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ON SOME SHARP THEOREMS ON DISTANCE FUNCTION IN HARDY TYPE, BERGMAN TYPE AND HERZ TYPE ANALYTIC CLASSES

R. F. Shamoyan, S. P. Maksakov

Bryansk State University, 241036, Bryansk, Bezhitskaya st., 14, RussiaIran University of Science and Technology.

E-mail: rsham@mail.ru, msp222@mail.ru

We present some new sharp estimates concerning distance function in some new mixed norm and Lizorkin-Triebel type spaces in the unit ball.This leads at the same time to direct generalizations of our recent results on extremal problems in such Bergman type spaces. In addition new sharp results in this direction in Hardy-Morrey, and some new weighted Hardy classes in the unit ball and mixed norm Hardy type spaces and Herz type spaces in polyball will be provided and discussed.

Key words: extremal problems, analytic functions, Bergman type spaces, unit ball, Hardy type spaces, Herz type spaces, Mockenhoupt type weights, Hardy-Morrey type spaces.

УДК 517.53+517.55

О НЕКОТОРЫХ ТОЧНЫХ ТЕОРЕМАХ, СВЯЗАННЫХ С ФУНКЦИЕЙ ДИСТАНЦИИ В ПРОСТРАНСТВАХ ТИПА ХАРДИ, БЕРГМАНА И ГЕРЦА

Р. Ф. Шамоян, С. П. Максаков

Брянский государственный университет, 241036, Брянск, ул. Бежицкая., 14, Россия

E-mail: rsham@mail.ru, msp222@mail.ru

Мы приводим новые точные результаты о дистанциях в новых классах со смешанной нормой и в пространствах типа Лизоркина-Трибелиа в единичном шаре. Тем самым, в частности, обощены недавние точные результаты, полученные первым автором в пространствах типа Бергмана. Новые результаты такого типа получены для нескольких первых шкал пространств типа Харди, в частности, для пространств типа Харди-Морреи в единичном шаре. Так же, рассмотрены новые шкалы пространств типа Харди на декартовом произведении шаров со смешанной нормой и получены новые теоремы подобного рода. Введено несколько новых шкал пространства типа Герца в шаре на произведении шаров и доказаны точние рездьтаты подобного типа для них.

Ключевые слова: экстремальные задачи, аналитические функции, пространства типа Бергмана, единичный шар, пространства типа Харди, пространства типа Герца, веса типа Макенхаупта,пространства типа Харди-Морреи

Introduction

In this paper we continue our recent intensive research on extremal problems in analytic function spaces. The intention of this note is to extend our previous results on extremal problems in the unit ball and polyball from [15] and [16]. Namely, we study the mixed norm spaces of analytic fnctions of Bergman type in the unit ball and polyball and we provide some direct generalization of our previous results. Then in addition we provide similar extensions for some new Hardy type spaces and Herz type spaces in the unit ball. As a base of proofs we put new uniform estimates for Hardy-Morrey and weighted Hardy spaces and new projection theorems in some new Bergman type spaces obtained recently (see [1] - [3], [11], [22]).

We alert the reader that we provide some of these results without proofs, since proofs from our point of view can be recovered easily by reader from a detailed remarks which will be provided .The base of all proofs is well-known Forelli-Rudin type estimate for Bergman kernel and Bergman representation formula and Bergman type projection theorem together with chain of transparent arguments combined with classical inequalities of function theory. We shortly remind the history of this problem to readers. After the appearance of [6] various papers appeared where arguments which can be seen in [6] were extended and modified in various directions (see [12], [14] [15], [16]).

In particular in mentioned papers various new results on distances for analytic function spaces in higher dimension (unit ball and polydisk) were obtained. Namely new results for large scales of analytic mixed norm spaces in higher dimension were proved.

Later, several new sharp results for harmonic functions of several variables in the unit ball and upperhalfplane of Euclidean space were also obtained (see, for, example, [1] and references there).

We mention separately [14] and [9], [13] where the case of higher dimension was considered in harmonic and analytic spaces and new similar results in domains with smooth boundary were also provided.

The motivation of this problem related with distance function is to find a concrete formula which will help to calculate this function more concretely via the well-known Bergman kernel.

The goal of this note is to develop further some ideas from recent mentioned papers and present new sharp assertions. The plan of this paper is the following.First we present a simple case with complete proof then turn to more complicated function spaces in various domains.

In some cases additional conditions in formulation of our theorems will be posed.We formulate it as a problem to remove these additional conditions. We denote by $C, C_1, C_2, C_\alpha, C_\beta$ various positive constants in this paper.

Notations and preliminaries

For formulations of our main results we need various standard definitions and preliminaries from function theory in the unit ball and polydisk. All material concerning definitions in the unit ball and polydisk can be seen in [4], [5], [10], [17], [21].

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let

$$\mathbb{C}^n = \{z = (z_1, \dots, z_n) : z_k \in \mathbb{C}, 1 \leq k \leq n\}$$

be the n-dimensional space of complex coordinates.

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbf{B} = \{z \in \mathbb{C}^n \mid |z| < 1\}.$$

The boundary of $\mathbf{B}$ will be denoted by $\mathbf{S}$,

$$\mathbf{S} = \{z \in \mathbb{C}^n \mid |z| = 1\}.$$

As usual, we denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$.

Let dV denote the Lebesgue measure on unit ball $\mathbf{B}$ normalized such that $V(\mathbf{B}) = 1$ and let $d\sigma$ denote the surface measure on $\mathbf{S}$ normalized such that $\sigma(\mathbf{S}) = 1$. For any real number α, let

$$dV_\alpha(z) = c_\alpha (1 - |z|)^{\alpha - 1} dV(z)$$

for $|z| < 1$. Here, if $\alpha \leq -1$, $c_\alpha = 1$ and if $\alpha > -1$,

$$c_\alpha = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)}$$

is the normalizing constant so that V_α has unit total mass.

For $\alpha > -1$ and $p > 0$ the weighted Bergman space $A^p_\alpha(\mathbf{B})$ consists of holomorphic functions f in $L^p(\mathbf{B}, dV_\alpha)$, that is,

$$A^p_\alpha(\mathbf{B}) = L^p(B, dV_\alpha) \cap H(\mathbf{B}),$$

$$A^p_\nu(\mathbf{B}) = \{f \in H(\mathbf{B}) : \int_{\mathbf{B}} |f(z)|^p (1 - |z|)^{\nu - 1} dV(z) < \infty\}, 0 < p < \infty, \nu > 0.$$

See [12] for more details of weighted Bergman spaces.

Let χ_G be below a characteristic function of the G set, as usual.

We denote the unit polydisk by

$$\mathbf{U}^n = \{z \in \mathbb{C}^n : |z_k| < 1, 1 \leq k \leq n\}$$

and the distinguished boundary $\mathbf{U}^n$ of by

$$\mathbf{T}^n = \{z \in \mathbb{C}^n : |z_k| = 1, 1 \leq k \leq n\}.$$

By m_{2n} we denote the volume measure on U^n and by m_n we denote the normalized Lebesgue measure on $\mathbf{T}^n$. Let $H(\mathbf{U}^n)$ be the space of all holomorphic functions on $\mathbf{U}^n$.

The Hardy spaces, denoted by $H^p(\mathbf{U}^n)$ $(0 < p \leq \infty)$, are defined by

$$H^p(\mathbf{U}^n) = \{f \in H(\mathbf{U}^n) : \sup_{0 \leq r < 1} M_p(f, r) < \infty\},$$

where

$$(M_p(f, r))^p = \int_{\mathbf{T}^n} |f(r\xi)|^p dm_n(\xi), r \in (0, 1), \tag{1}$$

$$M_\infty(f,r) = \max_{\xi \in \mathbf{T}^n} |f(r\xi)|, r \in (0,1), f \in H(\mathbf{U}^n), r \in (0,1), \tag{2}$$

$$(M_p(f,r))^p = \int_{\mathbf{S}} |f(r\xi)|^p d\sigma(\xi); 0 < p \leq \infty, r \in (0,1). \tag{3}$$

As usual, we denote by $\overrightarrow{\alpha}$ the vector $(\alpha_1, \ldots, \alpha_n)$. Let also $\vec{r} = (r_1, ..., r_n), r_j \in (0,1), j = 1, ..., n$, $\vec{z} = (z_1, ..., z_m), z_j \in \mathbb{C}, j = 1, ..., m$.

We define weighted Hardy space in the unit ball as follows

$$A_s^{\infty,\delta}(\mathbf{B}) = \{f \in H(B) : \sup_{r<1} (\int_{\mathbf{S}} |f(r\xi)|^\delta d\sigma(\xi))^{\frac{1}{\delta}} (1-r)^s < \infty\}, \tag{4}$$

where $s \geq 0$, $\delta > 0$.

Now we define weighted Hardy space in the polydisk

$$A_s^{\infty,\delta}(\mathbf{U}^n) = \{f \in H(\mathbf{U}^n) : \sup_{r_1<1,\ldots,r_n<1} (\int_{\mathbf{T}^n} |f(\overrightarrow{r}\xi)|^\delta dm_n(\xi))^{\frac{1}{\delta}} \prod_{k=1}^n (1-r_k)^s < \infty\}, \tag{5}$$

where $s > 0$, $\delta > 0$..

