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SOME PROPERTIES OF HORN TYPE SECOND ORDER DOUBLE HYPERGEOMETRIC SERIES

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Horn [1931, Hypergeometrische Funktionen zweier Veranderlichen, Math. Ann., 105(1), 381-407], (corrections in Borngasser [1933, Uber hypergeometrische funkionen zweier Veranderlichen, Dissertation, Darmstadt], defined and investigated ten second order hypergeometric series of two variables). In the course of further investigation of Horn's series, we noticed the existence of hypergeometric double series H_2^* analogous to Horn's double series H_2. The principal object of this paper is to present a natural further step toward the mathematical properties and presentations concerning the analogous hypergeometric double series H_2^* Indeed, motivated by the important role of the Horn's functions in several diverse fields of physics and the contributions toward the unification and generalization of the hyper-geometric functions, we establish a system of partial differential equations, integral representations, expansions, analytic continuation, transformation formulas and generating relations. Also, we discuss the links for the various results, which are presented in this paper, with known results.

Key words: Gauss hypergeometric function, Horn double series, partial differential equations, integral representations, transformation, generating functions.

1. Introduction

The various interpretations of Gauss hypergeometric function

$$ {}_2F_1\left(a,b;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}, \tag{1.1} $$

with parameters $a,\ b,\ c\ \in\ C$ and $c\ \notin\ Z\ \leqslant 0$, where the Pochhammer symbol $(a)_m$ is defined by $(a)_m=\Gamma(a+m)/\Gamma(a)=a(a+1)...(a+m-1)$ for $m\geqslant 1$, $(a)_0=1$ and Γ: Gamma function, have challenged mathematicians to generalize this function. The series (1.1) is easily seen to be convergent when $|x|<1$. The great successes of hypergeometric series theory in one variable has stimulated the development of a corresponding theory in two and more variables (cf. [1], [2], [3], [5], [6], [7], [10], [11], [14]). The class of Horn's double Gaussian second order hypergeometric series consists of ten complete series (cf. [8], [9], [17]: $G_1,G_2,G_3,H_1,...,H_7$. In the course of further investigation of Horn's series, we noticed the existence of double series analogous to the Horn's double series H_2 of the form:

$$ \sum_{m,n=0}^{\infty}\frac{(a)_{2m}(b_1)_n(b_2)_{n-m}}{(c)_n m!n!}x^m y^n. \tag{1.2} $$

The series (1.2) is analogous to and modified form of the Horn's double series H_2 [[17], p. 24(10)]:

$$ H_2\left[a,b,e,f;c;x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m(e)_n(f)_n}{(c)_m}\frac{x^m y^n}{m!n!}. \tag{1.3} $$

In this paper, we will study some of the properties of the series (1.2) involving differential equations, integral representations, expansions, generating functions and its relations with other known classical functions and hypergeometric series. We shall denote series (1.2) symbolically as

$$ H_2^*\left(a,b_1,b_2;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m}(b_1)_n(b_2)_{n-m}}{(c)_n m!n!}x^m y^n. \tag{1.4} $$

According to the Horn's theory [13], it can be easily verified that the series (1.4) is the second order hypergeometric series. Clearly, in view of the formula

$$ (\lambda)_{2m}=2^{2m}(\tfrac{\lambda}{2})_m(\tfrac{\lambda}{2}+\tfrac{1}{2})_m,\ \ m=0,1,2,..., \tag{1.5} $$

from (1.3) and (1.4) we obtain:

$$ H_2^*\left(a,b_1,b_2;c;x,y\right)=H_2\left[b_2,b_1,\tfrac{a}{2},\tfrac{a}{2}+\tfrac{1}{2};c;y,4x\right]. \tag{1.6} $$

Analogous and symmetric functions from the set of different hypergeometric functions of two or more variables have attracted the greatest attention because these kinds of hypergeometric series have simple and elegant results.

2. System of partial differential equations

According to the theory of multiple hypergeometric functions (see [2]), the system of partial differential equations for Horn-type function H_2^* is readily seen to be given as:

$$\begin{aligned}&\left[\left(b_1+y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}-1\right)\left(1+x\frac{\partial}{\partial x}\right)x^{-1}-\left(a+2x\frac{\partial}{\partial x}+1\right)\left(a+2x\frac{\partial}{\partial x}\right)\right]u=0\\&\left[\left(c+y\frac{\partial}{\partial y}\right)\left(1+y\frac{\partial}{\partial y}\right)y^{-1}-\left(b_1+y\frac{\partial}{\partial y}\right)\left(b_2+y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}\right)\right]u=0\end{aligned}\tag{2.1}$$

where $u=H_2^*\left(a,b_1,b_2;c;x,y\right)$. Now by making use of some elementary calculations, we find the following system of second-order partial differential equations:

$$\begin{aligned}&x\left(1+4x\right)u_{xx}-yu_{xy}-\left[b_2-1-2\left(2a+3\right)x\right]u_x+a\left(a+1\right)u=0,\\&y\left(1-y\right)u_{yy}+xyu_{xy}+\left[c-\left(b_1+b_2+1\right)y\right]u_y+b_1xu_x-b_1b_2u=0.\end{aligned}\tag{2.2}$$

