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CONSERVATION LAWS AND SYMMETRY ANALYSIS OF (1+1)-DIMENSIONAL SAWADA-KOTERA EQUATION

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The paper addresses an extended (1+1)-dimensional Sawada-Kotera (SK) equation. The Lie symmetry analysis leads to many plethora of solutions to the equation. The non-linear self-adjointness condition for the SK equation established and subsequently used to construct simplified independent conserved vectors. In particular, we also get conservation laws of the equation with the corresponding Lie symmetry.

Key words: Fluid mechanics, Lie symmetry, Partial differential equation, Shear stress, Optimal system, Partial differential equation; KdV equation, Lie symmetry; Conservation Laws

УДК 548.71

ЗАКОНЫ СОХРАНЕНИЯ И АНАЛИЗ СИММЕТРИИ (1 + 1)-МЕРНОГО УРАВНЕНИЯ САВАДА-КОТЕРА

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В работе рассматривается расширенное (1 + 1)-мерное уравнение Савада-Котера (СК). Анализ симметрии Ли приводит к множеству решений уравнения. Условие нелинейной самосопряженности для уравнения СК, установленное и впоследствии используемое для построения упрощенных независимых консервативных векторов. В частности, мы также получаем законы сохранения уравнения с соответствующей симметрией Ли.

Ключевые слова: механика жидкости, симметрия Ли, уравнение с частными производными, напряжение сдвига, оптимальная система, уравнение с частными производными; уравнение КдФ, симметрия Ли, законы сохранения

Introduction

In this paper, we study the following equation

$$\Delta_{SK} := u_t - 20u^2u_x - 25u_xu_{xx} - 10uu_{xxx} - u_{xxxxx} = 0, \tag{1}$$

which is a 5-th order PDE or a type of 5-th order KdV-equation found first by Sawada and Kotera and then by Caunderey, Dodd and Gibbon, which describes long waves in water of relatively shallow depth [2].

The symmetry group of a system of differential equations (DEs) transform solution of the system to other solutions of the system. For constructing the solutions of non-linear PDEs, Lie symmetry group theory can be regarded as one of the most powerful methods in the theory on non-linear PDEs [8, 9, 10, 12, 18, 19].

In the study of DEs, conservation laws play signeficant roles not only in obtaining in-depth understanding of physical properties of various systems, but also in constructing of their exact solutions [1, 3, 4, 7, 11, 14, 15]. They described physical conserved quantities sucha s mass, energy, momentum and angular momentum, as well as charge and other constant of motion. They are important for investigating integribility and linearization mapping and for stablishing exsistence and uniqeness of solutions. They are also used in the analysis of stability and global behaviour of solutions. In addition they play an essential role in the developement of numerical methods and provide an essential starting point for find non-locally related systems and potential variables. Moreover, the structure of conservation laws is coordinate-free, as a point or contact transformation maps a conservation laws into a conservation laws. A systematic way of constructing the conservation laws of a sytem of DEs that admits a variational principle is via Noether's theorem. Its application allows physicists to gain powerful insights into any general theory in physics just by analyzing the various transformations that would make bthe form of the laws involved invariant. For instance, the invariant of physical systems with respect to spatial translation, rotation and time translation respectively give rise to the well known conservation laws of linear momentum, angular momentum and energy. Among the generalization of Noether's theorem an Ibragimov's theorem, based on the self-adjontness concepy of DEs allows to find independent conservation laws for a system of PDEs. Thus, in this paper the adjointness of the SK equation is first established, then Lie point symmetries are applied to find non-trivial conservation laws for the equation (1).

Adjoint equation

In accordance to [13], the formal Lagrangian of the equation (1) is given by:

$$\mathscr{L} = v(u_t - 20u^2u_x - 25u_xu_{xx} - 10uu_{xxx} - u_{xxxxx}), \tag{2}$$

where v in a new dependent variable. The adjoint equation for SK equation is

$$\Delta^*_{SK} = \frac{\delta \mathscr{L}}{\delta u}, \tag{3}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \tag{4}$$

is the variational derivative with respect to u.

Taking into account Eq. (3) and (4) we obtain the following adjoint equation for SK equation:

$$\Delta^*_{SK} = v_{xxxxx} - 20u_{xx}v_x + 5u_xv_{xx} + 10uv_{xxx} + 25u_xv_x + 20u^2v_x - v_t.$$

This equation will be used for the derivation of non-linearity self-adjoint condition later on.