For $\alpha_j > 0$, $j = 1, \ldots, n$, $0 < p \leq \infty$, recall that the weighted Bergman space $A^p_{\overrightarrow{\alpha}}(\mathbf{U}^n)$ consists of all holomorphic functions on the polydisk such that

$$\|f\|^p_{A^p_{\overrightarrow{\alpha}}} = \int_{U^n} |f(z)|^p \prod_{i=1}^n (1-|z_i|^2)^{\alpha_i - 1} dm_{2n}(z) < \infty. \tag{6}$$

For $\alpha > 0$, let

$$\widetilde{A}^\infty_\alpha(\mathbf{B}) = \{f \in H(\mathbf{B}) : \sup_{r<1} (\int_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)|)(1-r)^\alpha < \infty\}. \tag{7}$$

$$\widetilde{A}^\infty_\alpha(\mathbf{U}^n) = A^{\infty,1}_\alpha \tag{8}$$

Extremal problem in A^1_α Bergman space in the unit ball and polydisk

This section provides the simple model case for our further work. Namely, these results serve as model for results of next section where *far* reaching extensions are given based fully on these proofs of this section on the unit ball, polydisk.

In this section, first we deal with simplest case Bergman A^1_α class in the unit ball (see [13] and [15]).

We now define a new subset of the unit interval and then using its characteristic function we will give a sharp assertion concerning distance function.

For $\varepsilon > 0$, $f \in H(\mathbf{B})$, let

$$L_{\varepsilon,\alpha}(f) = \{r \in (0,1) : (1-r)^\alpha \int_S |f(r\xi)||d\sigma(\xi)| \geq \varepsilon\}. \tag{9}$$

Theorem 1. *Let $f \in \widetilde{A}^{\infty}_{\alpha}(\mathbf{B})$, $\alpha > 0$. Then the following are equivalent:*
(a) $s_1 = dist_{\widetilde{A}^{\infty}_{\alpha}(\mathbf{B}^n)}(f, A^1_{\alpha}(\mathbf{B}))$;
(b) $s_2 = inf\{\varepsilon > 0 : \int\limits_0^1 (1-r)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r)dr < \infty\}$.

Proof. First we prove $s_1 \geq s_2$. Let as assume that $s_1 < s_2$. Then we can find two numbers ε, ε_1 such that $\varepsilon > \varepsilon_1 > 0$, and a function

$$f_{\varepsilon_1} \in A^1_{\alpha}(\mathbf{B}), \|f - f_{\varepsilon_1}\|_{\widetilde{A}^{\infty}_{\alpha}(\mathbf{B})} \leq \varepsilon_1,$$

and

$$\int\limits_0^1 (1-r)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r)dr = \infty.$$

Hence we have

$$(1-r)^{\alpha}\int\limits_{\mathbf{S}} |f_{\varepsilon_1}(r\xi)||d\sigma(\xi)| \tag{10}$$

$$\geq (1-r)^{\alpha}\int\limits_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)| - \sup_{r<1}(1-r)^{\alpha}\int\limits_{\mathbf{S}} |f(r\xi) - f_{\varepsilon_1}(r\xi)||d\sigma(\xi)|$$

$$\geq (1-r)^{\alpha}\int\limits_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)| - \varepsilon_1.$$

Hence for any $s \in [-1, \infty)$,

$$(\varepsilon - \varepsilon_1)\int\limits_0^1 (1-r)^s\chi_{L_{\varepsilon,\alpha}(f)}(r)dr \leq C_1\int\limits_0^1 (\int\limits_{\mathbf{S}} |f_{\varepsilon_1}(r\xi)||d\sigma(\xi)|)(1-r)^{\alpha+s}dr. \tag{11}$$

Thus we have a contradiction.

It remains to show $s_1 \leq Cs_2$. Let $I = [0,1)$. We argue as above and obtain from the classical Bergman representation formula (see [19]).

$$f(\rho\zeta) = f(z) = C_2(t)\int\limits_{L_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{S}} \frac{f(r\xi)(1-r)^t}{(1-r\overline{\xi}\rho\zeta)^{t+2}}d\sigma(\xi)dr+$$

$$+C_3(t)\int\limits_{I\backslash L_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{S}} \frac{f(r\xi)(1-r)^t}{(1-r\overline{\xi}\rho\zeta)^{t+2}}d\sigma(\xi)dr = f_1(z) + f_2(z),$$

where t is large enough. Then we have

$$(1-\rho)^{\alpha}\int\limits_{\mathbf{S}} |f_2(\rho\zeta)||d\sigma(\zeta)| \leq C_4(1-\rho)^{\alpha}\int\limits_{\mathbf{S}}\int\limits_{I\backslash L_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{S}} \frac{|f(r\xi)|(1-r)^t}{|1-r\overline{\xi}\rho\zeta|^{t+2}}|d\sigma(\xi)|dr|d\sigma(\zeta)|$$

$$\leq C_5(1-\rho)^{\alpha}\int\limits_{I\backslash L_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{S}} |f(r\xi)|(1-r)^t(\int\limits_{\mathbf{S}} \frac{1}{|1-r\overline{\xi}\rho\zeta|^{t+2}}|d\sigma(\zeta)|)|d\sigma(\xi)|dr \tag{12}$$

$$\leq C_6(1-\rho)^{\alpha}\int\limits_{I\setminus L_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{S}}|f(r\xi)||d\sigma(\xi)|\frac{(1-r)^t}{(1-r\rho)^{t+1}}dr\leq C_7\varepsilon(1-\rho)^{\alpha}\int\limits_0^1\frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}}dr\leq C_8\varepsilon.$$

For $\alpha>0$ we have

$$\int\limits_{\mathbf{B}}(1-\rho)^{\alpha-1}|f_1(\rho\zeta)|dV(\rho\zeta) \tag{13}$$

$$\leq C_9\int\limits_{\mathbf{B}}(1-\rho)^{\alpha-1}\int\limits_{L_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{S}}\frac{|f(r\xi)|(1-r)^t}{|1-r\overline{\xi}\rho\zeta|^{t+2}}|d\sigma(\xi)|drdV(\rho\zeta)$$

$$\leq C_{10}\sup_{r<1}((1-r)^{\alpha}\int\limits_{\mathbf{S}}|f(r\xi)||d\sigma(\xi)|)\int\limits_{L_{\varepsilon,\alpha}(f)}\frac{(1-r)^{t-\alpha}}{(1-r)^{t+1-\alpha}}dr$$

$$\leq C_{11}\sup_{r<1}((1-r)^{\alpha}\int\limits_{\mathbf{S}}|f(r\xi)||d\sigma(\xi)|)\int\limits_{L_{\varepsilon,\alpha}(f)}\frac{1}{(1-r)}dr.$$

Note that the implication

$$\|f_1\|_{A^1_{\alpha}(\mathbf{B})}<\infty$$

follows directly from the known estimate for $\alpha>0$, $f_1\in H(\mathbf{B})$ (see [9])

$$\int\limits_0^1(1-\rho)^{\alpha-1}(\int\limits_{\mathbf{S}}|f_1(\rho\xi)|d\sigma(\xi))d\rho\leq C_{12}\int\limits_{\mathbf{B}}(1-\rho)^{\alpha-1}|f_1(\rho\xi)|dV(\rho\xi). \tag{14}$$

Hence

$$\inf_{g\in A^1_{\alpha}(\mathbf{B})}\|f-g\|_{\widetilde{A^{\infty}_{\alpha}}(\mathbf{B})}\leq C_{13}\|f-f_1\|_{\widetilde{A^{\infty}_{\alpha}}(\mathbf{B})}=\|f_2\|_{\widetilde{A^{\infty}_{\alpha}}(\mathbf{B})}\leq C_{14}\varepsilon.$$

The theorem is proved.

Note very similar sharp theorem with same proof can be obtained for pair $(A^{\infty}_{\widetilde{\alpha}},A^1_{\alpha})$, where

$$A^{\infty}_{\widetilde{\alpha}}=\{f\in H(\mathbf{B}):\sup_{z\in\mathbf{B}}|f(z)|(1-|z|)^{\widetilde{\alpha}}<\infty\},\widetilde{\alpha}=\alpha+n+1;\alpha>-1.$$

This is very vital observation for us.

Now we will consider the analogous of previous theorem for the polydisk case.

For $I^n=[0,1]^n,\varepsilon>0$, $s>0$, $f\in H(\mathbf{U}^n)$, let

$$\widehat{L}_{s,\varepsilon}(f)=\{r\in\mathbf{I}^n:(\int\limits_{\mathbf{T}^n}|f(\overrightarrow{r}\xi)|dm_n(\xi))\prod_{k=1}^n(1-r_k)^s\geq\varepsilon\}. \tag{15}$$

Theorem 2. *Let $f\in\widetilde{A^{\infty}_{\alpha}}(\mathbf{U}^n)$, $\alpha>0$. Then the following are equivalent:*
(a) $\hat{s}_1=dist_{A^{\infty}_{\alpha}(\mathbf{U}^n)}(f,A^1_{\alpha}(\mathbf{U}^n))$;
(b) $\hat{s}_2=inf\{\varepsilon>0:\int\limits_0^1\ldots\int\limits_0^1\chi_{\hat{L}_{\alpha,\varepsilon}}(y)\prod\limits_{j=1}^n(1-y_j)^{-1}dy_1\ldots dy_n<\infty\}$.

