DOI: 10.18454/2079-6641-2017-17-1-22-32

MSC 76W05, 86A25

ABOUT A PROBLEM FOR THE DEGENERATING MIXED TYPE EQUATION FRACTIONAL DERIVATIVE

B. I. Islomov[1], N. K. Ochilova[2]

[1] National University of Uzbekistan, 100125, Tashkent, Vuzgorodok, Universitetskaya str.4, Uzbekistan
[2] Tashkent financial institute,100000, Tashkent, Amir Temur-57.Uzbekistan
E-mail: nargiz.ochilova@gmail.com

The existence and the uniqueness of solution of local problem for degenerating mixed type equation is investigated. Considering parabolic-hyperbolic equation involve the Caputo fractional derivative. The uniqueness of solution is proved using the method of the extremume principle and integral energy, the existence is proved by the method of integral equations.

Keywords: boundary value problem, degenerating equation, parabolic-hyperbolic type, Gauss hypergeometric function, Cauchy problem, existence and uniqueness of solution, a principle an extremum, method of integral equations, Caputo fractional derivative.

УДК 517.956

О ЗАДАЧЕ ДЛЯ ВЫРОЖДАЮЩЕГОСЯ УРАВНЕНИЯ СМЕШАННОГО ТИПА С ДРОБНОЙ ПРОИЗВОДНОЙ

Б. И. Исломов[1], Н. К. Очилова[2]

[1] Национальный университетет Узбекистана, 100125, Ташкент, Вуз-городок, ул. Университетская 4, Республика Узбекистан
[2] Ташкенский финансовый институт, 100000, Ташкент, ул. Амира Тимура, 57, Республика Узбекистан
E-mail: nargiz.ochilova@gmail.com

Исследуется существование и единственность решения локальной задачи для вырождающегося уравнения смешанного типа. Рассматривается параболическо-гиперболическое уравнение с дробной производной Капуто. Единственность решения доказана с использованием экстремального принципа и интеграла энергии, существование доказано методом интегральных уравнений.

Ключевые слова: краевая задача, вырождающееся уравнение, параболо-гиперболический тип, гипергеометрическая функция Гаусса, задача Коши, существование и единственность решения, принцип экстремума, метод интегральных уравнений, дробная производная Капуто.

Introduction

In the works [1],[2],[3],[4] we can see significant development in the fractional differential equations in recent years. The fractional calculus is widely applied to investigation of partial differential equations of mixed type and hyperbolic type with degenerations (see [4],[5],[6]). In a series of papers (see [7], [8],[9]) the authors considered some classes of boundary value problems for mixed type non degenerating and degenerating differential equations involving Caputo and Riemann-Liouville fractional derivatives of order $0 < \alpha \leq 1$.

Preliminaries

Definition. Caputo fractional derivatives ${}_CD^{\alpha}_{ax}f$ and ${}_CD^{\alpha}_{xb}f$ of order $\alpha > 0$, $(\alpha \notin N \cup \{0\})$ are defined by [1.p.92]:

$$({}_CD^{\alpha}_{ax}f)x = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, n = [\alpha]+1, x > a; \tag{1}$$

$$({}_CD^{\alpha}_{xb}f)x = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, n = [\alpha]+1, x < b; \tag{2}$$

respectively.
From (1), (2), as a conclusion we will have: $k-1 < \alpha \leq k$, $k \in N$;
consequently, while for $\alpha \in N \cup \{0\}$ we have

$$\left({}_CD^{0}_{ax}f\right)x = f(x),\ \left({}_CD^{0}_{xb}f\right)x = f(x),\ ({}_CD^{n}_{ax}f)x = f^{(n)}(x);$$

$$({}_CD^{n}_{xb}f)x = (-1)^n f^{(n)}(x), n \in N.$$

Gauss hypergeometric function $F(a,b,c,z)$is defined in the unit desk as the sum of the hypergeometricseries (see [1. p.27]):

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},. \tag{3}$$

where

$$|z| < 1,$$

$a, b \in \mathbb{C} c \in \mathbb{C} \backslash Z_0^-$ and $(a)_0 = 1, \quad (a)_n = a(a+1)...(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$, $(n = 1,2,...)$.

