(\int\limits_{D(a_k,r)}|f|^{p}(1-|z|)^{\widetilde{\alpha}}dV(z)\Big)^{\frac{q}{p}};$$

for $0<q,p<\infty$.

The second projection theorem for another Herz type function space can be shown similarly easily based on first and last estimate which was provided above by us and again Forelly-Rudin estimate. The proof of the unit disk case is complete.

Note following the presented proof step by step and using technique well developed in [9] for the unit ball case it can be easily shown that these projection theorems are valid also in case of the unit ball in $\mathbb{C}^n$ based on properties of r-lattices there.We omit easy details.

The most interesting question is to extend (under some additional condition on Bergman kernel) these projection type results to the case of bounded strongly pseudoconvex domains or tube domains over cones .

This will directly lead to new sharp theorems for distance function in tube and pseudoconvex domains (under some additional conditions on Bergman kernel). Previously such sharp results for other spaces were obtained in [7]-[8] and [13]-[15].

First we note that the uniform estimates are probably known in the unit ball B in C^n.We denote by dV the Lebegues measure on B. and by dV_α the weighted Lebegues measure.(see[9])

$$\sup_{z\in B}|f(z)|(1-|z|)^\tau \le C||f||_X$$

(or $||f||_{\widetilde{X}}$); for some fixed $\tau>0, \tau=\tau(p,q,\alpha,\alpha_1,\widetilde{\alpha})$ large enough depending on parameters of X (or $\widetilde{X}$) space and easy proofs of these estimates are based on the following known facts

$$\sup_{z\in D(a_k,r)}|f(z)|^p \le C\int\limits_{D(a_k,r)}|f(z)|^p(1-|z|)^{-\alpha-(n+1)}dV_\alpha(z);\alpha>-1;0<p\le\infty$$

(where $D(a_k,r)$ is a Bergman ball in the unit ball see [9]) and standard properties of r-lattice in the unit ball (see [9]). Hence we can now pose a dist problem for these pairs of analytic spaces involved in these embeddings.

This leads with projection theorem we provided to a sharp dist theorem for analytic Herz type (X type) spaces in the unit ball in $\mathbb{C}^n$ which under some conditions on kernel are even valid also in the bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$ and in tube domains. First however we give the unit ball result for Herz-type X spaces. The same result practically with very similar proof is valid for $\widetilde{X}$ spaces simply with another τ.

Theorem 1. *Let $f\in(A^\infty_\tau)(B)$. Let also $\alpha,\alpha_1>(-1),\widetilde{\alpha}\ge 0, q\le p\le 1$. Then*

$$dist_{A^\infty_\tau}(f,X)\asymp K=||\int\limits_{\Lambda_{\varepsilon,\tau}}\frac{(1-|z|)^{\beta-\tau}dV(z)}{|1-\overline{z}\omega|^{\beta+n+1}}||_X$$

for all $\beta>\beta_0$, where β_0 large enough, for some $\tau=\tau(p,q,\alpha,\alpha_1),\Lambda_{\varepsilon,t}=\{z\in B:|f(z)|(1-|z|)^t>\varepsilon\}$.

Proof. We refer for estimate $dist_{A^\infty_\tau}(f,X)\ge K=K(X)$ to [15] for the unit disk case where less general case of classical Bergman spaces was considered, the proof is the same. Our projection theorem we obtained above gives immediately the estimate we need (see also [14], [15],[13],[11] for very similar type arguments.)

Here is the proof of the fact that $(dist)_{A^\infty_\tau}(f,X)\le K$ for simplest case of the unit disk. In the unit ball arguments are the same. For $\beta>\beta_0,\beta_0$ is large enough we have from classical Bergman representation formula that

$$f(z) = c(\beta)\Big(\int\limits_{D_1/\Lambda_{\varepsilon,t}} \frac{f(\omega)(1-|\omega|)^{\beta}}{(1-\overline{\omega}z)^{\beta+2}} dV(\omega) +$$

$$+ \int\limits_{\Lambda_{\varepsilon,t}} \frac{f(\omega)(1-|\omega|)^{\beta}}{(1-\omega z)^{\beta+2}} dV(\omega)\Big) = (f_1)(z) + (f_2)(z), z \in D_1$$

where $c(\beta)$ is a Bergman representation constant.