Proof. First we prove $\hat{s}_1\geq\hat{s}_2$. Let as assume that $\hat{s}_1<\hat{s}_2$. Then we can find two numbers ε, ε_1 such that $\varepsilon>\varepsilon_1>0$, and a function

$$f_{\varepsilon_1}\in A^1_{\alpha}(\mathbf{U}^n),\|f-f_{\varepsilon_1}\|_{\tilde{A}^{\infty}_{\alpha}(\mathbf{U}^n)}\leq\varepsilon_1,$$

and

$$\int_0^1 \dots \int_0^1 \prod_{k=1}^n (1-r_k)^{-1} \chi_{\hat{L}_{\alpha,\varepsilon}(f)}(r) dr_1 \dots dr_n = \infty.$$

Hence we have

$$\prod_{k=1}^n (1-r_k)^{\alpha} \int_{\mathbf{T}^n} |f_{\varepsilon_1}(\vec{r}\xi)||dm_n(\xi)| \tag{16}$$

$$\geq \prod_{k=1}^n (1-r_k)^{\alpha} \int_{\mathbf{T}^n} |f(\vec{r}\xi)||dm_n(\xi)| - \sup_{r_1<1,\dots,r_n<1} \prod_{k=1}^n (1-r_k)^{\alpha} \int_{\mathbf{T}^n} |f(\vec{r}\xi) - f_{\varepsilon_1}(\vec{r}\xi)||dm_n(\xi)|$$

$$\geq \prod_{k=1}^n (1-r_k)^{\alpha} \int_{\mathbf{T}^n} |f(\vec{r}\xi)||dm_n(\xi)| - \varepsilon_1.$$

Hence for any $s \in [-1,\infty)$,

$$(\varepsilon - \varepsilon_1) \int_0^1 \dots \int_0^1 \prod_{k=1}^n (1-r_k)^s \chi_{\hat{L}_{\alpha,\varepsilon}(f)}(r) dr_1 \dots dr_n \tag{17}$$

$$\leq C_{15} \int_0^1 \dots \int_0^1 \left(\int_{\mathbf{T}^n} |f_{\varepsilon_1}(\vec{r}\xi)||dm_n(\xi)|\right) \prod_{k=1}^n (1-r_k)^{\alpha+s} dr_1 \dots dr_n.$$

Thus we have a contradiction.

It remains to show $\hat{s}_1 \leq C\hat{s}_2$. Let $I = [0,1)$. We argue as above and obtain from the classical Bergman representation formula (see [19]).

$$f(\rho\zeta) = f(z) = C_1(t) \int_{\hat{L}_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}^n} \frac{f(\vec{r}\xi) \prod_{k=1}^n (1-r_k)^t}{(1-r\overline{\xi}\rho\zeta)^{t+2}} dm_n(\xi) dr_1 \dots dr_n$$

$$+C_2(t) \int_{I\backslash\hat{L}_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}^n} \frac{f(\vec{r}\xi) \prod_{k=1}^n (1-r_k)^t}{(1-r\overline{\xi}\rho\zeta)^{t+2}} dm_n(\xi) dr_1 \dots dr_n = f_1(z) + f_2(z),$$

where t is large enough. Then we have

$$(1-\rho)^{\alpha} \int_{\mathbf{T}^n} |f_2(\rho\zeta)||dm_n(\zeta)| \leq C_3 (1-\rho)^{\alpha} \int_{\mathbf{T}^n} \int_{I\backslash\hat{L}_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}^n} \frac{|f(\vec{r}\xi)| \prod_{k=1}^n (1-r_k)^t}{|1-r\overline{\xi}\rho\zeta|^{t+2}} |dm_n(\xi)| dr |dm_n(\zeta)| \tag{18}$$

$$\leq C_4 (1-\rho)^{\alpha} \int_{I\backslash\hat{L}_{\varepsilon,\alpha}(f)} \dots \int_{I\backslash\hat{L}_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}^n} |f(\vec{r}\xi)| \prod_{k=1}^n (1-r_k)^t \left(\int_{\mathbf{T}^n} \frac{1}{|1-r\overline{\xi}\rho\zeta|^{t+2}} |dm_n(\zeta)|\right) |dm_n(\xi)| dr_1 \dots dr_n \tag{19}$$

$$\leq C_5(1-\rho)^{\alpha}\int\limits_{I\backslash\hat{L}_{\varepsilon,\alpha}(f)}\dots\int\limits_{I\backslash\hat{L}_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{T}^n}|f(\vec{r}\xi)||dm_n(\xi)|\frac{\prod\limits_{k=1}^{n}(1-r_k)^t}{(1-\vec{r}\rho)^{t+1}}dr_1\dots dr_n \tag{20}$$

$$\leq C_6\varepsilon(1-\rho)^{\alpha}\int\limits_0^1\dots\int\limits_0^1\frac{\prod\limits_{k=1}^{n}(1-r_k)^{t-\alpha}}{(1-r\rho)^{t+1}}dr_1\dots dr_n\leq C_7\varepsilon.$$

For $\alpha>0$ we have

$$\int\limits_{\mathbf{U}^n}(1-\rho)^{\alpha-1}|f_1(\rho\zeta)|dm_{2n}(\rho\zeta) \tag{21}$$

$$\leq C_8\int\limits_{\mathbf{U}^n}(1-\rho)^{\alpha-1}\int\limits_{\hat{L}_{\varepsilon,\alpha}(f)}\int\limits_{\mathbf{T}^n}\frac{|f(\vec{r}\xi)|\prod\limits_{k=1}^{n}(1-r_k)^t}{|1-r\overline{\xi}\rho\zeta|^{t+2}}|dm_n(\xi)|dr_1\dots dr_n dm_{2n}(\rho\zeta)$$

$$\leq C_9\sup_{r_1<1,\dots,r_n<1}(\prod_{k=1}^{n}(1-r_k)^{\alpha}\int\limits_{\mathbf{T}^n}|f(\vec{r}\xi)||dm_n(\xi)|)\int\limits_{\hat{L}_{\varepsilon,\alpha}(f)}\frac{\prod\limits_{k=1}^{n}(1-r_k)^{t-\alpha}}{\prod\limits_{k=1}^{n}(1-r_k)^{t+1-\alpha}}dr_1\dots dr_n$$

$$\leq C_{10}\sup_{r_1<1,\dots,r_n<1}(\prod_{k=1}^{n}(1-r_k)^{\alpha}\int\limits_{\mathbf{T}^n}|f(\vec{r}\xi)||dm_n(\xi)|)\int\limits_{\hat{L}_{\varepsilon,\alpha}(f)}\frac{1}{\prod\limits_{k=1}^{n}(1-r_k)}dr_1\dots dr_n.$$

Note that the implication

$$\|f_1\|_{A^1_{\alpha}(\mathbf{U}^n)}<\infty$$

follows directly from the known estimate for $\alpha>0,\ f_1\in H(\mathbf{U}^n)$

$$\int\limits_0^1(1-\rho)^{\alpha-1}(\int\limits_{\mathbf{T}^n}|f_1(\rho\xi)dm_n(\xi))d\rho\leq C_{11}\int\limits_{\mathbf{U}^n}(1-\rho)^{\alpha-1}|f_1(\rho\xi)|dm_{2n}(\rho\xi). \tag{22}$$

Hence

$$\inf_{g\in A^1_{\alpha}(\mathbf{U}^n)}\|f-g\|_{\widetilde{A^{\infty}_{\alpha}}(\mathbf{U}^n)}\leq C_{12}\|f-f_1\|_{\widetilde{A^{\infty}_{\alpha}}(\mathbf{U}^n)}=\|f_2\|_{\widetilde{A^{\infty}_{\alpha}}(\mathbf{U}^n)}\leq C_{13}\varepsilon.$$

The theorem is proved.

New results on distances in some function spaces in higher dimension

In this section we show that theorems of previous section can be extended based on recent results from [2], [3], [11] in various directions to some new Bergman and Hardy type function spaces in the unit ball and polyball in particular to so called mixed norm spaces $\mathbf{F}^{p,q}_{\alpha}$ and $\mathbf{A}^{p,q}_{\alpha}$, $0<p,\ q\leq\infty,\ \alpha>0$.

These results also extend some our previous results on distances from [1], [6], [7], [8], [15].

Let

$$(\mathbf{F}^{p,q}_{\alpha})(\mathbf{B}) = \{f \in H(\mathbf{B}) : \int_{\mathbf{S}} (\int_0^1 |f(z)|^q (1-|z|)^{\alpha q - 1} d|z|)^{\frac{p}{q}} d\sigma(\xi) < \infty\},$$

$$(\mathbf{A}^{p,q}_{\alpha})(\mathbf{B}) = \{f \in H(\mathbf{B}) : \int_0^1 [M_p^q(f,r)](1-r)^{\alpha q - 1} dr < \infty\}.$$

We also define new analytic weighted Hardy and Morrey type spaces in context of the unit ball as follows (see [2], [3]). These are weighted Hardy spaces with Muckenhoupt A_p weights.