It is noted that the two equations of the system (2.2) are linearly dependent, because the hypergeometric function H_2^* satisfies the system. Now, in order to find the linearly independent solutions of the system (2.2), we consider u as in the form $u=x^{\tau}y^{\nu}w$ an unknown functions, and τ and ν are constants which are to be determined. So, substituting $u=x^{\tau}y^{\nu}w$ into the system (2.2), we obtain

$$\begin{aligned}&x\left(1+4x\right)w_{xx}-yw_{xy}-\left[\beta_2-1-2\tau+\nu-\left(4\alpha+6+8\tau\right)x\right]w_x-\tau\frac{y}{x}w_y\\&+\left[\frac{\tau\left(\tau-\nu-\beta_2\right)}{x}+4\tau\left(\tau-1\right)+\alpha\left(\alpha+1\right)+\left(4\alpha+6\right)\tau\right]w=0\\&y\left(1-y\right)w_{yy}+xyw_{xy}+\left[\gamma+2\nu-\left(2\nu-\tau+\beta_1+\beta_2+1\right)y\right]w_y+\left(\nu+\beta_1\right)xw_x\\&+\left[\frac{\nu\left(\nu+\gamma-1\right)}{y}+\left(\tau-\beta_1-\beta_2-\nu\right)\nu+\beta_1\tau-\beta_1\beta_2\right]w=0\end{aligned}\tag{2.3}$$

It is noted that the system (2.3) is analogical to the system (2.2) and it is not difficult to see that the system satisfies the following solutions: for $\tau=0,\ \nu=0,\ u_1=H_2^*\left(\alpha,\beta,\gamma;\delta;x,y\right)$ and for $\tau=0,\ \nu=1-\gamma,\ u_2=y^{1-\gamma}H_2^*\left(\alpha,1-\gamma+\beta_1,1-\gamma+\beta_2;2-\gamma;x,y\right)$.

3. Integral representations

Integral representations for hypergeometric series are very useful. Change of variables in these integrals leads to equivalent integrals. This provides an effective and easy method to derive certain mathematical properties for series H_2^*. First, if we use the integral relation:

$$\frac{(b_1)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}B\left(b_1+n,c-b_1\right)=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}\int\limits_0^1\xi^{b_1+n-1}(1-\xi)^{c-b_1-1}d\xi,$$

$$\operatorname{Re}\left(c\right)>\operatorname{Re}\left(b_1\right)>0,\tag{3.1}$$

where $B(a,b)$ is Beta function (see e.g. [[8], pp. 9-11], [16] and [[18], p. 26 and p. 86, Problem 1]) defined by

$$B\left(a,\,b\right)=\begin{cases}\int_0^1 t^{a-1}(1-t)^{b-1}dt,\ \left(R\left(a\right)>0,\,R\left(b\right)>0\right),\\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\qquad\left(a,\,b\,\in\,C\backslash\,Z_0^-\right),\end{cases}\tag{3.2}$$

the definition (1.4) and the identity

$$(a)_{-m}=\frac{(-1)^m}{(1-a)_m}. \tag{3.3}$$

Then we get the following integral representation for H_2^*

$$H_2^*(a,b_1,b_2;c;x,y)=\frac{2\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}$$
$$\times\int_0^1 \xi^{2b_1-1}\left(1-\xi^2\right)^{c-b_1-1}\left(1-y\xi^2\right)^{-b_2}{}_2F_1\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};1-b_2;-4x\left(1-y\xi^2\right)\right)d\xi, \tag{3.4}$$
$$\mathrm{Re}\,(c)>\mathrm{Re}\,(b_1)>0,\ b_2\neq 0,\pm1,\pm2,\ldots .$$

Next, if in (3.4) we employ the result [8]

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \eta^{a-1}(1-\eta)^{c-a-1}(1-z\eta)^{-b}d\eta,$$
$$\mathrm{Re}\,(c)>\mathrm{Re}(a)>0,$$

then formula (3.4) can be rewritten in the more elegant form:

$$H_2^*(a,b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}\frac{\Gamma(1-b_2)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(1-b_2-\frac{a}{2}\right)}$$
$$\times\int_0^1\int_0^1 \xi^{b_1-1}\eta^{\frac{a}{2}-1}(1-\xi)^{c-b_1-1}(1-\eta)^{-b_2-\frac{a}{2}}(1-y\xi)^{-b_2}[1+4x\eta\,(1-y\xi)]^{-\frac{1+a}{2}}d\xi d\eta,$$
$$\mathrm{Re}\,(c)>\mathrm{Re}\,(b_1)>0,\ \mathrm{Re}\,(1-b_2)>\mathrm{Re}\left(\frac{a}{2}\right)>0. \tag{3.5}$$

Again, since

$$\frac{(b_1)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}B(b_1+n,c-b_1)=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}2^{1-n-c}$$
$$\times\int_0^1\left[(1+\xi)^{b_1-1+n}(1-\xi)^{c-b_1-1}+(1+\xi)^{c-b_1-1}(1-\xi)^{b_1-1+n}\right]d\xi, \tag{3.6}$$
$$\mathrm{Re}\,(c)>\mathrm{Re}\,(b_1)>0,$$

we find that

$$H_2^*(a,b_1,b_2;c;x,y)$$
$$=\frac{2^{1-c}\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}\int_0^1(1+\xi)^{b_1-1}(1-\xi)^{c-b_1-1}d\xi\sum_{m=0}^{\infty}\frac{(a)_{2m}(b_2)_{-m}}{m!}x^m\left(1-\frac{y(1+\xi)}{2}\right)^{-b_2+m}$$
$$+\frac{2^{1-c}\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}\int_0^1(1+\xi)^{c-b_1-1}(1-\xi)^{b_1-1}d\xi\sum_{m=0}^{\infty}\frac{(a)_{2m}(b_2)_{-m}}{m!}x^m\left(1-\frac{y(1-\xi)}{2}\right)^{-b_2+m},$$
$$\mathrm{Re}\,(c)>\mathrm{Re}\,(b_1)>0, \tag{3.7}$$