Then for $t<0$ using classical Forelli-Rudin estimate in the unit disk we have.

$$|f_1(z)| \leq c \int\limits_{D_1/\Lambda_{\varepsilon,-t}} \frac{|f(\omega)|(1-|\omega|)^{\beta}}{|1-\omega z|^{\beta+2}} dV(\omega) \leq \Big(\frac{c\varepsilon}{(1-|z|)-t}\Big);$$

So we have $(\sup\limits_{z\in D_1})|f_1(z)|(1-|z|)^{-t} \leq c\varepsilon$. For $s<0, t<0$, we have

$$\int\limits_{D_1} |f_2(z)|^q (1-|z|)^{-sq-1} dV(z) \leq c \int\limits_{D_1}\Big(\int\limits_{\Lambda_{\varepsilon,-t}} \frac{(1-|\omega|)^{\beta+t} dV(\omega)}{|1-\overline{\omega}z|^{\beta+2}}\Big)^q (1-|z|)^{-sq-1} dV(z) \leq C.$$

So for $\beta > \beta_0$ we have what we need, for $t = sq-1$,since we have easily from here that the A^{∞}_{τ} norm of $f - f_2$ function is less than $C\varepsilon$. The proof now is complete. □

The same result is valid for $\widetilde{X}$ space with very similar proof based on projection theorem without any additional condition on Bergman kernel in the unit disk and in the unit ball.

We turn now to more complicated domains.The simple idea is to follow this proof in the unit disk and to replace estimates used in this proof in the unit disk by their complete analogues in bounded pseudoconvex domains having smooth boundary or in tubular domains over symmetric cones.

First we have that $|f(z)|(1-|z|)^{\tau} \leq C||f||_X$ in bounded pseudoconvex D domain with smooth boundary ,namely

$$\sup_{z\in D} |f(z)|\delta(z)^{\tau} \leq C||f||_X \tag{5}$$

(X type spaces can be defined similarly via Kobayashi balls in pseudoconvex domains or via Bergman balls in tube).

The proof of (5) is based on simple known facts that

$$(\sup_{z\in D})(|f(z)|\delta(z)^{\frac{\alpha+n+1}{p}}) \leq C||f||_{A^p_{\alpha}} \leq C_1 \sum_{k\geq 0}\Big(\int\limits_{B(a_k,r)} |f(z)|^p \delta(z)^{\alpha} dV(z)\Big)^{\frac{q}{p}}; q \geq 1, \alpha > -1$$

similarly we have

$$||f||_{A^{\infty}_{\tau}} \leq C_2 ||f||_{A^q_{\widetilde{\alpha_1}}} \leq C_3 \int\limits_D \Big(\int\limits_{B(z,r)} |f(\widetilde{z})|^p \delta(\widetilde{z})^{\alpha} dV(\widetilde{z})\Big)^{\frac{q}{p}} \delta(z)^{\alpha_1} dV(z); q \leq p, p \in (0,\infty)$$

for some $\tau = (\frac{\alpha q}{p}) + \alpha_1 + (n+1)(\frac{q}{p}) + (n+1), \alpha, \alpha_1 > (-1)$, for some $\widetilde{\alpha_1}$; based on estimates in D (see ,for example, [23],[12])

$$|f(z)|^q (\delta(z))^{\tau} \leq C_4 \Big(\int\limits_{B(z,r)} |f(\widetilde{z})|^p dV_{\alpha}(\widetilde{z})\Big)^{\frac{q}{p}}$$

for some $\tau = (n+1) + (\frac{\alpha q}{p})$. These embeddings lead directly to distance problem for A^{∞}_{α} and Herz-type spaces in these domains.