One such analytic continuation is given by Eyler integral representation:

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx, \tag{4}$$

$$0 < Reb < Rec, \quad |\arg(1-z)| < \pi.$$

The Gauss hypergeometric function $F(a,b,c,z)$ allows the following estimation:

$$F(a,b,c;z) \leq \begin{cases} c_1, & if \quad c-a-b > 0, \quad 0 \leq z \leq 1 \\ c_2(1-z)^{c-a-b}, & if \quad c-a-b < 0, \quad 0 < z < 1 \\ c_3\left(1+|\ln(1-z)|\right), & if \quad c-a-b = 0 \end{cases}. \tag{5}$$

$$F(a,1-a,c,z)=(1-z)^{c-1}F\left(\frac{c-a}{2},\frac{c+a-1}{2},c,4z(1-z)\right) \tag{6}$$

Generalized fractional integro-differential operators with Gauss hypergeometric function$F(a,b,c;z)$defined for real $a,b,c$and $x>0$will be givenby formulate:

$$F_{ox}\left[\begin{array}{c} a,b \\ c,x^k \end{array}\right]f(x)=\frac{1}{\Gamma(c)}\int_0^x f(t)(x^k-t^k)^{c-1}F\left(a,b,c;\frac{x^k-t^k}{x^k}\right)kt^{t-1}dt,$$

c>0, k>0

(7)

The elementary definition of the Wright type function at $\alpha>\beta$, $\alpha>0$ and for all $z\in\mathbb{C}$, is [10]

$$e_{\alpha,\beta}^{\mu,\delta}(z)=\sum_{k=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\mu)\Gamma(\delta-\beta n)}. \tag{8}$$

If $\alpha=\mu=1$, then owing to (3) from (8) we have:

$$e_{1,\beta}^{1,\delta}(z)=\sum_{k=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\beta n)}. \tag{9}$$

Problem formulation and main functional relation

This work deals the existence and uniqueness of solution of the problem for the mixed type equation with two lines and different order of degenerating which involve the Caputo fractional derivative.

We consider equation:

$$0=\left\{\begin{array}{l} u_{xx}-{}_CD_{oy}^{\alpha}u, \text{ at } y>0 \\ (-y)^m u_{xx}-x^n u_{yy}, \text{ at } y<0 \end{array}\right. \tag{10}$$

with operators (*see* (1)):

$${}_CD_{oy}^{\alpha}u=\frac{1}{\Gamma(1-\alpha)}\int_0^y (y-t)^{-\alpha}u_t(x,t)dt, \tag{11}$$

where $0<\alpha<1$, $m,n=const$.

Let's Ω domain, bounded with segments:

$$A_1A_2=\{(x,y):\ x=0,\ 0<y<h_2\},$$

$B_1B_2=\{(x,y):\ x=h_1,\ 0<y<h_2\}$,$A_2B_2=\{(x,y):\ y=h_2,\ 0<x<h_1\}$at the $y>0$,

and by the characteristics: $A_1C:\ \frac{1}{q}x^q-\frac{1}{p}(-y)^p=0$,$B_1C:\ \frac{1}{q}x^q+\frac{1}{p}(-y)^p=1$; of equation (10) at $y<0$, where $A_1\,(0;0)$,$A_2\,(0;h_2)$,$B_1\,(h_1;0)$,$B_2\,(h_1;h_2)$and $C\left(\left(\frac{q}{2}\right)^{1/q},-\left(\frac{p}{2}\right)^{1/p}\right)$. Here $2q=n+2$, $2p=m+2$, $h_1=q^{1/q}$, $h_2>0$, and that $m>n$. Introduce designations: $2\alpha_1=n/(n+2)$, $2\beta_1=m/(m+2)$,

$$0<\alpha_1<\beta_1<\frac{1}{2}, \tag{12}$$

$\Omega^+ = \Omega \cap (y>0), \Omega^- = \Omega \cap (y<0), I_1 = \{x : 0 < x < h_1\}, I_2 = \{y : 0 < y < h_2\}$.

For the equation (10), we consider the following problem: Find a solution $u(x,y)$of equation (10) from the following class of functions:
$\Delta = \left\{ u(x,y) : \ u(x,y) \in C(\bar{\Omega}) \cap C^2(\Omega^-) \ u_{xx} \in C(\Omega^+), {}_C D^{\alpha}_{oy} u \in C(\Omega^+) \right\}$
satisfies boundary conditions:

$$u(x,y)\Big|_{A_1A_2} = \varphi_1(y), 0 \le y \le h_2, \tag{13}$$

$$u(x,y)\Big|_{B_1B_2} = \varphi_2(y), h_1 \le y \le h_2, \tag{14}$$

$$u(x,y)|_{A_1C} = h(x), \ x \in I_1. \tag{15}$$

and gluing condition:

$$\lim_{y \to +0} y^{1-\alpha} u_y(x,y) = u_y(x,-0), (x,0) \in A_1B_1 \tag{16}$$

where $\varphi_1(y)$, $\varphi_2(y)$, $h(x)$ are given functions.

In fact, that functional relation between $\tau(x)$ and $\nu(x)$ transferred from the parabolic part Ω^+ (hyperbolic part $\Omega^-$to the line $y=0$) is played important role on the proved unique and existence of solution.