which in view of (3.3) , the result (1.5) and after a little simplification, gives us the integral formula

$$
\begin{gathered}
H_2^*(a,b_1,b_2;c;x,y)=\frac{2^{1+b_2-c}\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}\\
\times\int_0^1(1+\xi)^{b_1-1}(1-\xi)^{c-b_1-1}(2-y-y\xi)^{-b_2}{}_2F_1\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};1-b_2;2(xy+xy\xi-2x)\right)d\xi\\
+\frac{2^{1+b_2-c}\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}\\
\times\int_0^1(1+\xi)^{c-b_1-1}(1-\xi)^{b_1-1}(2-y+y\xi)^{-b_2}{}_2F_1\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};1-b_2;2(xy-xy\xi-2x)\right)d\xi,\\
\mathrm{Re}\,(c)>\mathrm{Re}\,(b_1)>0,\ b_2\neq 0,\pm1,\pm2,...
\end{gathered}\tag{3.8}
$$

Similarly, if we use the expression

$$
\begin{gathered}
\frac{(\beta)_n}{(\delta)_n}=\frac{\Gamma(\delta)}{\Gamma(\beta)\Gamma(\delta-\beta)}B(\beta+n,\delta-\beta),\\
B(x,y)=(1+\lambda)^x\int_0^1\xi^{x-1}(1-\xi)^{y-1}(1+\lambda\xi)^{-x-y}d\xi,\\
\mathrm{Re}(x)>0,\mathrm{Re}\,(y)>0,\lambda>-1,
\end{gathered}\tag{3.9}
$$

one can show that

$$
\begin{gathered}
H_2^*(a,b_1,b_2;c;x,y)=\frac{\Gamma(c)(1+\lambda)^{b_1}}{\Gamma(b_1)\Gamma(c-b_1)}\int_0^1\xi^{b_1-1}(1-\xi)^{c-b_1-1}(1+\lambda\xi)^{b_2-c}\\
\times[1-y\xi+\lambda(1-y)\xi]^{-b_2}{}_2F_1\left(\frac{a}{2},\frac{1+a}{2};1-b_2;-4x\frac{1-y\xi+\lambda(1-y)\xi}{1+\lambda\xi}\right)d\xi,\\
\mathrm{Re}\,(c)>Re\,(b_1)>0,\ \lambda>-1,\ b_2\neq 0,\pm1,\pm2,...
\end{gathered}\tag{3.10}
$$

Moreover, by using the following well-known integral representation for Beta function (see, for example [[18], p. 86, Problem 1]):

$$
\begin{gathered}
B(x,y)=\frac{(b-c)^x(a-c)^y}{(b-a)^{x+y-1}}\int_a^b(b-\xi)^{y-1}(\xi-a)^{x-1}(\xi-c)^{-x-y}d\xi,\\
\mathrm{Re}\,(x)>0,\ \mathrm{Re}\,(y)>0,\ c<a<b,
\end{gathered}\tag{3.11}
$$

it is not difficult to show that

$$
\begin{gathered}
H_2^*(a,b_1,b_2;c;x,y)=\frac{\Gamma(c)(\beta-\gamma)^{b_1}(\alpha-\gamma)^{c-b_1}}{\Gamma(b_1)\Gamma(c-b_1)(\beta-\alpha)^{c-b_2-1}}\int_\alpha^\beta(\beta-\xi)^{c-b_1-1}(\xi-\alpha)^{b_1-1}\\
\times(\xi-\gamma)^{b_2-c}[(\beta-\alpha)(\xi-\gamma)-y(\beta-\gamma)(\xi-\alpha)]^{-b_2}\\
\times{}_2F_1\left(\frac{a}{2},\frac{1+a}{2};1-b_2;-4x\frac{(\beta-\alpha)(\xi-\gamma)-y(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)}\right)d\xi,\\
\mathrm{Re}\,(c)>\mathrm{Re}(b_1)>0,\ \gamma<\alpha<\beta,\ b_2\neq 0;\pm1,\pm2,...
\end{gathered}\tag{3.12}
$$

Further, in view of [[16], p. 26, Equation 49]):

$$B(x,y) = 2\int\limits_0^{\frac{\pi}{2}} \left(\sin^2\xi\right)^{x-\frac{1}{2}} \left(\cos^2\xi\right)^{y-\frac{1}{2}} d\xi, \ \text{Re } x > 0, \ \text{Re } y > 0, \tag{3.13}$$

we obtain

$$H_2^*(a,b_1,b_2;c;x,y) = \frac{2\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)} \int\limits_0^{\frac{\pi}{2}} \left(\sin^2\xi\right)^{b_1-\frac{1}{2}} \left(\cos^2\xi\right)^{c-b_1-\frac{1}{2}}$$
$$\times \left(1-y\sin^2\xi\right)^{-b_2} d\xi \sum_{m=0}^{\infty} \frac{(a)_{2m}(b_2)_{-m}}{m!} \left[x\left(1-y\sin^2\xi\right)\right]^m,$$
$$\text{Re } (b_1) > 0, \ \text{Re } (c-b_1) > 0,$$

which after a little simplification, gives us the result

$$\begin{aligned} H_2^*(a,b_1,b_2;c;x,y) &= \frac{2\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)} \int\limits_0^{\frac{\pi}{2}} \left(\sin^2\xi\right)^{b_1-\frac{1}{2}} \left(\cos^2\xi\right)^{c-b_1-\frac{1}{2}} \\ &\times \left(1-y\sin^2\xi\right)^{-b_2} {}_2F_1\left(\frac{a}{2}, \frac{1+a}{2}; 1-b_2; -4x\left(1-y\sin^2\xi\right)\right) d\xi, \\ &\text{Re } (b_1) > 0, \ \text{Re } (c-b_1) > 0, \ b_2 \neq 0, \pm 1, \pm 2, ... \neq 0, \pm 1, \pm 2, ... \end{aligned} \tag{3.14}$$