Note first a carefull inspection shows that the proofs of estimates $(dist)_{A^{\infty}_{\tau}}(f,X) \le K$ in the unit ball,pseudoconvex and tubular domains are the same as in the unit disk based on lemmas we listed above in these domains.

Note then adding a condition on weighted Bergman kernel $K_\tau(z,\omega), \tau \ge 0$ we can now easily extend our last theorem even to bounded pseudoconvex domains with smooth boundary in $\mathbb{C}^n$. This gives us first sharp result on distance function in such Herz type spaces in such type domains. Indeed the core of proof is the projection theorem and our projection theorem is valid in this context under one additional condition on Bergman kernel.

Note in pseudoconvex domains we have for $p \le 1, \alpha > -1, \tau > 0$

$$\Big(\int_D |f(\omega)||K_\tau(z,\omega)|(\delta(z)^\alpha)dV(z)\Big)^p \le C_5 \int_D |f(\omega)|^p |K_\tau(z,\omega)|^p \delta(z)^{\alpha p+(n+1)p-(n+1)} dV(z) \quad (see[23])$$

and the rest is repetition of proof of the unit disk case and the unit ball case based on integral representation and Forelli -Rudin estimate with one additional assumption that for weighted Bergman kernel $(K_\tau(z,\omega))$ the following estimate

$$\int_{B(z,\tau)} (\delta(\widetilde{z})^\alpha |K_\tau(\widetilde{z},\omega)|dV(\widetilde{z}) \le C_6 |K_{\tau+n+1}(\omega,z)|\delta(z)^\alpha; \omega, z \in D \tag{6}$$

is valid for $\alpha > 0, \tau \ge 0$ (and related to the last estimate (6) another estimate for the integral by $B(a_k,r)$ for $\widetilde{X}$ spaces is also valid,both are valid in the ball). As a result we have the following sharp result in bounded strongly pseudoconvex domains. This extends also our previous result in the unit disk(see also [23] for less general case) formulated in theorem 1 under one condition (6) on Bergman kernel.

The fact that for any function from A^{∞}_{τ} spaces the Bergman representation formula with large index is valid in tube and pseudoconvex domains is well-known (see ,for example, [23]) This as we can see from one dimensional proof is also important for our proofs in higher dimension. Let further $\Lambda_{\varepsilon,t} = \{z \in D : |f(z)|(\delta(z))^t > \varepsilon\}; t > 0, \varepsilon > 0;$

Theorem 2.

Let $f \in (A^{\infty}_{\tau})(D)$. Let also $\alpha, \alpha_1 > -1, \widetilde{\alpha} \ge 0; q \le p \le 1$ and let condition (6) holds. Then

$$(dist_{A^{\infty}_{\tau}})(f,X) \asymp (\inf_{\varepsilon>0})\{\varepsilon > 0 : \Big|\Big| \int_{\Omega_{\varepsilon,\tau}} (\delta(z)^{\beta-\tau})|K_{\beta+N+1}(z,\omega)|dV(z)\Big|\Big|_X < \infty\}$$

for all $\beta > \beta_0$ where (β_0) is large enough for some $\tau = \tau(p,q,\alpha,\alpha_1); X = X(p,q,\alpha,\alpha_1)$.

The similar result is valid with very similar proof for $\widetilde{X}$ space. We omit easy details leaving them to readers.

Note the proof of theorem 2 is based heavily on projection theorems in pseudoconvex domains and theorem 1 is based on same type theorem in the unit ball,such projection results can be seen in [12], [17 -22] in pseudoconvex domains.

In [2], [3], [4], [9], [10] in the unit ball we also can see such type results.

The short scheme of proof of such type distance results based directly on projection theorems can be seen in [7-8]; [11], [13-15]. We omit here easy details. We have based on arguments we provided in the unit disk and these projection theorems in Herz type spaces in the ball as corollary the dist theorem 2 for such type spaces in pseudoconvex domains.