Let θ be $L^1(S)$ function $\theta(E) = \int_E \theta d\sigma$. Let $z = r\xi$, $\xi \in S$, $r \in (0,1)$ consider

$$\widetilde{\theta}(z) = \frac{\theta(I_z)}{|I_z|}, I_z = I_{r\xi} = \{\eta \in S : |1-\eta\overline{\xi}| < (1-r)\}.$$

Let $1 < p < \infty$

$$A_p = \{\theta \in L^1(d\sigma) : \sup_{z \in B} \widetilde{\theta}(z)^{\frac{1}{p}} \widetilde{\theta}'(z)^{\frac{1}{p'}} < \infty\},$$

$$\theta' = \theta^{-\frac{p'}{p}}, \widetilde{\theta}'(z) = \frac{\theta'(I_z)}{|I_z|}, \frac{1}{p} + \frac{1}{p'} = 1.$$

For $1 < p < \infty$, $\theta \in A_p$ the Hardy space $H^p(\theta)$ is a space of analytic functions on B with norm

$$\|f\|_{H^p(\theta)} = \sup_{r \in (0,1)} (\int_{\mathbf{S}} |f(r\xi)|^p (\theta(\xi)) d\sigma(\xi))^{\frac{1}{p}} < \infty.$$

For $1 < p < \infty$, $s \in (-1, \frac{n}{p}]$ we define Morrey-Campanato space $M^{p,s}$ on S as follows

$$M^{p,s}(S) = \{f \in L^p(S) : \|f\|_{M^{p,s}} < \infty\}$$

where

$$\|f\|_{p,s} = \|f\|_{L^p} + \sup_{I_{\xi,\varepsilon}} (\varepsilon^{sp-n} \int_{I_{\xi,\varepsilon}} |f(\eta) - f(\zeta)|^p d\sigma(\eta))^{\frac{1}{p}}$$

where σ as usual is a Lebeques measure on S

$$I_{\xi,\varepsilon} = \{\eta \in S : |1 - \overline{\xi}\eta| < \varepsilon\}.$$

Note $M^{p,\frac{n}{p}} = L^p(S)$, $M^{p,0} = B^{p,0}$ (non isotropic BMO space), for $-1 < s < 0$, $M^{p,s} = I_s$ (non isotropic Lipschitz space)(see [2], [3]). We define $HM^{p,s} = M^{p,s} \cap H^p$ – as Morrey class.

In this section we define new mixed norm analytic Herz type spaces on product domains discussing some issues related with distance problem on such type spaces. We also define and provide a new sharp distance theorem for BMOA type spaces in the context of the unit ball. New projection theorem in Herz type spaces will be also given which also leads to new sharp results in this direction.

Let also

$$A^{\vec{p}}_{\vec{\alpha}}(B\times\ldots\times B)=\{f\in H(\underbrace{B\times\ldots\times B}_{m}):(\int_B(\int_B\ldots(\int_B|f(\vec{z})|^{p_1}dV_{\alpha_1}(z_1))^{\frac{p_1}{p_2}}\ldots dV_{\alpha_m}(z_m))^{\frac{1}{p_m}}<\infty\},$$

$$p_i\in(0,\infty),\alpha_i>-1,j=1,\ldots,m,dV_\beta(z)=(1-|z|)^\beta dV(z)$$

(see [1]). Let X be one of these spaces.

These are Banach spaces if $\min_j(p_j)\geq 1$ and complete metric spaces for other values of p. Such spaces in $\mathbb{R}^n$ was studied in [18], in polydisk in [22].

All these analytic spaces admit uniform estimates like

$$\sup_{z\in B}|f(z)|(1-|z|)^{\alpha_0}\leq c_0\|f\|_X,$$

or

$$(|f(z_1,\ldots,z_n)|)\prod_{j=1}^n(1-|z_j|)^{\alpha_j^0}\leq\tilde{c}_0(\|f\|_X);$$

for some α_0, α_j^0 for $z_j\in B$, $z\in B$, $j=1,\ldots,m$ for some fixed $\alpha_0=\alpha_0(X)$, $\alpha_j^{(0)}(X)$, $j=1,\ldots,n$ depending on X space and for some positive constants c_0, $\tilde{c}_0$. For $A^{p,q}_\alpha$, $F^{p,q}_\alpha$ this fact can be seen in [2]. For $H^p(\theta)$, $(HM^{p,s})$ spaces this fact can be seen in [3]. For $A^{\vec{p}}_{\vec{\alpha}}$ spaces in [1]. So, we can pose a dist problem since, for example, we have a natural question to estimate

$$dist_{A^\infty_{\alpha_0}}(f,X),\ f\in A^\infty_{\alpha_0}.$$

Next it is well-known for any f, $f\in A^\infty_{\alpha_0}$ the Bergman formula with large enough index is valid.

These facts are crucial for the proof of our theorems. The carefull inspection of the proof of results of previous section shows that only four tools were used: integral representation, Forelli-Rudin estimate, Bergman projection theorem, and uniform estimates.

To show assertion on distance function we need to use the following classical facts first for any f, $f\in A^\infty_\alpha$

$$f(z)=\int_B\frac{f(\omega)(1-|\omega|)^\beta dV(\omega)}{(1-\overline{\omega}z)^{\beta+n+1}},$$

$\beta>-1$ (representation formula), $z\in B$ and that the classical Bergman P_{β_0} projection for large β_0 index maps $\mathbf{A}^{p,q}_\alpha$ into $A^{p,q}_\alpha$ and $\mathbf{F}^{p,q}_\alpha$ into $F^{p,q}_\alpha$ (see [2], [11]).

The Bergman projection P_β for $f\in L^1(dV)$ is defined as follows

$$(P_\beta f)(z)=\int_B\frac{f(\omega)(1-|\omega|)^\beta dV(\omega)}{(1-z\overline{\omega})^{\beta+n+1}};z\in B$$

similarly for product domain via kernels

$$\prod_{j=1}^m\frac{(1-|\omega_j|)^{\beta_j}}{(1-z_j\overline{\omega_j})^{\beta_j+n+1}}.$$

We define P_{β}^{+} similarly taking modulus of kernel.

This operator for large enough β is acting from $F_{\alpha}^{p,r}$ into $F_{\alpha}^{p,r}$ and from $A_{\alpha}^{p,r}$ into $A_{\alpha}^{p,r}$. These assertions can be seen in [2], [11] for all p and r bigger than one. These assertions are important part of proof of theorems of this section. A simple inspection shows that these type assertions (indirectly) serve as a base of proofs of our theorems in previous section. This observation is important for us.

The last results are valid even for some weightes ω and for $A_{\omega}^{p,q}$ (general) spaces (see, for example, [21], [23]) for projection theorems in these spaces in the ball.

The same projection result is valid for $A_{\vec{\alpha}}^{\vec{p}}(B_n^m)$ classes (see [1] and references there) (mixed norm analytic spaces in polyball).

For $HM^{p,s}$ and H_{θ}^{p} spaces to prove a sharp theorem we put this assertion on projection in our theorem as some additional condition which probably can be even removed.

To prove these assertions we have to repeat step by step the proof of model case of previouus section based on these remarks we just did.

We have the following sharp results among other things (theorems on $A_{\alpha}^{p,q}$ can be partially even extended to $A_{\omega}^{p,q}$ spaces (more general weighted Bergman classes spaces) based on results on projections [21], [23].

Theorem 1. *Let $1 < p,\ q < \infty$ then for all $\beta > \beta_0$ for same large enough β_0 and all $f\ f \in A_{\alpha}^{\infty}$, $\alpha > 0$ we have $l_1 \asymp l_2$; where*

$$l_1 = dist(f, \mathbf{A}_{\alpha}^{p,q});$$

$$l_2 = \inf\{\varepsilon > 0 : \int\limits_0^1 (\int\limits_S (\int\limits_{\Omega_{\varepsilon,\alpha}} \frac{(1-|\omega|)^{\beta-\alpha} dV(z)}{|1-\bar{z}\omega|^{\beta+n+1}})^q d\sigma(\xi))^{\frac{p}{q}} dr < \infty\}.$$

Theorem 2. *Let $1 < p,\ q < \infty$ then for all $\beta > \beta_0$ for some large enough β_0 and all $f\ f \in A_{\alpha}^{\infty}$, $\alpha > 0$ we have $l_1 \asymp l_2$ where*

$$l_1 = dist(f, \mathbf{F}_{\alpha}^{p,q});$$

$$l_2 = \inf\{\varepsilon > 0 : \int\limits_S (\int\limits_0^1 (\int\limits_{\Omega_{\varepsilon,\alpha}} \frac{(1-|\omega|)^{\beta-\alpha} dV(z)}{|1-\bar{z}\omega|^{\beta+n+1}})^q dr)^{\frac{p}{q}} d\sigma(\xi) < \infty\},$$

where

$$\Omega_{\varepsilon,\alpha} = \{z \in B : |f(z)|(1-|z|)^{\alpha} > \varepsilon\};$$

Remark 1. Theorems 1,2 for $p = q$ case can be seen in [6], [7], [8], [15], [16].

The following results are for $H^p(\theta)$ and $HM^{p,s}$ spaces. For $s > 0$, $1 < p < \infty$ there is α_0 so that $HM^{p,s} \subset A_{\alpha_0}^{\infty\infty}$ see [2], [3], [11]. This pose a dist problem.

Theorem 3. *Let $1 < p < \infty$, $s \in (0, \frac{n}{p})$. Then there is large enough β_0 and some fixed α_0, $\alpha_0 = \alpha_0(s,n,p)$, so that for all $\beta > \beta_0$ we have $l_1 \asymp l_2$, where*

$$l_1 = dist_{A_{\alpha_0}^{\infty\infty}}(f, HM^{p,s});$$

$$l_2 = \inf\{\varepsilon > 0 : \| \int\limits_{\Omega_{\varepsilon,\alpha_0}} \frac{|f(z)|(1-|z|)^{\beta-\alpha_0} dV(z)}{|1-\bar{\omega}z|^{\beta+n+1}} \|_{HM^{p,s}} < \infty\}$$

if

$$\|P^+_\beta |g|\|_{(HM^{p,s})} \le C_1\|g\|_{(HM^{p,s})}.$$

There is β_0 so that $H^p(\theta)\subset A^\infty_{\beta_0}$, $\beta_0=\beta_0(p,\theta)$, (see [2], [3]). So we pose dist problem.