Similarly, if we consider the relations

$$\frac{\Gamma(b_1)\Gamma(c-b_1)}{\Gamma(c)} \frac{(b_1)_n}{(c)_n} = B(b_1+n, c-b_1)$$

$$\begin{aligned} &= 2(1+\lambda)^{b_1+n} \int\limits_0^{\frac{\pi}{2}} \frac{\left(\sin^2\xi\right)^{b_1+n-\frac{1}{2}} \left(\cos^2\xi\right)^{c-b_1-\frac{1}{2}}}{\left(1+\lambda\sin^2\xi\right)^{n+c}} d\xi, \\ &\text{Re } (b_1+n) > 0, \ \text{Re } (c-b_1) > 0, \ \lambda > -1, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &= 2\lambda^{b_1+n} \int\limits_0^{\frac{\pi}{2}} \frac{\left(\sin^2\xi\right)^{b_1+n-\frac{1}{2}} \left(\cos^2\xi\right)^{c-b_1-\frac{1}{2}}}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^{n+c}} d\xi, \\ &\text{Re } (b_1+n) > 0, \ \text{Re } (c-b_1) > 0, \ \lambda > 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &= 2\alpha^{2b_1+2n}\beta^{2c-2b_1} \int\limits_0^{\frac{\pi}{2}} \sin^{2b_1+2n-1}\varphi \cos^{2c-2b_1-1}\varphi \left[\alpha^2\sin^2\varphi + \beta^2\cos^2\varphi\right]^{-n-c} d\varphi, \\ &\text{Re } (c) > Re\,(b_1) > 0, \ \alpha > 0, \ \beta > 0, \end{aligned} \tag{3.17}$$

we can easily derive the following integral representations

$$
\begin{aligned}
&H_2^*(a,b_1,b_2;c;x,y)=\frac{2\Gamma(c)\,(1+\lambda)^{b_1}}{\Gamma(b_1)\,\Gamma(c-b_1)}\int\limits_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{b_1-\frac{1}{2}}\left(\cos^2\xi\right)^{c-b_1-\frac{1}{2}}}{\left(1+\lambda\sin^2\xi\right)^{c-b_2}}\\
&\times\left[1+\lambda\sin^2\xi-y\,(1+\lambda)\sin^2\xi\right]^{-b_2}\\
&\times{}_2F_1\left(\frac{a}{2},\frac{1+a}{2};1-b_2;-4x\frac{1+\lambda\sin^2\xi-y\,(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi}\right)d\xi,\\
&\mathrm{Re}\,(b_1)>0,\ \mathrm{Re}\,(c-b_1)>0,\ \lambda>-1,
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&H_2^*(a,b_1,b_2;c;x,y)=\frac{2\Gamma(c)\,\lambda^{b_1}}{\Gamma(b_1)\,\Gamma(c-b_1)}\\
&\times\int\limits_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{b_1-\frac{1}{2}}\left(\cos^2\xi\right)^{c-b_1-\frac{1}{2}}}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^{c-b_2}}\left(\cos^2\xi+\lambda\sin^2\xi-y\lambda\sin^2\xi\right)^{-b_2}\\
&\times{}_2F_1\left(\frac{a}{2},\frac{1+a}{2};1-b_2;-4x\frac{\cos^2\xi+\lambda\sin^2\xi-y\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi}\right)d\xi,\\
&\mathrm{Re}\,(b_1)>0,\ \mathrm{Re}\,(c-b_1)>0,\ \lambda>0,
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&H_2^*(a,b_1,b_2;c;x,y)=\frac{2\alpha^{2b_1}\beta^{2c-2b_1}\Gamma(c)}{\Gamma(b_1)\,\Gamma(c-b_1)}\int\limits_0^{\frac{\pi}{2}}\sin^{2b_1-1}\varphi\cos^{2c-2b_1-1}\varphi\\
&\times\left[\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi\right]^{b_2-c}\left[(1-y)\,\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi\right]^{-b_2}\\
&\times{}_2F_1\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};1-b_2;-4x\frac{(1-y)\,\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi}{\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi}\right)d\varphi,\\
&\mathrm{Re}\,(c)>\mathrm{Re}\,(b_1)>0,\ \alpha>0,\ \beta>0,\ b_2\neq 0,\pm1,\pm2,...
\end{aligned}
\tag{3.20}
$$

respectively.
Finally, in view of the well-known integral representation for Gamma function [8]:

$$
\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt,\ \ \mathrm{Re}(z)>0,
\tag{3.21}
$$

we have

$$
(\alpha)_{2m}=\frac{\Gamma(\alpha+2m)}{\Gamma(\alpha)}=\frac{1}{\Gamma(\alpha)}\int\limits_0^{\infty}e^{-\xi}\xi^{\alpha+2m-1}d\xi,\mathrm{Re}(\alpha)>0.
\tag{3.22}
$$