We assume that (6) is valid. Then we have the following projection theorem which is interesting also as a separate statement.

Theorem 3. *The P^+_β integral operator (the Bergman projection with positive kernel) for $\beta > \beta_0$, where β_0 is large enough is mapping from X to X and from $\widetilde{X}$ to $\widetilde{X}$ for all $q \le p \le 1, \alpha > (-1), \alpha_1 > 1, \widetilde{\alpha} \ge 0$*

Probably this result can be extended also to other values of p and q parameters. We leave these problems to intereted readers. We will now discuss also similar type result in context of tubular domain over symmetric cones and we will formulate similar type result also in that context below.

Note it is easy to see that if we assume that $||P^+_\beta(f)||_X \le c||f||_X$, for $\beta > \beta_0, where \beta_0$ large enough then the same distance theorem very similar to theorem 2 for $p,q > 1$ in tube (sharp result on distances) can be also easily formulated.

Let further $\widetilde{\Lambda}_{\varepsilon,t} = \{z \in T_\Lambda : |f(z)|\Delta^t(Imz) > \varepsilon\}, t > 0, \varepsilon > 0$

Note because of the following embeddings (see [1],[5],[6])

$$\sup|f(z)|\Delta^\tau(Imz) \le C||f||_X; \sup|f(z)|\Delta^{\tau_1}(Imz) \le C_1||f||_{\widetilde{X}}$$

for some $\tau, \tau_1 > 0$ the very similar distance problems in tube can be posed also and solved by similar methods. (Short proofs of last two estimates are similar to the proof which we have in the unit disk.)

We omit the proof of the following result(assuming the boundedness of Bergman projection with positive Bergman kernel with large index in Herz-type spaces we discuss)since no new idea here is involved.

Theorem 4. *Let $f \in (A^\infty_\tau)(T_\Lambda)$. Let also $\alpha, \alpha_1 > -1, \widetilde{\alpha} \ge 0; q > 1, p > 1$. Then*

$$(dist_{A^\infty_\tau})(f,X) \asymp (\inf_{\varepsilon>0})\{\varepsilon > 0 : \Big|\Big| \int\limits_{\widetilde{\Lambda}_{\varepsilon,\tau}} (\Delta^{\beta-\tau}(Imz))|B_{\beta+\frac{n}{r}}(z,\omega)|dm(z)\Big|\Big|_X < \infty\}$$

for all $\beta > \beta_0$ where (β_0) is large enough for some $\tau = \tau(p,q,\alpha,\alpha_1)$.

The similar result is valid for $\widetilde{X}$ type Herz spaces in tubular domains over symmetric cones. We omit easy details leaving them to readers since proofs of these results are very similar.

Our results maybe can be also extended to more general $M^{p,q}_\alpha$ and $N^{p,q}_\alpha$ type analytic function spaces,we pose this as a problem.

References

[1] Bekolle D., Bonami A., Garrigos G., Ricci F., Sehba B., "Hardy-type inequalities and analytic Besov spaces in tube domains over symmetric cones", *J. Reine Angew. Math.*, **647**:25 (2010).

[2] Ortega J., Fabrega J., "Holomorphic Lizorkin-Triebel type spaces", *Journal of Funct. Analysis*, 1997, 177-212.

[3] Rudin W., *Function theory in the unit ball*, Springer-Verlag, New York, 1980.

[4] Shamoyan F., Djrbashian A., *Topics in the theory of A^p_α spaces*, Teubner - Text zur Mathematics, Leipzig, 1988.

[5] Bekolle D., Kagou A. T., “Reproducing properties and L^p estimates for Bergman projections in Siegel domains of second type”, *Studia Math.*, **115**:3 (1995).

[6] Bekolle D., Bonami A., Garrigos G., Ricci F., Sehba B., “Analytic Besov spaces and symmetric cones”, *Jour. Fur seine and ang.*, **647** (2010), 25-56.