Theorem 4. *Let* $1<p<\infty$, $\theta\in A_p$, $f\in A^\infty_{\beta_0}$ *then* $l_1\asymp l_2$, *where*

$$l_1 = dist_{A^\infty_{\beta_0}}(f,H^p(\theta));$$

$$l_2 = \inf\{\varepsilon>0 : \|\int\limits_{\Omega_{\beta_0,\varepsilon}} \frac{|f(z)|(1-|z|)^{\beta-\beta_0}dV(z)}{|1-\bar{\omega}z|^{\beta+n+1}}\|_{H^p(\theta)}<\infty\}$$

for all $\beta>\widetilde{\beta}_0$ *for some large enough* $\widetilde{\beta}_0$ *if*

$$\|P^+_\beta |g|\|_{H^p(\theta)} \le C_2\|g\|_{H^p(\theta)}.$$

Similarly more general weighted Hardy classes $H^{\vec{p}}_{\vec{\theta}}$ can be defined and Hardy-Morrey spaces $HM^{\vec{p},\vec{s}}(B\times ...\times B)$ can be defined and estimates for distances can be given (not sharp).

Based on projection theorems on $A^{\vec{p}}_{\vec{\alpha}}$ spaces we following the model case of previous section can formulate the following sharp theorem (see [1], [22]).

Note that (see [1])

$$\sup_{z_k\in B;k=1,...,m.} |f(\vec{z})|\prod_{k=1}^m(1-|z_k|)^{\tau_k}\le C_3\|f\|_{A^{\vec{p}}_{\vec{\alpha}}},$$

$$\tau_k=\left(\frac{\alpha_k}{p_k}+\frac{n+1}{p_k}\right), j=1,...,m.$$

So, dist problem can be obviously posed.

Theorem 5. *Let* $1<p<\infty$, $\alpha_j>-1$, $j=1,\dots,m$ *and* $\gamma_j=\frac{\alpha_j}{p_j}+\frac{n+1}{p_j}$, $j=1,\dots,m$, *let*

$$f\in A^\infty_{\vec{\gamma}}(B\times\dots\times B)=\{f\in H(B\times\dots\times B): \sup_{z_1,...,z_m}|f(z_1,...,z_m)|\prod_{k=1}^m(1-|z_k|)^{\frac{\alpha_k}{p_k}+\frac{n+1}{p_k}}<\infty\}.$$

Then we have that $l_1(f)\asymp l_2(f)$ *where*

$$l_1(f)=dist_{A^\infty_{\vec{\gamma}}}(f,A^{\vec{p}}_{\vec{\alpha}});$$

$$l_2(f)=\inf\{\varepsilon>0:\int\limits_B(\int\limits_B\dots(\int\limits_{\widetilde{\Omega}^1_{\varepsilon,\alpha}}\frac{\prod_{k=1}^m(1-|\omega_k|)^{\beta+\gamma_k}d(V(\omega_k))}{\prod_{k=1}^m|1-\overline{\omega}_k z_k|^{\beta+n+1}})^{\frac{p_1}{p_2}}$$

$$\times(1-|z_1|)^{\alpha_1}dV(z_1)(1-|z_m|)^{\alpha_m}dV(z_m))^{\frac{1}{p_m}}<\infty\}, where$$

$$\widetilde{\Omega}^1_{\varepsilon,\gamma}=\{z\in B\times\dots\times B: |f(\vec{z})|(1-|z|)^\gamma\ge\varepsilon\},$$

for all β, $\beta>\beta_0$ *for some large enough* β_0.

We follow previous section and we provide one proof for all spaces. Let X be one of these spaces. Note since $f \in A_t^\infty(B)$ then by integral representation theorem we have

$$f(z) = f_1(z) + f_2(z), z \in B \tag{23}$$

where

$$f_1(z) = C_4(\beta) \int\limits_{B \backslash \Omega_{\varepsilon,t}} \frac{f(\omega)(1-|\omega|)^\beta}{(1-\overline{\omega}z)^{\beta+n+1}} dV(\omega), \beta > \beta_0, where \tag{24}$$

β_0 is large enough and also

$$f_2(z) = C_4(\beta) \int\limits_{\Omega_{\varepsilon,t}} \frac{f(\omega)(1-|\omega|)^\beta dV(\omega)}{(1-\overline{\omega}z)^{\beta+n+1}}, where\, z \in B. \tag{25}$$

It is easy to show now following previous section using the Forelli-Rudin estimate and definition of $\Omega_{\varepsilon,t}$ set that the following estimate is valid

$$\sup_{z \in B}(1-|z|)^t |f_1(z)| \le C_5 \varepsilon;$$

and also

$$\|f_2\|_X < \infty.$$

The last estimate follows from condition in formulation of theorem immediately. Hence we have that

$$dist_{A_t^\infty}(f, X) \le C_6 \|f - f_2\|_{A_t^\infty} = C_7 \|f_1\|_{A_t^\infty} \le C_8 \varepsilon \tag{26}$$

This gives one part of theorem. For the proof of the second part we refer to the previous section. The proof follows from there and from Bergman type projection theorem, that is

$$\|P_\beta^+ f\|_X \le C_9 \|f\|_X;$$

estimate $\beta > \beta_0$.

Some of these our results can be extended even to spaces on product domains by very similar methods.

Let further

$$H^{p_1,\dots,p_m}(B \times \dots \times B) = \left\{ g \in H(B \times \dots \times B) : \sup_{r_m<1} \left(\int_S \left(\dots \sup_{r_2<1} \int_S \left(\sup_{r_1<1} \int_S |g(r_1\zeta_1, \dots, r_m\zeta_m)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \right.\right.\right.$$

$$\left. d\sigma(\zeta_2) \dots \right)^{\frac{p_m}{p_{m-1}}} d\sigma(\zeta_m) \Big)^{\frac{1}{p_m}} < \infty;$$

$0 < p_i < \infty,\ i = 1, \dots, m.$

Let also

$$\tilde{H}^{p_1,\dots,p_m}(B \times \dots \times B) = \left\{ g \in H(B \times \dots \times B) : \sup_{r_j<1; j=1\dots m} \left(\int_S \left(\dots \int_S \left(\int_S |g(r_1\zeta_1, \dots, r_m\zeta_m)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \right.\right.\right.$$

$$\left. d\sigma(\zeta_2) \dots \right)^{\frac{p_m}{p_{m-1}}} d\sigma(\zeta_m) \Big)^{\frac{1}{p_m}} < \infty;$$

$0 < p_i < \infty,\ i = 1,\ldots,m,\ j = 1,..,m$ where σ is a Lebegues measure on S. These are a new mixed norm analytic Hardy class on products of unit balls (polyball). These are direct analogues of mixed norm Bergman type function spaces which we defined above in produts of unit balls.The study of these types of Hardy function spaces is a new and interesting problem. Some one side (not sharp) estimates for a distance function in these type new Hardy spaces can be also provided based on approaches used in this paper.

Note by induction based on classical result in product domain and Holders inequality we have

$$\sup_{z_j \in B; j=1...m} |f(z_1,...,z_m)|(1-|z_1|)...(1-|z_m|) \le C_{10}\|f\|_{H^{\vec{p}}};$$

$p_j > 1, j = 1,...,m$, and the same is valid for $\tilde{H}^{\vec{p}}$ spaces, $p_j > 1, j = 1,...,m$.

Putting boundedness condition on Bergman projection similar to theorems 5, 6 can be easily formulated and proved for $H^{\vec{p}},\tilde{H}^{\vec{p}}$. We leave this to interested readers.

Note that the following uniform estimate is true:

$$\sup|f(\vec{z})| \times \prod_{j=1}^{m}\left(1-|z_j|\right)^{\alpha_j+n+1} \le C_{11}\|f\|_{\tilde{F}^{\vec{p}}_{\vec{\alpha}}} =$$

$$= C_{12}\left(\int\limits_{S}\left(\int\limits_{\Gamma_t(\xi_m)}\cdots\int\limits_{S}\left(\int\limits_{\Gamma_t(\xi_1)}|f(\vec{z})|^{p_1}(1-|z_1|)^{\alpha_1-n}dV(z_1)\right)^{\frac{p_2}{p_1}}(1-|z_m|)^{\alpha_m-n}dV(z_m)\right)^{\frac{1}{p_m}}\right),$$

for $p_j > 1, j = 1,...,m, \alpha_j > -1, j = 1,...,m,\ \{\alpha_j\}_{j=1}^m$ is large enough, where

$$\Gamma_t(\xi) = \left\{z \in B : |1-\overline{\xi}z| < t(1-|z|)\right\}, t > 1.$$

This uniform estimate also provides directly some new estimate for distances in spaces with such quazinorms on product domains.

Similar to this estimates for $A^{\vec{p}}_{\vec{\alpha}}, F^{\vec{p}}_{\alpha}$ spaces (product analogues of $A^{p,q}_{\alpha}, F^{p,q}_{\alpha}$ spaces) can be provided and estimates for distances for such spaces can be also given based on methods of this paper.

In addition let $\theta_i \in A^{p_i}(S), 1 < p_i < \infty, i = 1,...,n$ be Muckenhoupt weight on S.