Thus for the Horn-type function H_2^*, we can show that

$$
H_2^*(a,b_1,b_2;c;x,y)=\frac{1}{\Gamma(a)}\int\limits_0^{\infty}e^{-\xi}\xi^{a-1}H_4\left(b_2,b_1;c;y,x\xi^2\right)d\xi,\mathrm{Re}(a)>0,
\tag{3.23}
$$

where

$$
H_4(\gamma,\beta;\delta;x,y)=\sum_{m,n=0}^{\infty}\frac{(\gamma)_{m-n}(\beta)_m}{(\delta)_m m!n!}x^m y^n,
$$

is Horn confluent series [[8], section 5.7.1]. Similarly, by considering the Horn confluent series

$$H_5(\alpha;\beta;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}}{(\beta)_m m!n!}x^m y^n,$$

we can easily derive the following integral representation

$$H_2^*(a,b_1,b_2;c;x,y)=\frac{1}{\Gamma(a)\Gamma(b_1)}\int_0^{\infty}\int_0^{\infty}e^{-\xi}e^{-\eta}\xi^{a-1}\eta^{b_1-1}H_5\left(b_2;c;y\eta,x\xi^2\right)d\xi d\eta, \tag{3.24}$$

$\mathrm{Re}\,(a)>0, \mathrm{Re}\,(b_1)>0.$

4. Expansion, analytic continuation and transformation formulas

First, in [4] the authors introduced the following inverse pair of symbolic operators:

$$H_x(a,c)=\sum_{i=0}^{\infty}\frac{(c-a)_i(-\delta_x)_i}{(c)_i i!},\ \bar{H}_x(a,c)=\sum_{i=0}^{\infty}\frac{(c-a)_i(-\delta_x)_i}{(1-a-\delta_x)_i i!}, \tag{4.1}$$

where $\delta_x=x\frac{\partial}{\partial x}$. Then

$$H_2^*(a,b_1,b_2;c;x,y)=\sum_{i=0}^{\infty}\frac{(c-b_1)_i(b_2)_i}{(c)_i i!}(-1)^i y^i\sum_{m,p=0}^{\infty}\frac{(a)_{2m}(b_2+i)_{n-m}}{m!n!}x^m y^n,$$

by using (3.3) and (1.5), gives us the expansion formula:

$$\begin{aligned}&H_2^*(a,b_1,b_2;c;x,y)\\&=(1-y)^{-b_2}\sum_{i=0}^{\infty}\frac{(c-b_1)_i(b_2)_i}{(c)_i i!}\left(\frac{y}{y-1}\right)^i{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};1-b_2-i;-4x(1-y)\right).\end{aligned} \tag{4.2}$$

On other hand, we have

$$(1-y)^{-c}{}_2F_1\left(\frac{a}{2},\frac{a}{2}+\frac{1}{2};1-b_2;-4x(1-y)\right)=\bar{H}(c,b_1)H_2^*(a,b_1,b_2;c;x,y).$$

Therefore, from the second operator in (4.1), we get

$$\begin{aligned}&(1-y)^{-c}{}_2F_1\left(\frac{a}{2},\frac{a}{2}+\frac{1}{2};1-b_2;-4x(1-y)\right)\\&=\sum_{i=0}^{\infty}\frac{(-1)^i(c-b_1)_i(b_1)_i(b_2)_i}{(1-b_1)_i(c)_i i!}y^i H_2^*(a,b_1,b_2+i;c+i;x,y).\end{aligned} \tag{4.3}$$

Secondly, starting from the series

$$H_2^*(2c-2b_2,b_1,b_2;c;x,y)=\sum_{m=0}^{\infty}\frac{(2c-2b_2)_{2m}(b_2)_{-m}}{m!}x^m{}_2F_1(b_1,b_2-m;c;y), \tag{4.4}$$

taking to account that the Gaussian hypergeometric function ${}_2F_1$ has the relation [[8], p. 108-109]

$$\begin{aligned}&F(a,b;c;x)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-x)^{-a}{}_2F_1\left[a,c-b;a-b+1;(1-x)^{-1}\right]\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-x)^{-b}{}_2F_1\left[b,c-a;b-a+1;(1-x)^{-1}\right],|\arg(1-x)|<\pi,\end{aligned} \tag{4.5}$$

and by employing (3.3) and (1.5), we obtain by means the routine calculations

$$\begin{aligned}
&H_2^*(2c-2b_2,b_1,b_2;c;x,y)\\
&=\frac{\Gamma(c)\Gamma(b_2-b_1)}{\Gamma(b_2)\Gamma(c-b_1)}(1-y)^{-b_1}\sum_{m,n=0}^{\infty}\frac{(c-b_2)_{m+n}\left(\frac{1}{2}+c-b_2\right)_m(b_1)_n}{(1-b_2+b_1)_{m+n}m!n!}(-4x)^m\left(\frac{1}{1-y}\right)^n\\
&+\frac{\Gamma(c)\Gamma(b_1-b_2)}{\Gamma(b_1)\Gamma(c-b_2)}(1-y)^{-b_2}\sum_{m,n=0}^{\infty}\frac{\left(\frac{1}{2}+c-b_2\right)_m(c-b_1)_n(b_2)_{n-m}}{(1-b_1+b_2)_{n-m}m!n!}(-4x)^m(1-y)^m\left(\frac{1}{1-y}\right)^n.
\end{aligned}\tag{4.6}$$