[7] Shamoyan R., Mihic O., “On a distance function in some New analytic Bergman type spaces in higher dimension”, *Journal of Function spaces*, **2014** (2014).

[8] Shamoyan R., “On some extremal problems in certain harmonic function spaces”, *Issues of Analysis*, **20(I)** (2013), 43-58.

[9] Zhu K., *Spaces of Holomorphic Functions in the unit ball*, Springer-Verlag, New York, 2005, 226 pp.

[10] Xu W., “Distances from Bloch functions to some Mobius invariant function spaces in the unit ball of C^n”, *Journal. of Funct. Spaces and Appl.*, **7**:1 (2009), 91-104.

[11] Kurilenko S., Shamoyan R., “On Extremal problems in tubular domains”, *Issues of Analysis*, **3(21)** (2013), 44-65.

[12] Ortega J. M., Fabrega J., “Mixed-norm spaces and interpolation”, *Studia Math.*, **109**:3 (1994), 233-254.

[13] Shamoyan R. F., Arsenovi’c M., “Some remarks on extremal problems in weighted Bergman spaces of analytic function”, *Commun. Korean Math. Soc.*, **27**:4 (2012), 753-762.

[14] Shamoyan R., Mihic O., “On new estimates for distances in analytic function spaces in higher dimension”, *Sib. Elektron. Mat. Izv.*, 2009, 514-517.

[15] R. Shamoyan and O. Mihic, “On new estimates for distances in analytic function spaces in the unit disk, the polydisk and the unit ball”, *Bol. Asoc. Mat. Venez.*, **17**:2 (2010), 89-103.

[16] Duren P., *Theory of H_p Spaces*, Academic Press, 1970.

[17] Beatrous F., “L_p-estimates for extensions of holomorphic function”, *Michigan Math. Journal*, **32**:3 (1985), 361-380.

[18] Ahern P., Schneider R., “Holomorphic Lipschitz function in pseudoconvex domains”, *Amer. Journal of Math.*, **101**:3 (1979), 543-565.

[19] Kerzman N. and Stein E. M., “The Szego kernel in terms of Cauchy-Fantappie kernels”, *Duke Math. J.*, **43**:2 (1978), 197-223..

[20] Ligocka E., “On the Forelli-Rudin construction and weighted Bergman projection”, *Studia Math.*, **94**:3 (1989), 257-272.

[21] Cohn W. S., “Weighted Bergman projection and tangential area integrals”, *Studia Math.*, **106**:1 (1993), 59-76.

[22] Abate M., Raissy J., Saracco A., “Toeplitz operator and Carleson measures in strongly pseudoconvex domains”, *Journal of. Funct. Anal.*, **263**:11 (2012), 3449-3491.

[23] Arsenovic M., Shamoyan R., “On some sharp estimates for distances in bounded strongly pseudocnvex domains”, *Bulletin Korean Math. Society*, **1** (2015).

References (GOST)

[1] Bekolle D., Bonami A., Garrigos G., Ricci F., Sehba B. Hardy-type inequalities and analytic Besov spaces in tube domains over symmetric cones // J. Reine Angew. Math. 2010. vol. 647. no. 25.

[2] Ortega J., Fabrega J. Holomorphic Lizorkin-Triebel type spaces // Journal of Funct. Analysis. 1997. pp. 177-212.

[3] Rudin W. Function theory in the unit ball. New York: Springer-Verlag, 1980.

[4] Shamoyan F., Djrbashian A. Topics in the theory of A^p_α spaces. Leipzig: Teubner - Text zur Mathematics, 1988

[5] Bekolle D., Kagou A.Ṫ. Reproducing properties and L^p estimates for Bergman projections in Siegel domains of second type // Studia Math., 1995. vol. 115. no. 3.