We define new weight Hardy class in polyball and we can show same type results for these classes

$$H^{\vec{p}}_{\vec{\theta}}(S) = \{f \in H(B^n) :$$

$$\sup_{r_j<1; j=1,...,n.}\left(\int\limits_{S}\cdots\left(\int\limits_{S}|f(r_1\xi_1,...,z_n\xi_n)|^{p_1}\theta_1(\xi_1)d\sigma(\xi_1)\right)^{\frac{p_2}{p_1}}\theta_n(\xi_n)d\sigma(\xi_n)\right)^{\frac{1}{p_n}} < \infty\},$$

and let

$$\tilde{H}^{\vec{p}}_{\vec{\theta}}(S) = \{f \in H(B^n) :$$

$$\sup_{r_j<1; j=1,...,n.}\left(\int\limits_{S}\cdots\left(\int\limits_{S}|f(r_1\xi_1,...,z_n\xi_n)|^{p_1}\theta_1(\xi_1)d\sigma(\xi_1)\right)^{\frac{p_2}{p_1}}\theta_n(\xi_n)d\sigma(\xi_n)\right)^{\frac{1}{p_n}} < \infty\}.$$

We can show by induction first easily that for some $\tau_j > 0, p_j > 1, j = 1,...,m$:

$$\|f\|_{A^{\infty}_{\vec{\tau}}(B\times...\times B)} \le C_{13} \begin{cases} \|f\|_{H^{\vec{p}}_{\vec{\theta}}}, \\ \|f\|_{\tilde{H}^{\vec{p}}_{\vec{\theta}}}. \end{cases} \tag{27}$$

Based on same estimates in one domain (see above), then we can prove similar results for these spaces under additional condition on boundedness of Bergman projection in these spaces.

Some results and remarks on distances in BMOA and Herz type mixed norm spaces on product domain

In this section we clofwe new analytic mixed norm Herz type spaces on product domains and discus on related issues on distances for them.

We will need Bergman ball in B, we define the Bergman ball in B in a standard way as follows (see [10]).

For $z \in B, r > 0$, the set

$$D(z,r) = \{\omega \in B : \beta(z,\omega) < r\},$$

where β is a Bergman metric on B,

$$\beta(z,\omega) = \left(\frac{1}{2}\right) log\left(\frac{1+|\varphi_z(\omega)|}{1-|\varphi_z(\omega)|}\right); z,\omega \in B$$

is called the Bergman metric ball at $z, z \in B$ (see [10]). We denote below these balls also by $B(z,r)$ sometimes also.

Let now $f \in H(B\times...\times B)$. Consider new Herz type spaces on product domains with quazinorms:

1) $$\sum_{k_1\ge 0}...\sum_{k_n\ge 0}\left(\int\limits_{D(a_{k_n},r)}...\left(\int\limits_{D(a_{k_1},r)}|f(\vec{z})|^{p_1}dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}dV_{\alpha_n}(z_n)\right)^{\frac{1}{p_n}},$$

$$0 < p_j < \infty,\ j = 1,...,n;$$

or

2) $$\sum_{k\ge 0}\left(\int\limits_{D(a_k,r)}...\left(\int\limits_{D(a_k,r)}|f(\vec{z})|^{p_1}dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_m}},$$

$$0 < p_j < \infty,\ j = 1,...,m;$$

or

3) $$\sum_{k_n\ge 0}\left(\int\limits_{D(a_{k_n},r)}...\left(\sum_{k_1\ge 0}\left(\int\limits_{D(a_{k_1},r)}|f(\vec{z})|^{p_1}dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}\right)dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_m}},$$

$$0 < p_j < \infty,\ j = 1,...,m;$$

or

$$4)\ \sum_{k\geq 0}\left(\int\limits_{D(a_k,r)}\dots\left(\sum_{k\geq 0}\left(\int\limits_{D(a_k,r)}|f(\vec{z})|^{p_1}\,dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}\right)dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_m}},$$

$$0<p_j<\infty,\ \alpha_j>-1, j=1,\dots,m.$$

With some restriction on $\{p_j\}$ for all these Y spaces we have that

$$|f(\vec{z})|\left(\prod_{j=1}^{n}\left(1-|z_j|\right)^{\tau_j}\right)\leq C\|f\|_Y; z_j\in B,$$

for some $\tau_j\geq\tau_0, j=1,\dots,m,\tau_0$ is large enough $\tau_0=\tau_0(\vec{p},\vec{\alpha})$.
This easily follows by induction from one domain result or by similar type arguments.
Similarly for Z type spaces and various similar spaces:

$$\int\limits_{B}\left(\int\limits_{B(z,r)}\dots\left(\int\limits_{B(z,r)}|f(\vec{z})|^{p_1}dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_n}};$$

$0<p_j<\infty,\ j=1,\dots,n.$

Having projection theorem for Y type, Z type analytic spaces gives easily a sharp distance theorem for $(A^{\infty}_{\tau}(B\times\dots\times B),Y)$ or $\left(A^{\infty}_{\tilde{\tau}}(B\times\dots\times B),Z\right)$ pair of spaces of functions. We need to prove only that here for all

$$\beta>\beta_0, \|P_\beta f\|_Y\leq C_0\|f\|_Y, \|P_\beta f\|_Z\leq C_1\|f\|_Z$$

to get a sharp dist theorem: β_0 is large enough.

Similar arguments can be applied for new analytic Herz type spaces with the following quazinorms in product domains.

$$1)\ \int\limits_{B}\dots\int\limits_{B}\left(\int\limits_{B(z_1,r)}\dots\left(\int\limits_{B(z_m,r)}|f(\vec{z})|^{p_1}\,dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_m}},$$

$$0<p_j<\infty,\ j=1,\dots,m;$$

or

$$2)\ \int\limits_{B}\left(\int\limits_{B(z,r)}\dots\left(\int\limits_{B}\left(\int\limits_{B(a_k,r)}|f(\vec{z})|^{p_1}\,dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}\right)dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_m}},$$

$$0<p_j<\infty,\ j=1,\dots,m;$$

or

$$3)\ \int\limits_{B}\left(\int\limits_{B(z,r)}\dots\left(\sum_{k\geq 0}\left(\int\limits_{D(a_k,r)}|f(\vec{z})|^{p_1}\,dV_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}}\right)dV_{\alpha_m}(z_m)\right)^{\frac{1}{p_m}},$$

$$0<p_j<\infty,\ j=1,\dots,m.$$

Note, for example, that since

$$\max_{z\in B(\tilde{z},r)}(1-|z|)^{\alpha}|f(z)|^{p}\le C_2\int\limits_{B(\tilde{z},r)}|f(\omega)|^{p}(1-|\omega|)^{\tilde{\alpha}}dV(\omega),\tilde{\alpha}=\alpha-(n+1),0<p<\infty,\alpha>0,$$

(or the same estimate with $B(a_k,r),\{a_k\}$ is r - lattice in B)(see for example [10]), we have that for $0<q<\infty,\alpha>-1,0<p<\infty$

$$\int\limits_{B}(1-|z|)^{\alpha\frac{q}{p}}|f(z)|^{q}dV(z)\le\int\limits_{B}\left(\int\limits_{B(\tilde{z},r)}|f(z)|^{p}(1-|z|)^{\tilde{\alpha}}dV(z)\right)^{\frac{q}{p}}dV(\tilde{z})=\|f\|_1.$$

Hence there are $\tau>0,\tau_0>0$, so that

$$\sup_{z\in B}|f(z)|(1-|z|)^{\tau}\le C_3\|f\|_1$$

$$\sup_{z\in B}|f(z)|(1-|z|)^{\tau_0}\le C_4\sum_{k\ge0}\left(\int\limits_{D(a_k,r)}|f(z)|^{p}(1-|z|)^{\alpha}dV(z)\right)^{\frac{q}{p}},0<p,q<\infty,\alpha>-1.$$

From these estimates if we apply them by each variable separately we easily obtain more complicated similar type estimates from below for various mentioned quazinorms of mixed norm Herz type spaces but in product domains since obviously for example

$$\sup_{z_1\in B,z_2\in B}|f(z_1,z_2)|(1-|z_1|)^{\tau_1}(1-|z_2|)^{\tau_2}\le$$
$$\le C_5\sup_{z_2\in B}\|f\|_{X(z_1)}(1-|z_2|)^{\tau_2}\le$$
$$\le C_6\|\sup_{z_2\in B}|f(\tilde{z})|(1-|z_2|)^{\tau_2}\|_{X(z_1)}\le$$
$$\le C_7\|\|f\|_{X(z_2)}\|_{X(z_1)},$$

$\tau_j>0,j=1,2;$

where $X(z_j)$ is a quazinorm of some Herz type space in B by z_j variable. It remains to apply induction by amount of variables.

Similar arguments are valid for Herz type spaces with sup instead of integration.

Let further

$$A^{p}_{\vec{\alpha},\beta}(B\times\ldots\times B)=\{f\in H(B^m):\int\limits_{B}\left(\sup_{z_1\in B(\omega,r)}\ldots\sup_{z_m\in B(\omega,r)}|f(z_1,\ldots,z_m)|^{p}(1-|z_1|)^{\alpha_1}\ldots(1-|z_m|)^{\alpha_m}\right)\times$$
$$\times(1-|\omega|)^{\beta}dV(\omega)<\infty\}.$$

$$\tilde{A}^{p}_{\vec{\alpha},\beta}=\{f\in H(B^m):\int\limits_{B}\ldots\int\limits_{B}\sup_{z_1\in B(\omega_1,r)}\ldots\sup_{z_m\in B(\omega_m,r)}|f(z_1,\ldots,z_m)|^{p}\prod_{j=1}^{m}(1-|z_j|)^{\alpha_j}\times$$

$$\times\prod_{j=1}^{m}(1-|\omega_j|)^{\beta_j}dV(\omega_1)...dV(\omega_m), \alpha_j>0, j=1,...,m, \beta>-1, \beta_j>-1, j=1,...,m.$$

Let further also

$$B^p_{\vec{\alpha},\beta}=\{f\in H(B\times...\times B):$$

$$\sum_{k\geq 0}\left(\sup_{z_1\in D(a_k,r);z_m\in D(a_k,r)}|f(z_1,...,z_m)|^p\times\prod_{j=1}^{m}(1-|z_j|)^{\alpha_j}(1-|a_k|)^{\beta}\right)<\infty\}.$$

$$\tilde{B}^p_{\vec{\alpha},\vec{\beta}}=\{f\in H(B^m):\left(\sum_{k_1\geq 0}...\sum_{k_m\geq 0}\right)(1-|a_{k_1}|^{\beta_1}...(1-|a_{k_m}|^{\beta_m}\times$$

$$\times\sup_{z_1\in D(a_{k_1},r);z_m\in D(a_{k_m},r)}|f(z_1,...,z_m)|^p\prod_{j=1}^{m}(1-|z_j|)^{\alpha_j}<\infty\},$$

$\beta\geq 0, \beta_j\geq 0, \alpha_j\geq 0, j=1,...,m.$

The natural question is to calculate distances of these classes. We mean pairs $\left(A^{\infty}_{\tau}, B^p_{\alpha,\beta}\right)$ or $\left(A^{\infty}_{\tilde{\tau}}, \tilde{B}^p_{\alpha,\beta}\right)$ and $\left(A^{\infty}_{\tau_1}, A^p_{\vec{\alpha},\beta}\right)$ or $\left(A^{\infty}_{\tilde{\tau}_1}, \tilde{A}^p_{\vec{\alpha},\beta}\right)$ also, there $\tau>\tau_0, \tau_1>\tau_0, \tilde{\tau}_1>\tau_0$, for some large enough positive τ_0.