Now, if we consider the definitions of the Appell's function F_1 and the Horn function G_2 then from relation (4.6) (See [8]), we infer the following

$$\begin{aligned}
&H_2^*(2c-2b_2,b_1,b_2;c;x,y)\\
&=\frac{\Gamma(c)\Gamma(b_2-b_1)}{\Gamma(b_2)\Gamma(c-b_1)}(1-y)^{-b_1}F_1\left(c-b_2;\frac{1}{2}+c-b_2,b_1;1-b_2+b_1;-4x,\frac{1}{1-y}\right)\\
&+\frac{\Gamma(c)\Gamma(b_1-b_2)}{\Gamma(b_1)\Gamma(c-b_2)}(1-y)^{-b_2}G_2\left(\frac{1}{2}+c-b_2,c-b_1,b_2,b_1-b_2;4x(1-y),-\frac{1}{1-y}\right).
\end{aligned}\tag{4.7}$$

Alternatively, starting from (4.4), employing the relation

$$\begin{aligned}
&{}_2F_1(a,b;c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}x^{-a}{}_2F_1\left[a,a-c+1;a+b-c+1;1-x^{-1}\right]\\
&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}x^{a-c}(1-x)^{c-a-b}{}_2F_1\left[c-a,1-a;c-a-b+1;1-x^{-1}\right],\\
&|\arg x|<\pi,
\end{aligned}\tag{4.8}$$

and following the method of the derivation of equation (4.6), we can establish the analytic continuation formula:

$$\begin{aligned}
&H_2^*(2c-2b_2,b_1,b_2;c;x,y)=\frac{\Gamma(c)\Gamma(c-b_1-b_2)}{\Gamma(c-b_1)\Gamma(c-b_2)}y^{-b_1}\\
&\times H_2\left(c-b_1-b_2;\frac{1}{2}+c-b_2,b_1,1+b_1-c;1-b_2;-4x,\frac{1-y}{y}\right)+\\
&+\frac{\Gamma(c)\Gamma(b_1+b_2-c)}{\Gamma(b_1)\Gamma(b_2)}y^{b_1-c}(1-y)^{c-b_1-b_2}\\
&\times F_3\left(c-b_2,c-b_1,c-b_2+\tfrac{1}{2},1-b_1;1-b_1-b_2+c;-4x(1-y),1-\frac{1}{y}\right),
\end{aligned}\tag{4.9}$$

where F_3 is Appell's double series defined by

$$F_3(a_1,b_1,a_2,b_2;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(b_1)_n(a_2)_m(b_2)_n}{(c)_{m+n}m!n!}x^my^n.$$

Next, if in (3.4) let $a=2-2b_2$ and by using the identity

$$1+4x(1-y\xi)=1+4x-4xy\xi=(1+4x)\left(1-\frac{4xy\xi}{1+4x}\right),$$

we get

$$H_2^*(2-2b_2,b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(b_1)\Gamma(c-b_1)}$$
$$\times(1+4x)^{-\frac{3-2b_2}{2}}\int_0^1\xi^{b_1-1}(1-\xi)^{c-b_1-1}(1-y\xi)^{-b_2}\left(1-\frac{4xy\xi}{1+4x}\right)^{-\frac{3-2b_2}{2}}d\xi. \tag{4.10}$$

Now, if we use (3.2) then we will obtain a transformation relation in the form

$$H_2^*(2-2b_2,b_1,b_2;c;x,y)=(1+4x)^{\frac{2b_2-3}{2}}F_1\left(b_1;b_2,\frac{3-2b_2}{2};c;y,\frac{4xy}{1+4x}\right). \tag{4.11}$$

Further, if we start from (4.2), use (1.1) and simplify, we obtain

$$H_2^*(a,b_1,b_2;c;x,y)=(1-y)^{-b_2}\sum_{i,j=0}^{\infty}\frac{(c-b_1)_i(b_2)_i(a)_{2j}}{(c)_i(1-b_2-i)_j i!j!}\left(\frac{y}{y-1}\right)^i[-x(1-y)]^j$$

which on making use the identity $(1-b-i)_j=\frac{(-1)^j(b)_i}{(b)_{i-j}}$, gives us

$$H_2^*(a,b_1,b_2;c;x,y)=(1-y)^{-b_1}H_2^*\left(a,c-b_1,b_2;c;x(1-y),\frac{y}{y-1}\right). \tag{4.12}$$

5. Generating relations via operational identities

The principle of the operational techniques provides a powerful and flexible means to deal with hypergeometric functions of one, two and multiple variables. In fact, an appropriate combination of methods, relevant to operational calculus and to special functions, can be a very useful tool to establish and treat operational identities for hypergeometric functions. In this regard, the following two formulas are the well-known consequences of the derivative operator $\hat{D}_x$ and the integral operator $\hat{D}_x^{-1}$ [15]:

$$\hat{D}_x^n\, x^\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\,x^{\alpha-n}, \tag{5.1}$$

$$\hat{D}_x^{-n}\, x^\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\,x^{\alpha+n}, \tag{5.2}$$

where $\alpha\in\mathbb{C}\backslash\{-1,-2,...\}, n\in\mathbb{N}\cup\{0\}$. Based on the operational relations (5.1) and (5.2), we first prove the following Lemma.