Non-linear self-adjointness

This concept has significantly expanded the notion of adjointness with respect to construction of canservation laws. It incorporates all the previous concepts of adjointness and thus enables more consereved vectors for DEs to be constructed. According to the definition of non-linearly self-adjointness the Eq. (1) is non-linear self-adjoint if the equation obtained from the adjoint Δ^*_{SK} after the substituition $v = \varphi(x,t,u)$ is identical with original Eq. (1) where φ is an arbitrary function. That is, if

$$\Delta^*_{SK}\Big|_{v=\varphi(x,t,u)} = \zeta(x,t,u)\Delta_{SK}, \tag{5}$$

for some indeterminate variable coefficient ζ.

Take the substituition v written together with the necessary derivatives

$$\begin{aligned}
v_x &= \varphi_x + \varphi_u u_x,\\
v_t &= \varphi_t + \varphi_u u_t,\\
v_{xx} &= \varphi_{xx} + 2\varphi_{xu}u_x + \varphi_{uu}u_x^2 + \varphi_u u_{xx},\\
v_{xxx} &= \varphi_{xxx} + (3\varphi_{xxu} + 2\varphi_{xuu})u_x + 3\varphi_{xu}u_{xx}\\
&\quad + \varphi_{xuu}u_x^2 + \varphi_{uuu}u_x^3 + 3\varphi_{uu}u_xu_{xx} + 2\varphi_{xu}u_{xxx},\\
v_{xxxxx} &= \varphi_{xxxxx} + (5\varphi_{xxxxu} + 2\varphi_{xxuuu})u_x + 8\varphi_{xxxuu}u_x^2\\
&\quad + \cdots + 2\varphi_{xuu}u_xu_{xxxx} + 2\varphi_{xu}u_{xxxxx} + 2\varphi_{xxxuu},
\end{aligned}$$

to Eq. (5) we conclude that v is just a constant non-zero function.

Conservation laws

In this section we use both Noether's method and direct method to construct canservation laws for Eq. (1).

Conservation laws constructed by symmetries

Recall thet the following general result on construction on conserved vectors associated with Lie point symmetries of any system of DEs holds [13, 17].

Theorem. *Any symmetry (Lie point symmnetry, Lie-Backlund symmetry, non-local symmetry)*

$$V = \xi^i(x,u,\partial u,\ldots)\frac{\partial}{\partial x^i} + \phi_\sigma(x,u,\partial u,\ldots)\frac{\partial}{\partial u^\sigma},$$

of a system of q−differential equations

$$\Delta_\sigma(x,u,\partial u,...,\partial^s u)=0, \qquad \sigma=1,...,q, \tag{6}$$

with p−independent variables $x=(x^1,...,x^p)$ *and q−dependent variables* $u=(u^1,...,u^q)$ *is inherited by the adjoint system. Specifically the operator*

$$V^*=\xi^i\frac{\partial}{\partial x^i}+\phi_\sigma\frac{\partial}{\partial u^\sigma}+\phi^*_\sigma\frac{\partial}{\partial v^\sigma} \tag{7}$$

with appropriately chosen coefficients ϕ^*_σ*, is admitted by the system of equations consisting of Eq. (6) and adjoint equation*

$$\Delta^*_\sigma(x,u,v,\partial u,...,\partial^s u,\partial^s v)\equiv\frac{\delta(v^\alpha\Delta_\alpha)}{\delta u^\sigma}=0, \qquad \sigma=1,...,q. \tag{8}$$

Furthermore, the combined system (6) and (8) has the conservation law $\mathrm{D}_iC^i=0$*, where*

$$\begin{aligned} C^i=\xi^i\mathscr{L}+Q^\sigma&\left(\frac{\partial\mathscr{L}}{\partial u^\sigma_i}-\mathrm{D}_j\left(\frac{\partial\mathscr{L}}{\partial u^\sigma_{ij}}\right)+\mathrm{D}_j\mathrm{D}_k\left(\frac{\partial\mathscr{L}}{\partial u^\sigma_{ijk}}\right)-\cdots\right)\\ &+\mathrm{D}_j(Q^\sigma)\left(\frac{\partial\mathscr{L}}{\partial u^\sigma_{ij}}-\mathrm{D}_k\left(\frac{\partial\mathscr{L}}{\partial u^\sigma_{ijk}}\right)+\cdots\right)\\ &+\mathrm{D}_j\mathrm{D}_k(Q^\sigma)\left(\frac{\partial\mathscr{L}}{\partial u^\sigma_{ijk}}-\cdots\right)+\cdots, \end{aligned} \tag{9}$$