Note also that dist of mixed version of these classes can also be calculated using approaches. We provided in this paper in proofs of previous section. These are new interesting generalization of so called mixed norm. Herz type spaces in the polyball of analytic functions on product domains $B\times...\times B$ (polyballs).

Indeed it is easy to show that for for all this classes with sup we have:

$$\prod_{j=1}^{m}|f(\vec{z})|(1-|z_j|)^{\tau_j}\leq C_8\|f\|_{\tilde{S}}$$

for some $\tau_j>0, j=1,...,n$, where $\tilde{S}$ is one of these just mentioned quazinorms of Herz type spaces.

Based on basic properties of r - lattices in the unit ball we have for example in B:

$$\sup_{z\in B}|f(z)|^p(1-|z|)^{\tau}\leq C_9\sum_{k\geq 0}\left(\sup_{D(a_k,r)}|f(z)|^p(1-|z|)^{\tau_1}\right)(1-|a_k|)^{\tau_2}\leq$$

$$\leq\sum_{k\geq 0}\left(\int\limits_{D(a_k,r)}|f(z)|^p(1-|z|)^{\tau_1-(n+1)}dV(z)\right)(1-|a_k|)^{\tau_2}\leq C_{10}\|f\|^p_{A^p_s};$$

$s=\tau_1+\tau_2-(n+1), \tau_1+\tau_2=\tau, \tau>0, 0<p<\infty.$

Hence projection theorem will give us a sharp theorem similarly as above.

The Bergman representation is very vital for this paper. We add below a remark on weighted Hardy-Sobolev spaces and another integral representations.

We will consider weights ω in $\tilde{A}_p$ classes in $\mathbb{S}$, $1<p<+\infty$, that is, weights in $\mathbb{S}$ satisfying that there exists $C>0$ such that for any nonisotropic ball

$$\tilde{B}\subset\mathbb{S}, \tilde{B}=\tilde{B}(\zeta,r)=\{\eta\in\mathbb{S};|1-\zeta\overline{\eta}|<r\},$$

$$(\frac{1}{|\tilde{B}|}\int_{\tilde{B}}\omega(\zeta)d\sigma(\zeta))(\frac{1}{|\tilde{B}|}\int_{\tilde{B}}(\omega(\zeta))^{\frac{-1}{p-1}}d\sigma(\zeta))^{p-1}\leq C,$$

where σ is the Lebesgue measure on $\mathbb{S}$ and $|\tilde{B}|$ the Lebesgue measure of $\tilde{B}$.

The weighted Hardy-Sobolev space $H_s^p(\omega)$, $0 < s,\ p < +\infty$, consists of those functions f holomorphic in $\mathbb{B}$ such that if

$$f(z) = \sum_k f_k(z)$$

is its homogeneous polynomial expansion, and

$$(I+R)^s f(z) = \sum_k (1+k)^s f_k(z),$$

we have that

$$\|f\|_{H_s^p}(\omega) = \sup_{0<r<1} \|(I+R)^s f_r\|_{L^p(\omega)} < +\infty$$

where $f_r(\zeta) = f(r\zeta), r \in (0,1), \zeta \in \mathbb{S}$.

If $0 < s < n$, and function f in $H_s^p(\omega)$ can be expressed as

$$f(z) = C_s(g)(z) := \int_{\mathbb{S}} \frac{g(\zeta)}{(1-z\bar{\zeta})^{n-s}} d\sigma(\zeta),$$

where $d\sigma$ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}$ and $g \in L^p(\omega)$. These type not Bergman type integral representations also can be useful for distance theorem in $H_s^p(\omega)$ (see [2],[3]).

Let $Q_r(\xi)$ be Carleson tube

$$Q_r(\xi) = \{z \in B : |1 - \langle z, \xi\rangle|^{\frac{1}{2}} < r\}, r > 0, \xi \in S.$$

Let

$$QM^t = \{f \in H(B) : \|f\|_t^2 = \sup_{z\in B}\left(\frac{\mu_f(Q_s(\xi))}{s^{2n}}\right); \xi \in S, 0 \le s < 1\}, d\mu_f(z) = |f(z)|^2(1-|z|^2)^t dV(\omega)$$

(see [10]). Let again as above

$$(P_\alpha^+ f)(z) = \int_B \frac{(1-|\omega|)^\alpha f(\omega) dV(\omega)}{|1-\langle z,\omega\rangle|^{\alpha+n+1}}, z \in B.$$

Then for $\alpha > \alpha_0$ for large enough α_0 and for all $t > -1, P_\alpha^+$ is bounded on QM^t (see [10]).

Note for Bergman ball $B(a,R) \subset Q_r(\xi), \xi \in S, 0 < r < 1, R > 0, a = (1-\sigma z^2)\xi, \sigma \in (0,1), \sigma = \sigma(R)$ (see [10]). Hence there is $\tau_0, \tau_0 = \tau_0(t)$. So that

$$\|f\|_{A_{\tau_0}^\infty} = \sup_{z\in B} |f(z)|(1-|z|)^{\tau_0} \le C_0 \|f\|_t^2,$$

and proof is based on known fact

$$\sup_{z\in D(a_k,r)} |f(z)|^p \le C_1 \int_{D(a_k,r)} |f(z)|^p (1-|z|)^{-\alpha-(n+1)} dV_\alpha(z); \alpha > (-1); 0 < p \le \infty$$

(or $B(w,z)$ instead of $D(a_k,r)$) and standard properties of r-lattice in the ball (see [10]).

And using this projection theorem we easily have hence a new sharp dist theorem for $\left(QM^t, A^\infty_{\tau_0}\right)$ spaces (BMOA type space) in terms of $Q_s(\xi)$ sets.

Theorem 6. *Let $t > -1$, $f \in A^\infty_{\tau_0}$ then we have the following*

$$dist_{A^\infty_{\tau_0}}\left(f, QM^t\right) \asymp$$

$$\asymp inf\{\varepsilon > 0 : \| \int\limits_{\Omega_{\varepsilon,\tau_0}} \frac{(1-|\omega|)^{\beta-\tau_0} dV(\omega)}{|1-\bar{z}\omega|^{\beta+n+1}} \|_t < \infty\};$$

for all $\beta, \beta > \beta_0$, where β_0 is large enough.

The same type result similarly can be proved for (QM^t) in product domains.

Similar results with similar proofs are also valid for $F^\infty_{q,s}(B)$ spaces of BMOA type in the unit ball (see [2], [11] for definition of these spaces and uniform estimates for them and projection theorem for them.we omit here details leaving them for readers).

A new projection theorem in Herz spaces and a sharp distance theorem

In this section we provide a new projection theorem for Herz spaces in the unit ball and then based on it we provide a new sharp distance theorem for such type spaces.

We provide the proof in the unit disk the same proof can be given in the ball by repetition of arguments.

Let $f \in H(D)$, where D is a unit disk then let $q \le p \le 1$, then for $\beta > \beta_0$, where β_0 is large enough we have that

$$\|P^+_\beta(|f|)\|_M \le C_0 \|f\|_M$$

or

$$\|P^+_\beta(|f|)\|_{\tilde{M}} \le C_1 \|f\|_{\tilde{M}}$$

for all $\alpha > -1, \alpha_1 > 1, \widetilde{\alpha} \ge 0$ if

$$M(D) = M_{p,q,\tilde{\alpha}}(D) = \{f \in H(D) : \sum_{k\ge 0} (\int\limits_{D(a_k,r)} |f(w)|^p dV_{\widetilde{\alpha}}(w))^{\frac{q}{p}} < \infty\};$$

or

$$\tilde{M}(D) = M_{p,q,\alpha,\alpha_1}(D) = \{f \in H(D) : \int\limits_D (\int\limits_{B(z,r)} |f(w)|^p dV_\alpha(w))^{\frac{q}{p}} dV_{\alpha_1}(z) < \infty\}$$

where

$$P^+_\beta(|f|)(z) = \int\limits_D \frac{|f(w)|(1-|w|)^\beta}{|(1-\overline{w}z)^{\beta+2}|} dV(w).$$