[6] Bekolle D., Bonami A., Garrigos G., Ricci F., Sehba B. Analytic Besov spaces and symmetric cones // Jour. Fur seine and ang. 2010. vol. 647. pp. 25-56.
[7] Shamoyan R., Mihic O. On a distance function in some New analytic Bergman type spaces in higher dimension // Journal of Function spaces. 2014. vol. 2014
[8] Shamoyan R. On some extremal problems in certain harmonic function spaces // Issues of Analysis. 2013. vol. 20(I), pp. 43-58.
[9] Zhu K.Spaces of Holomorphic Functions in the unit ball. New York: Springer-Verlag, 2005. 226 p.
[10] Xu W. Distances from Bloch functions to some Mobius invariant function spaces in the unit ball of C^n // Journal. of Funct. Spaces and Appl. 2009. vol.7. no. 1. pp. 91-104.
[11] Kurilenko S., Shamoyan R. On Extremal problems in tubular domains // Issues of Analysis. 2013. vol 3(21). pp. 44-65.
[12] Ortega J. M., Fabrega J. Mixed-norm spaces and interpolation // Studia Math. 1994. vol. 109. no. 3 .pp. 233-254.
[13] Shamoyan R. F., Arsenovi'c M. Some remarks on extremal problems in weighted Bergman spaces of analytic function // Commun. Korean Math. Soc. 2012. vol. 27. no. 4. pp. 753-762.
[14] Shamoyan R., Mihic O. On new estimates for distances in analytic function spaces in higher dimension // Sib. Elektron. Mat. Izv. 2009. pp. 514-517.
[15] Shamoyan R., Mihic O. On new estimates for distances in analytic function spaces in the unit disk, the polydisk and the unit ball // Bol. Asoc. Mat. Venez. 2010. vol. 17. no 2. pp. 89-103.
[16] Duren P. Theory of H_p Spaces. Academic Press, 1970.
[17] Beatrous F. L_p-estimates for extensions of holomorphic function // Michigan Math. Journal. 1985. vol. 32. no. 3. pp. 361-380.
[18] Ahern P., Schneider R. Holomorphic Lipschitz function in pseudoconvex domains // Amer. Journal of Math. 1979. vol. 101. no. 3. pp. 543-565.
[19] Kerzman N., Stein E. M. The Szego kernel in terms of Cauchy-Fantappie kernels // Duke Math. J.1978. vol. 43, no. 2. pp. 197-223.
[20] Ligocka E. On the Forelli-Rudin construction and weighted Bergman projection // Studia Math. 1989. vol. 94. no. 3. pp. 257-272.
[21] Cohn W. S. Weighted Bergman projection and tangential area integrals // Studia Math. 1993. vol. 106. no. 1. pp. 59-76.
[22] Abate M., Raissy J., Saracco A. Toeplitz operator and Carleson measures in strongly pseudoconvex domains // Journal of. Funct. Anal. 2012. vol. 263. no. 11. pp. 3449-3491.
[23] Arsenovic M., Shamoyan R. On some sharp estimates for distances in bounded strongly pseudocnvex domains // Bulletin Korean Math. Society. 2015. vol. 1.

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О НЕКОТОРЫХ НОВЫХ ТЕОРЕМАХ И ОЦЕНКАХ В ПРОСТРАНСТВАХ ТИПА ГЕРЦА, ОГРАНИЧЕННЫХ В ПСЕВДОВЫПУКЛЫХ ОБЛАСТЯХ

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Мы доказываем новые проекционные теоремы для новых пространств типа Герца в различных областях в C^n в единичном круге, единичном шаре, ограниченных псевдовыпуклыми областями и на основе этих результатов даем точные оценки расстояний в таких пространствах при одном дополнительном условии на ядре Бергмана. Аналогичный точный результат при одном дополнительном условии будет установлен для пространств подобного типа в трубчатых областях над симметрическими конусами.

Ключевые слова: псевдовыпуклые и трубчатые области, единичный шар, проекционная теорема, пространства Герца