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MATHEMATICAL MODELING

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MATHEMATICAL MODELING OF THE LAW OF CLOUD DROPLET CHARGE CHANGE IN FRACTAL ENVIRONMENT

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The paper proposes a new mathematical model of cloud droplet charge change in storm clouds. The model takes into account the fractal properties of storm clouds, and the solution was obtained using the apparatus of fractional calculus.

Key words: fractal dimension, the mathematical model, operator Riemann-Liouville, operator Caputo

Introduction

During the last decay many geophysicists study intensively the fractality of environment structures and its effect on different geophysical processes. A cloud also refers to such natural phenomena where the question on electric charge formation and separation is a topical one. Many researches are devoted to the investigation of the regularities of electric charge separation in clouds. The main results are summarized in classical papers [1]-[9] where many explanations are presented not taking into account environment fractality. The results of the study in this area show that one of the important prerequisites for electric charge separation in clouds are the ice phase (ice crystals, small hail and hailstones) and supercooled water droplets [10].

It is known that clouds with intensive convective currents have fractal structure and a cloud is a fractal environment [11]. Thus, we may state that the processes occurring in such an environment are well described by the apparatus of fractional calculus.

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Problem definition and solution

From Frenckel's theory [13] in the paper [12], average charge q_r which is generated by one cloud droplet with radius r was obtained for cloud droplets in slightly ionized air environment in the form

$$q_r = 4\pi\varepsilon_0 n\zeta a, \tag{1}$$

where ε_0 is the electric constant; a is the bubble radius; ζ is the electrokinetic potential; n is the number of bubbles with radius a formed in a cloud droplet with radius r. Thus, relying upon the Frenckel's theory, the droplet total charge may be written in the following form:

$$q(x,t) = 4\pi\varepsilon_0\zeta R(x,t), \tag{2}$$

where $R(x,t)$ is the droplet radius.

The law of droplet charge change may have the form:

$$\frac{\partial q(x,t)}{\partial t} = 4\pi\varepsilon_0\zeta\frac{R(x,t)}{\partial t}. \tag{3}$$

In equation (3) the value $j(x,t) = \dfrac{\partial q(x,t)}{\partial t}$ is the charge flux which depends on the velocity of droplet radius change $\dfrac{R(x,t)}{\partial t}$ coinciding with the diffusive flux by the droplet surface if they grow due to the diffusion from the surrounding environment [14].

Since the process takes place in a fractal environment, than instead of the model (3) we consider the law of droplet charge change taking into account the fractality. But before the consideration of the law of droplet charge change, it is necessary to consider the droplet size change taking into account the fractality as long as charge change on the whole occurs due to the drop size change.

It is known [15] that flux equation is expressed by the formula

$$q(x,t) = -kD_{ax}^{\alpha}u(x,t),\ 0 < \alpha < 1, \tag{4}$$

where k is «diffusion» coefficient; $u(x,t)$ is the concentration (temperature and so on), D_{ax}^{α} is the integrodifferentiating operator in the sense of Riemann-Liouville of fraction order α with the initial point a which is determined as follows [16]:

$$D_{ax}^{\alpha}u(\xi,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int\limits_a^x \frac{u(\xi,t)\,d\xi}{(x-\xi)^{\alpha}}.$$

The substitution of $\partial/\partial t$ by D_{ax}^{α} in differential equations includes implicitly the additional factors of physical system interaction. Thus, we may state that equation (4) describes a fractal process [15].

Taking into account the relations $j(x,t) = \dfrac{\partial q(x,t)}{\partial t}$ from (3) and (4) we obtain:

$$j(x,t) = -\frac{k}{4\pi\varepsilon_0\zeta}D_{0t}^{\alpha}R(x,t). \tag{5}$$

Denoting by $\lambda = -\dfrac{k}{4\pi\varepsilon_0\zeta}$ and substituting the flux value $j(x,t)$, formula (5) с учетом (3) has the form:

$$\frac{\partial R(x,t)}{\partial t} - \lambda D_{0t}^{\alpha}R(x,t) = 0. \tag{6}$$

Formula is the partial differential equation of the first order. We add the starting and edge values to equation (6) [11]:

$$R(x,0) = r_1(x), x \in [0,L], \tag{7}$$

$$\lim_{x\to 0} D_{0x}^{\alpha-1} R(x,t) = r_2(t), t \in [0,T], \tag{8}$$

Solution of the problems (6)-(8) has the following form [17]:

$$R(x,t) = \int_0^x \frac{r_1(s)}{x-s} e_{1,\alpha}^{1,0}\left(\frac{-\lambda t}{(x-s)^\alpha}\right) ds + \lambda \int_0^t \frac{r_2(\eta)}{x} e_{1,\alpha}^{1,0}\left(\frac{-\lambda(t-\eta)}{x^\alpha}\right) d\eta. \tag{9}$$

where $e_{\alpha,\beta}^{\nu,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \nu)\Gamma(\delta - \beta n)}$ is the Wright-type function.