with the characteristics

$$Q^\sigma=\phi_\sigma-\xi^k u^\sigma_k, \qquad \sigma=1,...,q, \tag{10}$$

and the formal Lagrangian

$$\mathscr{L}=v^\alpha\Delta_\alpha(x,u,\partial u,...,\partial^s u). \tag{11}$$

Now the theorem can be applied as follows. Let us

$$V=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}, \tag{12}$$

be an arbitrary symmetry of the SK, then the generator gives the conservation law

$$\mathrm{D}_x(C^1)+\mathrm{D}_t(C^2)|_{(1)}=0, \tag{13}$$

with the conserved vector components

$$\begin{aligned} C^1=\xi^1\mathscr{L}+Q&\left(\frac{\partial\mathscr{L}}{\partial u_x}-\mathrm{D}_x\left(\frac{\partial\mathscr{L}}{\partial u_{xx}}\right)+\mathrm{D}^2_x\left(\frac{\partial\mathscr{L}}{\partial u_{xx}}\right)+\mathrm{D}^4_x\left(\frac{\partial\mathscr{L}}{\partial u_{xxxxx}}\right)\right)\\ &+\mathrm{D}_x(Q)\left(\frac{\partial\mathscr{L}}{\partial u_{xx}}-\mathrm{D}_x\left(\frac{\partial\mathscr{L}}{\partial u_{xxx}}\right)-\mathrm{D}^3_x\left(\frac{\partial\mathscr{L}}{\partial u_{xxxxx}}\right)\right)\\ &+\mathrm{D}^2_x(Q)\left(\frac{\partial\mathscr{L}}{\partial u_{xxx}}+\mathrm{D}^2_x\left(\frac{\partial\mathscr{L}}{\partial u_{xxxxx}}\right)\right)+\mathrm{D}^3_x(Q)\left(-\mathrm{D}_x\left(\frac{\partial\mathscr{L}}{\partial u_{xxxxx}}\right)\right)\\ &+\mathrm{D}^4_x\left(\frac{\partial\mathscr{L}}{\partial u_{xxxxx}}\right), \end{aligned} \tag{14}$$

$$C^2=\xi^2\mathscr{L}+Q\frac{\partial\mathscr{L}}{\partial u_t}, \tag{15}$$

where $\mathscr{L}$ is as given in Eq. (2) and the characteristic (10) become

$$Q = \phi - \xi^1 u_x - \xi^2 u_t. \qquad (16)$$

The explicit form for the infinitesimal (12) provide the following 3-dimensional Lie algebra for the Eq. (1) spanned by the perators

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = \frac{x}{5}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{5}\frac{\partial}{\partial u}. \qquad (17)$$

In what follows, we use each of the Lie algebra (17) and the general non-linearity self -adjoint conditions of above theorem to find non-trivial conservation laws for the Eq. (1).

Conservation laws through V_1

For this symmetry operator the, the characteristic is

$$Q = -u_x.$$

Effecting this value into the vector components (14) and (15), yields

$$C^1 = u_t + 10u_x(u_x - u_{xx}), \qquad C^2 = -u_x.$$

Conservation laws through V_2

For this symmetry operator the, the characteristic is

$$Q = -u_t.$$

Effecting this value into the vector components (14) and (15) as well as above, yields

$$\begin{aligned} C^1 &= u_{xxxxt} + u_x u_{xxxx} + 20u^2 u_x^2 + 15u_x u_{xt} + 25u_x^2 u_{xx} + 10uu_{xxt} + u_x u_{xxxxx}, \\ C^2 &= -u_{xxxxx} - 10uu_{xxx} - 25u_x u_{xx} - 20u^2 u_x. \end{aligned}$$