Lemma 1. Let $\mathrm{Re}(a)>0$, $\mathrm{Re}(b_1)>0$, $\mathrm{Re}(b_2)>0$, $\mathrm{Re}(c)>0$ and , $\mathrm{Re}(e)>0$, then

$$H_2^*(a,b_1,b_2;c;x,y)\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}$$
$$=\left(1-yt^{-1}\hat{D}_t^{-1}\hat{D}_y y\right)^{-b_1}\times\exp\left[x\hat{D}_x^2x^2\hat{D}_y^{-1}y^{-1}\right]\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}, \tag{5.3}$$

$$H_2^*(a,b_1,b_2;c;x,y)\left\{x^{e-1}y^{b_2-1}t^{b_2-1}\right\}$$
$$={}_2F_1\left(\tfrac{a}{2},\tfrac{a}{2}+\tfrac{1}{2};e;4x\hat{D}_x x\,\hat{D}_t^{-1}t^{-1}\right)\times{}_2F_1\left(b_1,b_2;c;\hat{D}_y^{-1}\hat{D}_t t\right)\left\{x^{e-1}y^{b_2-1}t^{b_2-1}\right\}. \tag{5.4}$$

Proof. Denote, for convenience, the right-hand side of assertion (5.3) by I. Then as a consequence of the binomial theorem and the exponential function e^x, it is easily seen that:

$$I=\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\tfrac{(b_1)_s}{s!r!}\,y^st^{-s}x^r\,\left(\hat{D}_t^{-s}t^{c-1}\right)\left(\hat{D}_x^{2r}x^{a+2r-1}\right)\left(\hat{D}_y^{s-r}y^{b_2+s-r-1}\right). \tag{5.5}$$

Upon using (5.1), (5.2) and considering the definition (1.4), we are led finally to the left-hand side of the assertion (5.3). The proof of the operational representation (5.4) runs parallel to that of (5.3) then we skip the details.
Now, we will explore the formal properties of the operational identities (5.3) and (5.4) to derive some generating functions for H_2^*. First of all, in the identity (5.3) put $b_1=-m,\ m\in\mathbb{N}\cup\{0\}$, multiply throughout by $t^m/m!$ and then sum to get the generating relation

$$\begin{aligned}&\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}\sum_{m=0}^{\infty}H_2^*\left(a,-m,b_2;c;x,y\right)\frac{u^m}{m!}\\&=\exp\left[\left(1-yt^{-1}\hat{D}_t^{-1}\hat{D}_yy\right)u\right]\times\exp\left[x\hat{D}_x^2x^2\hat{D}_y^{-1}y^{-1}\right]\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}.\end{aligned} \tag{5.6}$$

In the same manner, from the operational identity in (5.3) one can derive the following generating function

$$\begin{aligned}&\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}\sum_{m=0}^{\infty}H_2^*\left(a,-m,b_2;c;x,y\right)u^m\\&=\left(1-u+yt^{-1}\hat{D}_t^{-1}\hat{D}_yyu\right)^{-1}\times\exp\left[x\hat{D}_x^2x^2\hat{D}_y^{-1}y^{-1}\right]\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}.\end{aligned} \tag{5.7}$$

Further, to obtain more generating functions our starting point will be some of the bilinear generating functions presented in [[18], Chapter 5]. First, let us consider the bilinear generating function [[18], p. 308 (ii)]

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^r{}_2F_1\left(-r,\alpha;\beta;x\right){}_2F_1\left(-n+r,\gamma;\delta;y\right)\\&=\frac{(\alpha)_n}{(\beta)_n}x^n{}_3F_2\left(-n,1-\beta-n,\gamma;1-\alpha-n,\delta;\tfrac{y}{x}\right).\end{aligned} \tag{5.8}$$

If in (5.4), we put $a=-2r$ and $b_1=-n+r,\ \{r,n\}\in\mathbb{N}\cup\{0\}$, multiply throughout by $\binom{n}{r}(-1)^r$, take the sum of both sides and then compare the resulting equation with the generating function (5.8), we obtain

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^rH_2^*\left(-2r,r-n,b_2;c;x,y\right)\\&=\frac{(\frac{1}{2}-r)_n}{(1-b_2)_n}(-4x)^n{}_3F_2\left(-n,\tfrac{b_2-n}{2},\tfrac{b_2-n}{2}+\tfrac{1}{2};e-n+\tfrac{1}{2},c;\tfrac{-y}{x}\right).\end{aligned} \tag{5.9}$$

Similarly, by considering the following bilinear generating functions [[18], p. 299(5.3) (11)]

$$\begin{aligned}&\sum_{r=0}^{\infty}\tfrac{(\lambda)_n}{n!}{}_2F_1\left(\rho-n,\alpha;\beta;x\right){}_2F_1\left(\lambda+n,\gamma;\delta;y\right)v^n\\&=(1-v)^{-\lambda}F_M\left(\gamma,\alpha,\alpha,\lambda,\rho,\lambda;\delta,\beta,\beta;\tfrac{y}{1-v},x,\tfrac{xv}{v-1}\right).\end{aligned} \tag{5.10}$$

and [[18], p. 294(5.3) (1)]

$$\sum_{r=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\lambda+n,\alpha;\beta;x)\, {}_2F_1(-n,\gamma;\delta;y)\, v^n = (1-v)^{-\lambda} F_2\left(\lambda,\alpha,\gamma;\beta,\delta;\tfrac{x}{1-v},-\tfrac{yt}{1-v}\right). \tag{5.11}$$

we can establish the results

$$\sum_{r=0}^{\infty} \frac{(\lambda)_n}{n!} H_2^*(\rho-n,\lambda+n,b_2;c;x,y)\, v^n$$