Substituting (9) into formula (2), we obtain the expression for droplet charge taking into account the environment fractality.

$$q(x,t) = 4\pi\varepsilon_0\zeta \left[\int_0^x \frac{r_1(s)}{x-s} e_{1,\alpha}^{1,0}\left(\frac{-\lambda t}{(x-s)^\alpha}\right) ds + \lambda \int_0^t \frac{r_2(\eta)}{x} e_{1,\alpha}^{1,0}\left(\frac{-\lambda(t-\eta)}{x^\alpha}\right) d\eta\right]. \tag{10}$$

Considering(10) charged particle flux has the form:

$$j(x,t) = 4\pi\varepsilon_0\zeta \frac{\partial}{\partial t} \left[\int_0^x \frac{r_1(s)}{x-s} e_{1,\alpha}^{1,0}\left(\frac{-\lambda t}{(x-s)^\alpha}\right) ds + \lambda \int_0^t \frac{r_2(\eta)}{x} e_{1,\alpha}^{1,0}\left(\frac{-\lambda(t-\eta)}{x^\alpha}\right) d\eta\right]. \tag{11}$$

Applying the following rule:

$$I'(t) = \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t} f(x,t)\, dx + f(x_2(t),t)\, x_2'(t) - f(x_1(t),t)\, x_1'(t),$$

to 11), we obtain

$$\begin{gathered} j(x,t) = 4\pi\varepsilon_0\zeta \left[\int_0^x \frac{r_1(s)}{x-s} \frac{\partial}{\partial t} e_{1,\alpha}^{1,0}\left(\frac{-\lambda t}{(x-s)^\alpha}\right) ds + \lambda \frac{\partial}{\partial t} \int_0^t \frac{r_2(\eta)}{x} e_{1,\alpha}^{1,0}\left(\frac{-\lambda(t-\eta)}{x^\alpha}\right) d\eta\right] = \\ = 4\pi\varepsilon_0\zeta \int_0^x \frac{r_1(s)}{x-s} \frac{\partial}{\partial t} e_{1,\alpha}^{1,0}\left(\frac{-\lambda t}{(x-s)^\alpha}\right) ds + \lambda \frac{r_2(t)}{x} e_{1,\alpha}^{1,0}(0) + \\ + \lambda \int_0^t \frac{r_2(\eta)}{x} \frac{\partial}{\partial t} e_{1,\alpha}^{1,0}\left(\frac{-\lambda(t-\eta)}{x^\alpha}\right) d\eta. \end{gathered} \tag{12}$$

Considering properties [17]:

$$e_{1,\alpha}^{1,0}(0) = 1,\ D_{0t}^{\nu} z^{\delta-1} e_{\alpha,\beta}^{\mu,\delta}(\lambda z^\alpha) = z^{\delta-\nu-1} e_{\alpha,\beta}^{\mu-\nu,\delta}(\lambda z^\alpha)$$

Result in the final form:

$$j(x,t) = 4\pi\varepsilon_0\zeta\left(\frac{\lambda r_2(t)}{x} + \int\limits_0^x \frac{r_1(s)}{(x-s)t} e_{1,\alpha}^{0,0}\left(\frac{-\lambda t}{(x-s)^\alpha}\right) ds\right) + \tag{13}$$

$$+4\pi\varepsilon_0\zeta\lambda \int\limits_0^t \frac{r_2(\eta)}{xt} e_{1,\alpha}^{0,0}\left(\frac{-\lambda(t-\eta)}{x^\alpha}\right) d\eta.$$

Expression (13) is the law of cloud droplet charge change considering the environment fractality by the Wright-type function.

In the paper [11], an equation of (4) type with Caputo fractional derivative operator was obtained:

$$q(x,t) = \gamma\partial_{0t}^\alpha u(x,\tau),\ 0<\alpha<1, \tag{14}$$

where $\gamma>0$, $\partial_{0t}^\alpha u(x,\tau) = D_{0t}^{\alpha-1}\dfrac{du(x,\tau)}{d\tau}$ is the regularized fractional derivative of the order α from function $u(x,\tau)$ with initial and end points 0 and τ (Caputo derivative). Taking into account formula (14) and the law of droplet size change, formula (3) is written in the form:

$$\partial_{0t}^\alpha R(t) - kR(t) = 0, \tag{15}$$

where $k=\dfrac{1}{\gamma}$. Formula (15) is an ordinary differential equation of fractional order. Add an initial condition to equation (15):

$$R(x,0) = R_0. \tag{16}$$

Since $f(x)=0$, the solution of problem (16) for equation (15) has in general view the following form:

$$R(t) = R_0 E_{\alpha,1}(kt^\alpha), \tag{17}$$

where $E_{\alpha,\beta}(z) = \sum\limits_{k=0}^{\infty}\dfrac{z^k}{\Gamma(\alpha k+\beta)}$ is the Mittag-Leffler-type function [17]. Substituting (17) into the corresponding formulas for the charge and charged droplet flux, we obtain

$$q(t) = 4\pi\varepsilon_0\zeta R_0 E_{\alpha,1}(kt^\alpha),\ j(t) = 4\pi\varepsilon_0\zeta R_0 t^{\alpha-1}E_{\alpha,\alpha}(kt^\alpha). \tag{18}$$

Formula (18) is the law of droplet charge change in Frenkel's generalized theory in cloud environment by means of Mittag-Leffler function.