Conservation laws through V_3

For this symmetry operator the, the characteristic is

$$Q = -\frac{2u}{5} - tu_t - \frac{x}{5}u_x.$$

Effecting this value into the vector components (14) and (15), yields

$$\begin{aligned} C^1 &= 8u^3 + 6u_x^2 + 2uu_{xx} + 4(1 + 10tu^2)u^2 u_x + 2(2x + 100tu^2)uu_{xxx} \\ &\quad + 5(3t + 5x + 100tu^2)u_x u_{xx} + \frac{1}{2}u_{xxxx} + tu_{xxxxt} + (20tu^2 + x)u_{xxxxx}, \\ C^2 &= \frac{2u}{5} + \frac{x}{5}u_x - 20tu^2 u_x - 25tu_x u_{xxx} - 10tuu_{xxx} - tu_{xxxxx}. \end{aligned}$$

Conservation laws constructed by direct method

As the basic definition of conservation laws, if one of the independent variables of the system (6) is time t the conservation laws takes the form

$$\mathrm{D}_i\Phi(x,u,\partial u,...,\partial^\ell u)+\mathrm{D}_t\Psi^i(x,u,\partial u,...,\partial^\ell u)=0, \tag{18}$$

where Ψ is reffered to as a *density* and Φ^i is spatial *fluxes* of the conservation law (18).

Consider the system of differential equations defined in (6). A set of differentiable function $\{\Lambda_\sigma(x,u,\partial u,...,\partial^\ell u)\}_{\sigma=1}^q$ called *local multipliers* yields a divergence expression for the PDE system (6) if the identity

$$\Lambda_\sigma(x,u,\partial u,...,\partial^\ell u)\Delta_\sigma\equiv \mathrm{D}_i\Phi^i(x,u,\partial u,...,\partial^\ell u),$$

on the solutions of the system. The following theorem connects local multipliers and conservation laws [4]. A set on non-singular local multipliers $\{\Lambda_\sigma(x,u,\partial u,...,\partial^\ell u)\}$ yields a conservation law for the system (6) if and only if the set of identities

$$\frac{\delta}{\delta u^\sigma}(\Lambda_\sigma(x,u,\partial u,...,\partial^\ell u)\Delta_\sigma)\equiv 0, \tag{19}$$

holds for arbitrary functions $u(x)$.

For instance for investigating the zeroth order set of multipliers of the Eq. (1), we should take the function $\Lambda(x,t,u)$ in the theorem which satisfies the determining equation (19) in the form of

$$\frac{\delta}{\delta u}(\Lambda(x,t,u)\Delta_{SK})\equiv 0,$$

for the variational derivative

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-\mathrm{D}_x\frac{\partial}{\partial u_x}-\mathrm{D}_t\frac{\partial}{\partial u_t}+\mathrm{D}_x^2\frac{\partial}{\partial u_{xx}} +\cdots+\mathrm{D}_x^5\frac{\partial}{\partial u_{xxxxx}}. \tag{20}$$

Expanding the left hand side of the equation (20), yields the following determining equation:

$$\Lambda_t+\Lambda_{xxxxx}+20u^2\Lambda_x+2\Lambda_u u_{xxxxx}+\cdots+20u\Lambda_u u_{xxx}+135\Lambda_u u_x u_{xx}=0. \tag{21}$$

Solving the equation (21) with respect to Λ and its derivatives shows that Λ is just a constant number. Thus, the density and the flux are

$$\Phi(x,t,u)=F(x,t), \qquad \Psi(x,t,u)=-\int F(x,t)dt+G(x).$$

Similarly, if we take the multiplier Λ as the first order multiplier $\Lambda(x,t,u,u_x,u_t)$, we obtain that it is just a constant number too. In this case the density and flux are:

$$\Phi(x,t,u,u_x,u_t)=F(x,t,u)u_t+\int^u F_x(x,t,u)dw+G(x,t),$$
$$\Psi(x,t,u,u_x,u_t)=-F(x,t,u)u_x-\int^u F_x(x,t,u)dw-\int G_x(x,t)dt+H(x).$$

Doing as well as the above case for zeroth, first and etc. order density and flux, we can set the following list of conservation laws in the form of (18) in the more simple polynomial form. Thus, the set of density and fluxes of the SK equation up to fourth order in the polynomial form is comming in table 1:

Таблица 1. Fluxes and densities of the KS Eq.