$$= (1-v)^{-\lambda} F_{11d}^{(3)}\left(\tfrac{1}{2}(\rho-n+1),\lambda,\tfrac{1}{2}\rho,b_2;c;4x,\tfrac{4xv}{v-1},\tfrac{y}{1-v}\right). \tag{5.12}$$

where (see[[17], p. 77, Series (11d)])

$$F_{11d}^{(3)}(a_1,a_2,a_3,b;c;;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m (b)_{p-m-n}}{(c)_p} \frac{x^m y^n z^p}{m!n!p!}, \tag{5.13}$$

and

$$\begin{aligned}&\sum_{r=0}^{\infty} \frac{(\lambda)_n}{n!} H_2^*(2\lambda+2n,-n,b_2;c;x,y)\ v^n \\ &= (1-v)^{-\lambda} H_1\left(b_2,\lambda,\lambda+n+\tfrac{1}{2};c;\tfrac{-vy}{1-v},\tfrac{4x}{1-v}\right),\end{aligned} \tag{5.14}$$

where

$$H_1(a,b,c;e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m + n\,(c)_n}{(e)_m} \frac{x^m y^n}{m!n!}, \tag{5.15}$$

is the Horn series defined in [[17], p. 24, (9)], respectively. Finally, it is important to note that the operational representations (5.3) and (5.4) can, in turn, be used to state other needed properties of hypergeometric series H_2^*. For instance, if we let

$$\hat{M} = \left(1 - yt^{-1}\hat{D}_t^{-1}\hat{D}_y y\right)$$

then from relation (5.3), we can state that

$$\left[1-\hat{M}\right]^n \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1} y^{b_2-1} t^{c-1}\right\}$$

$$= \sum_{s=0}^{\infty} \frac{(-n)_s}{s!} \quad \left(1 - yt^{-1}\hat{D}_t^{-1}\hat{D}_y y\right)^s \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1} y^{b_2-1} t^{c-1}\right\}$$

which in view of (5.3), gives us the result

$$\left[1-\hat{M}\right]^n \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1} y^{b_2-1} t^{c-1}\right\}$$

$$= \sum_{s=0}^{n} \binom{n}{s} (-1)^s H_2^*(a,-s,b_2;c;x,y.) \tag{5.16}$$

On other hand, we have

$$\left[1-\hat{M}\right]^n \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1} y^{b_2-1} t^{c-1}\right\}$$

$$= \left[1-\left(1 - yt^{-1}\hat{D}_t^{-1}\hat{D}_y y\right)\right]^n \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1} y^{b_2-1} t^{c-1}\right\}$$

$$= \left(yt^{-1}\hat{D}_t^{-1}\hat{D}_y y\right)^n \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}$$
$$= \sum_{m=0}^{\infty}\frac{x^m y^n t^{-n}}{m!}\left(\hat{D}_t^{-n}t^{c-1}\right)\left(\hat{D}_y^{n-m}y^{b_2+n-m-1}\right)\left(\hat{D}_x^{2m}x^{a+2m-1}\right)$$

which in view of the formulas (5.1) and (5.2)and considering the definition of Gaussian hypergeomtric function ${}_2F_1$, we get

$$\left[1-\hat{M}\right]^n \times \exp\left[x\hat{D}_x^2 x^2 \hat{D}_y^{-1} y^{-1}\right]\left\{x^{a-1}y^{b_2-1}t^{c-1}\right\}$$

$$= \frac{y^n (b_2)_n}{(c)_n}{}_2F_1\left(\tfrac{a}{2}, \tfrac{a}{2}+\tfrac{1}{2}; 1-b_2-n; -4x\right),\ n = 0,1,2,... \tag{5.17}$$

Hence from (5.16) and (5.17), we obtain the following interesting summation formula:

$$\sum_{s=0}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)(-1)^s H_2^*\left(a,-s,b_2;c;x,y\right)$$

$$= \frac{y^n (b_2)_n}{(c)_n}{}_2F_1\left(\tfrac{a}{2}, \tfrac{a}{2}+\tfrac{1}{2}; 1-b_2-n; -4x\right),\ \{n = 0,1,2,...\}. \tag{5.18}$$

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НЕКОТОРЫЕ СВОЙСТВА ГИПЕРГЕОМЕТРИЧЕСКОГО РЯДА ГОРНА ВТОРОГО ПОРЯДКА

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В работах Горн [1931, Hypergeometrische Funktionen zweier Veranderlichen, Math. Ann., 105 (1), 381-407], (исправления в книге Борнгассера [1933, Uber hypergeometrische funkionen zweier Veranderlichen, Диссертация, Дармштадт]) были определены и исследованы десять гипергеометрических рядов двух переменных второго порядка. Исследуя ряды Горна, мы заметили существование гипергеометрических двойных рядов H_2^*, аналогичных двойному ряду Горна H_2. Основная цель настоящей статьи это представить дальнейшие шаги исследования математических свойств и представлений, относительно аналогичных гипергеометрических двойных рядов H_2. Действительно, воодушевленные важной ролью функций Горна в нескольких разнообразных областях физики и вкладом в унификацию и обобщение гипергеометрических функций, мы составляем систему уравнений в частных производных, интегральные представления, формул разложения, аналитическое продолжение, формулы преобразования. А также обсуждаются связи результатов, представленные в этой статье с уже известными.

Ключевые слова: гипергеометрическая функция Гаусса, двойные ряды Хорна, уравнения в частных производных, интегральные представления, преобразование, производящие функции