It is well know, that the solution of the Cauchy problem for the equation (10) in domain Ω^- satisfies conditions

$$u(x,-0) = \tau^-(x), \ 0 \le x \le 1 \ \ and \ \ u_y(x,-0) = \nu^-(x), \ 0 < x < 1, \tag{17}$$

presented on the form [11]:

$$u(x,y) = \frac{\Gamma(2\alpha_1)}{\Gamma^2(\alpha_1)} \left(\frac{1}{q}x^q\right)^{-\alpha_1} \int_0^1 \left[\tfrac{1}{p}(-y)^p(2z-1) + \frac{1}{q}x^q\right]^{\alpha_1} [z(1-z)]^{\beta_1 - 1} \times$$

$$\times \tau^- \left\{ \left[\frac{q}{p}(-y)^p \cdot 2z - 1 + x^q\right]^{\frac{1}{q}} \right\} F(\alpha_1, 1-\alpha_1, \beta_1, \rho) dz -$$

$$-\frac{\Gamma(1-2\alpha_1)}{\Gamma^2(1-\alpha_1)} p^{-2\beta_1} \left(\frac{1}{p}(-y)^p\right)^{1-2\beta_1} \int_0^1 \left[\frac{1}{p}(-y)^p(2z-1) + \frac{1}{q}x^q\right]^{\alpha_1} [z(1-z)]^{-\beta_1} \times$$

$$\times \nu^- \left\{ \left[\frac{q}{p}(-y)^p \cdot (2z-1) + x^q\right]^{\frac{1}{q}} \right\} F(\alpha_1, 1-\alpha_1, \beta_1, \rho) dz \tag{18}$$

where $\rho = \frac{q(-y)^{\frac{1}{p}} z(1-z)}{p^2 x^q \left[\frac{1}{p}(-y)^p(2z-1) + \frac{1}{q}x^q\right]}$.

Due to condition (15) from (18), by using formulate (6) and (7) we will get

$$h^*(x) = \gamma_1 (x^{2q})^{\frac{2-\alpha_1-3\beta_1}{2}} F_{0x} \left[\begin{array}{c} \frac{\beta_1-\alpha_1}{2}, \ \frac{\alpha_1+\beta_1-1}{2} \\ \beta_1, \ x^{2q} \end{array} \right] (x^{2q})^{\frac{\alpha_1+\beta_1-2}{2}} \tau^-(x) -$$

$$-\gamma_2 (x^{2q})^{\frac{\beta_1-\alpha_1}{2}} F_{0x} \left[\begin{array}{c} \frac{1-\beta_1-\alpha_1}{2}, \frac{\alpha_1-\beta_1}{2} \\ 1-\beta_1, \ x^{2q} \end{array} \right] (x^{2q})^{\frac{\alpha_1-\beta_1-1}{2}} \nu^-(x), (x,0) \in I_1, \tag{19}$$

where $h^*(x)=h\left[\left(\frac{x^q}{2}\right)^{1/q}\right]$, $\gamma_1=\frac{\Gamma(2\alpha_1)}{\Gamma^2(\alpha_1)}2^{\alpha_1-\beta_1}$, $\gamma_2=\frac{2^{\alpha_1+3\beta_1-2}\Gamma(1-2\alpha_1)}{\Gamma(1-\alpha_1)}\left(\frac{p}{q}\right)^{1-2\alpha_1}$.

Applying operator

$$\frac{d}{d(x^{2q})}(x^{2q})^{\frac{1-\alpha_1-\beta_1}{2}}F_{0x}\left[\begin{array}{c}\frac{\alpha_1+\beta_1-1}{2},\ \frac{\alpha_1+\beta_1}{2}\\ \beta_1,\ x^{2q}\end{array}\right](x^{2q})^{\frac{2\alpha_1-1}{2}}$$

to both parts of the equality (19), we obtain functional relation between $\tau^-(x)$ and $\nu^-(x)$ transferred from hyperbolic domain Ω^- on the line $y=0$:

$$\tilde{\nu}^-(x)=\frac{\gamma_1}{\gamma_2}x^{\frac{1-2\alpha_1}{2}}\frac{d}{dx}x^{\frac{1-2\beta_1}{2}}F_{0x}\left[\begin{array}{cc}\alpha_1+\beta_1, & \frac{2\beta_1-1}{2}\\ 2\beta_1, & x\end{array}\right]x^{\frac{2\alpha_1-1}{2}}\tilde{\tau}^-(x)-\frac{x^{\frac{1-\alpha_1+\beta_1}{2}}}{\gamma_2}h^*(x), 0<x<h_1, \tag{20}$$

where $\tilde{\tau}^-(x)=u\left[\left((qk)^2x\right)^{1/2q},0\right]$, $\quad\tilde{\nu}^-(x)=u_y\left[\left((qk)^2x\right)^{1/2q},-0\right]$.