Flux	Density
$-x-u_{xxxx}-10uu_{xx}-\frac{15}{2}u_x^2-\frac{20}{3}u^3-u_{xt}-u_t$	$t+u+u_{xx}+u_x$
$-u_{tttt}-u_{xtt}-tu_{xttt}+u_{ttt}-xu_{xttt}$	$u_{xttt}+tu_{xxtt}+xu_{xxtt}$
$-u_{xxx}u_{tttt}-u_{ttt}u_{xxxt}-u_{xttt}$	$u_{xxtt}+xu_{xxxt}$
$-u_{xxx}u_{tttt}-u_{ttt}u_{xxxt}$	$u_{xxx}u_{xttt}+u_{ttt}u_{xxxx}$
$-u_{xxx}u_{xttt}-u_{xtt}u_{xxxt}$	$u_{xxx}u_{xxtt}+u_{xtt}u_{xxxx}$
$-u_{xxt}u_{xxxt}-u_{xxx}u_{xxxt}-u_xu_{xxxt}+u_{xx}u_{xxt}-u_{xt}u_{xxx}$	$u_{xxx}u_{xxxt}+u_{xxt}u_{xxxx}+u_{xxx}u_{xxxx}$ $+u_xu_{xxxx}$
$-u_{xxx}-tu_{xxxt}+u_{xxt}-xu_{xxxt}-u_{ttt}^2-u_{tt}u_{tttt}$	$tu_{xxxx}+xu_{xxx}+u_{tt}u_{xttt}+u_{xtt}u_{ttt}$
$-u_{xt}u_{tttt}-u_{xtt}u_{ttt}-u_{xxtt}$	$u_{xt}u_{xttt}+u_{xxt}u_{ttt}+u_{xxxt}$
$-u_{xx}u_{tttt}-u_{xxt}u_{ttt}-u_tu_{tttt}-u_{tt}u_{ttt}$	$u_{xx}u_{xttt}+u_{xxx}u_{ttt}+u_tu_{xttt}+u_{xt}u_{ttt}$
$-u_xu_{tttt}-u_{xt}u_{ttt}-uu_{tttt}-u_tu_{ttt}$	$u_xu_{xttt}+u_{xx}u_{ttt}+uu_{xttt}+u_xu_{ttt}$
$-u_xu_{xttt}-u_{xt}u_{xttt}$	$u_{xtt}^2+u_{tt}u_{xxtt}$
$-u_{xx}u_{xttt}-u_{xxt}u_{xtt}$	$u_{xx}u_{xxtt}+u_{xxx}u_{xtt}$
$-u_xu_{xttt}-u_{xt}u_{xttt}-uu_{xttt}-u_tu_{xtt}$	$u_xu_{xxtt}+u_{xx}u_{xtt}+uu_{xxtt}+u_xu_{xtt}$
$-u_t^2-u_{tt}-tu_{ttt}-xu_{ttt}$	$tu_{xtt}+tu_{xtt}$
$-u_{xt}u_{xxtt}-u_{xxt}u_{xtt}$	$u_{xt}u_{xxxt}+u_{xxt}^2$
$-u_{xx}u_{xxtt}-u_{xxt}^2-u_{xtt}-u_{xt}u_{xtt}$	$u_{xx}u_{xxtt}+u_{xxx}u_{xxt}+u_{xxt}+u_{xt}u_{xxt}$
$-u_xu_{xxtt}-u_{xt}u_{xxt}$	$u_xu_{xxxt}+u_{xx}u_{xxt}$
$-u_{xxtt}-u_tu_{xxt}-uu_{xxt}+u_xu_{xt}-u_tu_{xx}$	$uu_{xxx}+uu_{xxxt}+u_xu_{xxt}$
$u_{tt}-xu_{xtt}-u_{xt}-tu_{xtt}$	$tu_{xxt}+xu_{xxt}$
$-uu_{xxxt}-u_tu_{xxx}$	$uu_{xxxx}+u_xu_{xxx}$
$-u_tu_{ttt}-u_{tt}^2-xu_{xxt}+u_{xt}-u_{xx}-tu_{xxt}$	$u_tu_{xtt}+u_{xt}u_{tt}+xu_{xxx}+tu_{xxx}$
$uu_{xttt}-u_xu_{ttt}+u_tu_{xtt}-u_{xt}u_{tt}$	$-uu_{xxtt}+u_{xx}u_{tt}$
$-uu_{ttt}-u_tu_{tt}-u_tu_{xtt}-u_{xt}u_{tt}$	$uu_{xtt}+u_xu_{tt}+u_tu_{xxt}+u_{xt}^2$
$uu_{xxtt}-u_xu_{xtt}+u_tu_{xxt}-u_{xt}^2$	$-uu_{xxxt}+u_{xx}u_{xt}$
$-uu_{xtt}-u_tu_{xt}-u_tu_{tt}$	$uu_{xxt}+u_xu_{xt}+u_tu_{xt}$
$-xu_{tt}-u_t-tu_{tt}+uu_{xtt}-u_xu_{tt}-u_{xxt}$	$u_t+xu_t+tu_{xt}-uu_{xxt}+u_tu_{xx}+u_{xxx}$
$uu_{xxxt}-u_xu_{xxt}+u_tu_{xxx}-uxxu_{xt}$	$-uu_{xxxx}+u_{xx}^2$
$uu_{xtt}-u_xu_{tt}-tu_{xt}-u_x-xu_{xt}+u_t$	$tu_{xx}+xu_{xx}-uu_{xxt}+u_tu_{xx}$
$-uu_{xt}-u_xu_t-uu_{tt}-uu_{tt}$	$uu_{xx}+u_x^2+uu_{xt}+u_xu_t$
$\frac{20}{3}u^3+10uu_{xx}+\frac{15}{2}u_x^2-xu_t+u_{xxxx}-u_{tt}-2xt$	$xu_x+u_{xt}+t^2$