The proof follows directly from following well known estimates (see for example [10])

$$(\int\limits_D \frac{|f(w)|(1-|w|)^\beta}{|1-\overline{w}z|^{\beta+2}} dV(w))^p \le C_2 \int\limits_D \frac{|f(w)|^p (1-|w|)^{\beta p+2p-2} dV(w)}{|1-\overline{w}z|^{(\beta+2)p}}; p \le 1; \beta > -1;$$

$$\int\limits_{D(a_k,r)} \frac{(1-|w|)^{\tau} dV(w)}{|1-\overline{w}z|^{v}} \leq \frac{C_3}{|1-\bar{z}a_k|^{v-(\tau+2)}}; \tau \geq 0; v > \tau+2; a_k, z \in D, k = 1,2,3....$$

$$\sum_{k\geq 0} [\frac{(1-|a_k|)^{\beta}}{|1-w\overline{a_k}|^{s}}] \leq \sum_{k\geq 0} (\int\limits_{D(a_k,r)} \frac{dV(z)}{|1-\overline{w}z|^{s}})((1-|a_k|)^{-2+\beta}) \leq \frac{\widetilde{C_4}}{(1-|w|)^{s-\beta}}; \beta > 1, s > \beta.$$

$$I(v,z) = \int\limits_{B(z,r)} \frac{(1-|w|)^{\tau} dV(w)}{|1-w\overline{v}|^{\alpha}} \leq C_5 (\frac{(1-|z|)^{\tau}}{|1-\overline{v}z|^{\alpha-2}}); \tau > -1; \alpha > 2; z, v \in D,$$

or

$$I(v,z) \leq \frac{C_6}{|1-\overline{v}z|^{\alpha-2-\tau}}; \alpha > 2; \tau \geq 0; z, v \in D.$$

Indeed we have by these estimates for example the following chain of inequalities

$$\sum_{k\geq 0} (\int\limits_{D(a_k,r)} |(P_{\beta}^{+})||f(z)|^{p} dV_{\widetilde{\alpha}}(z))^{q/p} \leq$$

$$\leq C_7 \sum_{k\geq 0} (\int\limits_{D(a_k,r)} (\int\limits_{D} \frac{|f(w)|^{p}(1-|w|)^{\beta p+2p-2}}{|1-\overline{w}z|^{(\beta+2)p}} dV(w)) dV_{\widetilde{\alpha}}(z))^{q/p} \leq$$

$$\leq \sum_{k\geq 0} (\int\limits_{D} \frac{|f(w)|^{p}(1-|w|)^{\beta p+2p-2}}{|1-\overline{w}a_k|^{(\beta+2)p} - 2} dV(w))^{q/p} (1-|a_k|)^{\frac{\widetilde{\alpha} q}{p}}.$$

It remains to use one more time first and third estimates for $\frac{q}{p} \leq 1$ and $\beta = \frac{\widetilde{\alpha} q}{p}, \widetilde{\alpha} > 1$. The second "projection theorem"can be shown similarly based on first and last estimate which was provided above by us.

Note following the proof step by step and using technique developed in [10] for the unit ball case it can be easily shown that these projection theorems are valid also in case of the unit ball in $\mathbb{C}^n$.

These two projection type theorems provide the following results in the unit ball in $\mathbb{C}^n$. (We define M (or $\tilde{M}$) space in the ball similarly).

These M (or $\tilde{M}$) type spaces in the ball defined by using Bergman balls $D(a_k, r)$ or $B(z, r)$ in B, $z \in B, a_k \in B$ (see [10] for example for definition of these objects).

First the uniform estimates are probably known

$$\sup_{z\in B} |f(z)|(1-|z|)^{\tau} \leq C_8 \|f\|_{M},$$

or

$$\sup_{z\in B} |f(z)|(1-|z|)^{\tilde{\tau}} \leq C_9 \|f\|_{\tilde{M}};$$

for some fixed $\tau > 0, \tau = \tau(p,q,\alpha,\alpha_1,\widetilde{\alpha})$, $\widetilde{\tau} > 0$, $\widetilde{\tau} = \widetilde{\tau}(p,q,\alpha,\alpha_1,\widetilde{\alpha})$ large enough depending on parameters of M (or $\tilde{M}$) space and proofs are based on known fact

$$\sup_{z\in D(a_k,r)} |f(z)|^{p} \leq C_{10} \int\limits_{D(a_k,r)} |f(z)|^{p}(1-|z|)^{-\alpha-(n+1)} dV_{\alpha}(z); \alpha > (-1); 0 < p \leq \infty$$

(or $B(w,z)$ instead of $D(a_k,r)$) and standard properties of r-lattice in the ball (see [10]).

This leads with projection type theorem we provided to a sharp dist theorem for analytic M (or $\tilde{M}$) type spaces in the ball in $\mathbb{C}^n$ which under some conditions on kernel is valid even also in the bounded strongly pseudoconvex domain in $\mathbb{C}^n$ under some condition on Bergman kernel. We formulate first the unit ball case.

Theorem 7. *Let* $f \in (A^{\infty}_{\tau})(B)$. *Let also* $\alpha, \alpha_1 > (-1), \widetilde{\alpha} \geq 0, q \leq p \leq 1$.

Then

1) $dist_{A^{\infty}_{\tau}}(f,M) \asymp \| \int\limits_{\Omega_{\varepsilon,\tau}} \frac{(1-|z|)^{\beta-\tau} dV(z)}{|1-\bar{z}w|^{\beta+n+1}} \|_M$

for all $\beta > \beta_0$, where β_0 large enough, for some $\tau = \tau(p,q,\alpha,\alpha_1)$;

2) $dist_{A^{\infty}_{\tilde{\tau}}}(f,\tilde{M}) \asymp \| \int\limits_{\Omega_{\varepsilon,\tilde{\tau}}} \frac{(1-|z|)^{\tilde{\beta}-\tilde{\tau}} dV(z)}{|1-\bar{z}w|^{\tilde{\beta}+n+1}} \|_{\tilde{M}}$

for all $\tilde{\beta} > \tilde{\beta}_0$, where $\tilde{\beta}_0$ large enough, for some $\tilde{\tau} = \tilde{\tau}(p,q,\alpha,\alpha_1)$;

Remark 2. We think cases when $p \leq q$ or $min(p,q) \geq 1$ can be considered and solved similarly. We pose this as problem.

Remark 3. The case of Herz type M (or $\tilde{M}$) spaces product domains (polyball) can be considered and solved by similar methods.

We finally add some remarks on strongly pseudoconvex domains with smooth boundary for this last theorem.

The most interesting question is to try to extend (under some additional conditions on Bergman kernel) these projection type results to the case of general bounded strongly pseudoconvex domains with smooth boundary Ω.

The fact that for all $z \in \Omega$

$$|f(z)|(\delta(z))^{\tau} \leq C_{11} \|f\|_M$$

or

$$|f(z)|(\delta(z))^{\tilde{\tau}} \leq C_{12} \|f\|_{\tilde{M}}$$

in bounded strictly pseudoconvex domain with smooth boundary Ω is also valid that is

$$\sup_{z\in\Omega} |f(z)| \delta(z)^{\tau} \leq C_{13} \|f\|_M; \delta(z) = dist(z, \partial\Omega), z \in \Omega \tag{28}$$

$$\sup_{z\in\Omega} |f(z)| \delta(z)^{\tilde{\tau}} \leq C_{14} \|f\|_{\tilde{M}}; \delta(z) = dist(z, \partial\Omega), z \in \Omega \tag{29}$$

$B(z,r)$ and $B(a_k,r)$ are Kobayashi balls where $\{a_k\}$ is r - lattice in Ω (see [20]) (M (or $\tilde{M}$) type spaces can be defined similarly via Kobayashi balls (we refer to [20] for definition of Kobayashi balls)).

The proof of (28) and (29) is based on facts that

$$\sup_{z\in\Omega} |f(z)| \delta(z)^{\frac{\alpha+n+1}{p}} \leq C_{15} (\|f\|_{A^p_\alpha}) \leq C_{16} \left(\sum_{k\geq 0} (\int\limits_{D(a_k,r)} |f(z)|^p \delta(z)^{\alpha} dV(z))^{\frac{p}{q}} \right)^{\frac{1}{q}} = \|f\|_M;$$
$$q \leq p, \alpha > -1$$

similarly

$$\sup_{z\in\Omega}|f(z)|(\delta(z)^{\tau}) \leq C_{17}\|f\|_{A^q_{\widetilde{\alpha_1}}} \leq C_{18}\left(\int_{\Omega}\Big(\int_{B(z,r)}|f(\widetilde{z})|^p\delta(\widetilde{z})^{\alpha}dV(\widetilde{z})\Big)^{\frac{p}{q}}\delta(z)^{\alpha_1}dV(z)\right)^{\frac{1}{q}} = \|f\|_{\widetilde{M}};$$
$$p\in(0,\infty), q\in(0,\infty)$$

for some
$\tau = \frac{\alpha}{p}+\frac{\alpha_1}{q}+\frac{n+1}{p}+\frac{n+1}{q}$, $\alpha,\alpha_1 > (-1)$, for some $\widetilde{\alpha_1}$; these are based on estimate in Ω (see [20])

$$|f(z)|^q(\delta(z))^{\tau} \leq C_{19}\Big(\int_{B(z,r)}|f(\widetilde{z})|^p dV_{\alpha}(\widetilde{z})\Big)^{\frac{q}{p}}, z\in\Omega$$

for $\tau = \frac{q}{p}(n+1)+\frac{\alpha q}{p}$.

So dist problem for these pair of spaces can be again posed and projection theorem with additional condition on Bergman kernel can be show probably.

Remark some results of this paper on Herz type spaces were partially extended by similar methods by authors to tubular domains over symmetric ones.

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