Conclusions

Considering the clouds which are known to have different structure and have different classification in origin and morphological features to which the data on their fractal structure may be added, formation of a more general view of cloud physics state is possible in the future. The paper suggests the mathematical model for the droplet charge change in fractal cloud environment generalizing Frenkel's theory. The solution of this model was obtained taking into account Write- and Mittag-Leffler-type functions.

References

1. Kachurin L. G., Morachevskii V.G. Kinetika fazovykh perekhodov vody v atmosfere [Kinetics of water phase transitions in the atmosphere]. Leningrad, Gidrometeoizdat, 1965, 114 p.
2. Mason B.J. Fizika oblakov [Cloud Physics]. Leningrad, Gidrometeoizdt, 1961, 542 p.
3. Muchnik V. M. Fizika grozy [Thunderstorm physics]. Leningrad, Gidrometeoizdt, 1974, pp. 252-257.
4. Chalmers J.A. Atmosphernoe electrichestvo [Atmospheric electricity]. 1974, 420 p.
5. Uman M. Molniya [Lightning]. Moscow, Mir, 1972, 328 p.
6. Ribeira J.C. On the thermo-dielectric effect. Ann. Acad. Brasil. Sci. 1950. vol. 22. №3. P. 547-556.
7. Workman E. J., Reynold S.E. Electrical phenomena occurring during the freezing of delute aqueous solution and their possible relationship to thunderstorm electricity. Phys.Rev. 1956. vol. 94. №4. P. 1073-1075.
8. Workman E. J. The possible role of ammonia in thunderstorm electrification. In: Proc. Intern. Conf. Cloud Phys.. Toronto. 1968. P. 653-656.
9. Imyanitov I. M. Elektricheskaya struktura konvektivnykh oblakov i ee svyaz' s dvizheniem vozdukha v oblakakh. Issledovanie oblakov, osadkov i grozovogo elektrichestva [Electric structure of convective clouds and its relation with air motion in clouds. Investigation of clouds, precipitation and lightning electricity]. Moscow, Gidrometeoizdat, 1961, pp. 225-238.
10. Adzhiev A.Kh. Kupovykh G.V. Atmosferno-elektricheskie yavleniya na Severnom Kavkaze [Atmospheric-electric phenomena in the Northern Caucasus]. Taganrog, 2004, p. 122.
11. Nakhushev A.M. Drobnoe ischislenie i ego primenenie [Fractional calculus and its application]. Moscow, Fizmatlit, 2003, 272 p.
12. Kumykov T.S., Zhekamukhov M.K., Karov B.G. Elektrizatsiya i prostranstvennoe razdelenie zaryadov pri vydelenii puzyr'kov vozdukha v protsesse koagulyatsionnogo rosta gradin v oblake II. Generirovanie grozovogo elektrichestva za schet vydeleniya zaryazhennykh puzyr'kov pri namerzanii pereokhlazhdennykh oblachnykh kapel' na poverkhnosti gradin [Electrification and spatial separation of charges during air bubbling in the process of hailstone coagulation growth in cloud II. Generation of thunderstorm electricity due to charged bubble formation during undercooled cloud droplet icing on hailstone surface]. Meteorologiya i gidrologiya – Meteorology and Hydrology, 2008, No.12, pp. 15-24.
13. Frenkel' Ya.I. Teoriya osnovnykh yavlenii atmosfernogo elektrichestva [Theory of main phenomena of atmospheric electricity]. Collected papers, Moscow, Nauka, 1958, I.2, pp.538-567.
14. Lifshits E.M., Pitaevskii L.P. Fizicheskaya kinetika [Physics kinetics]. Moscow, Nauka, 1979, 527
15. Shogenov V.Kh., Shkhanukov-Lafishev M.Kh., Kh.M. Beshtoev Drobnye proizvodnye: interpretatsiya i nekotorye primeneniya v fizike. [Fractional derivatives: interpretation and some applications in physics]. Papers of the Joint Institute for Nuclear Research, Dubna, 1997. 20 p.
16. Nakhushev A.M. Uravneniya matematicheskoi biologii [Equations of mathematical biology]. Moscow, Vysshaya shkola, 1995, 301 p.
17. Pskhu A.V. Kraevye zadachi dlya differentsial'nykh uravnenii s chastnymi proizvodnymi drobnogo i kontinual'nogo poryadka [Boundary problems for differential equations with partial derivatives of fractional and continual orders]. Nalchik, KBNTs RAN, 2005, p.185, p.22.

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