Similarity reduction

In this section we make some discussion on the SK equation based on the symmetries (17), [5, 6, 16, 17].

(I) V_1

For the generator V_1, we have

$$u = \varphi(r,s), \tag{22}$$

where $r = x, s = t$ are the group-invariants. Substituting (22) in (1), one can get

$$\varphi''''' + 10\varphi\varphi''' + 25\varphi\varphi'' + 20\varphi^2\varphi' = 0,$$

where $\varphi', \varphi'', \ldots$ are derivatives of φ with respect to r.

(II) V_2

For the generator V_2 we get

$$u = \varphi(r,s), \tag{23}$$

where $r = t, x = s$ are the group-invariants. Substituting (23) in (1), we reduce it to the following ODE

$$\varphi' = 0.$$

(II) V_3

For the generator V_2 we get

$$u = \exp\left(-\frac{2}{5}s\right)\varphi(r,s), \tag{24}$$

where

$$r = -\frac{x^2}{5} + t^2, \qquad s = -\ln\left(\sqrt{5}(\sqrt{5}x - 5t)\right).$$

are the group-invariants. Substituting (24) in (1), we reduce it to the following so complicated equation!

$$\exp\left(s(\sqrt{5}-\frac{6}{5})\right)\Bigg(15625000\exp\left(-\frac{4s}{\sqrt{5}}\right)\varphi''' + 187500000\exp\left(-\frac{2}{5}s(\sqrt{5}-1)\right)\varphi''$$
$$-25000\exp\left(-\frac{2}{5}s(3\sqrt{5}-2)r\right)\varphi'''' + 250\exp\left(-\frac{4}{5}s(2\sqrt{5}-1)\right)\varphi''''$$
$$+\cdots - 17187500\exp\left(-\frac{2}{5}s(2\sqrt{5}-1)\right)r\varphi\varphi'' - 234375000\sqrt{5}\exp\left(\frac{4}{5}s\right)r^2\varphi''$$
$$+460937500\sqrt{5}\exp\left(\frac{4}{5}s\right)r^3\varphi''' + 9765625\exp\left(\frac{4}{5}s\right)r^5\varphi'''''\Bigg) = 0.$$

(IV) $V_1 + V_2$

The symmetry $V_1 + V_2$ yields the following invariants

$$r = t - s, \qquad ,s = x, \qquad u = \varphi(r,s).$$

Treating φ as the new dependent variable and r as new independent variable, the KS equation transforms to

$$\varphi''''' + 10\varphi\varphi''' + 25\varphi\varphi'' + (20\varphi^2 + 1)\varphi' = 0.$$

Conclusion

In this paper, by using the Lie symmetry groups, we studied the symmetry properties and similarity reduction forms of the (1+1)-dimensional SK equation. Moreover, we also derived the non-linear self-adjonitness of Eq. (1), by virtue of this fact, some conservation laws through Lie symmetries are given. The direct method is used to find many new conservation laws for the Eq. (1) in the polynomial form.

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