On the other hand, considering designations (17) and $\lim\limits_{y\to+0}y^{1-\alpha}u_y(x,y)=\nu^+(x)$, $0<x<h_1$ from gluing condition (16) we have

$$\nu^+(x)=\nu^-(x) \tag{21}$$

For further supposes, from Eq. (10) at $y\to+0$ considering (11), (21) and

$$\lim_{y\to0}D_{0y}^{\alpha-1}f(y)=\Gamma(\alpha)\lim_{y\to0}y^{1-\alpha}f(y)$$

we get:

$$\tau''(x)-\Gamma(\alpha)\nu^+(x)=0 \tag{22}$$

Uniqueness of the solution

Theorem 1. *If satisfy conditions* $0<\alpha<1$*and* (12) *then, the solution is unique.*

Proof.

As usual we consider corresponding homogeneous problem $[\varphi_1(y)\equiv\varphi_2(y)\equiv0]$ and prove that $u(x,y)\equiv0$. With this aim we multiply to $\tau(x)$ equation (22) and integrate from 0 to h_1:

$$\Gamma(\alpha)\int_0^{h_1}\tau(x)\nu^+(x)dx=\int_0^{h_1}\tau''(x)\tau(x)dx. \tag{23}$$

Integrating by part and using the relations $\tau(0)=\tau(h_1)=0$,owing to (21) we obtain

$$\int_0^{h_1}\tau(x)\nu^-(x)dx=\int_0^{h_1}\tau(x)\nu^+(x)dx=-\int_0^{h_1}\left(\tau'(x)\right)^2dx\le0. \tag{24}$$

Now, we prove that $\int_0^{h_1}\tau(x)\nu^-(x)dx\ge0$.

At first, using by formulate (7) we make some simplifications in (20):

$$\nu^-(x)=\frac{\gamma_1\left(x^{2q}\right)^{\frac{1-2\alpha_1}{2}}}{\gamma_2\Gamma(2\beta_1)}\frac{d}{dx^{2q}}\left(x^{2q}\right)^{\frac{1-2\beta_1}{2}}\int_0^x\left(x^{2q}-t^{2q}\right)^{2\beta_1-1}\left(t^{2q}\right)^{\frac{2\alpha_1-1}{2}}\times$$

$$\times\tau^-(t)F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{x^{2q}-t^{2q}}{x^{2q}}\right)dt^{2q}-\frac{\left(x^{2q}\right)^{\frac{1-\alpha_1+\beta_1}{2}}}{\gamma_2}h^*(x), \tag{25}$$

Entering replacement$t=xz$ and after some simplifications we have:

$$\nu^{-}(x)=\frac{2q\gamma_1\left(\alpha_1+\beta_1\right)\left(x^{2q}\right)^{\frac{2\beta_1-1}{2}}}{\gamma_2\Gamma(2\beta_1)}\int_0^1\left(1-z^{2q}\right)^{2\beta_1-1}\left(z^{2q}\right)^{\frac{2\alpha_1+1}{2}}\times$$

$$\times\tau^{-}(xz)F\left(\beta_1-\alpha_1,\frac{2\beta_1+1}{2},2\beta_1,1-z^{2q}\right)dz+$$

$$+\frac{2q\gamma_1\left(x^{2q}\right)^{\frac{2\beta_1+1}{2}}}{\gamma_2\Gamma(2\beta_1)}\int_0^1\left(1-z^{2q}\right)^{2\beta_1-1}z^{2q}\,\tau'^{-}(xz)\times$$

$$\times F\left(\beta_1-\alpha_1,\frac{2\beta_1+1}{2},2\beta_1,1-z^{2q}\right)dz-\frac{\left(x^{2q}\right)^{\frac{1-\alpha_1+\beta_1}{2}}}{\gamma_2}h^{*}(x),$$

Consequently, using invers replacements $s=xz$ we can receive

$$\nu^{-}(x)=\frac{2q\gamma_1\left(\alpha_1+\beta_1\right)}{\gamma_2\Gamma(2\beta_1)}\left(x^{2q}\right)^{-\alpha_1-\beta_1}\int_0^x\left(x^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$

$$\times\tau^{-}(s)F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{x^{2q}-s^{2q}}{x^{2q}}\right)ds+$$

$$+\frac{2q\gamma_1}{\gamma_2\Gamma(2\beta_1)}\left(x^{2q}\right)^{\frac{1}{2}-\beta_1}\int_0^x\left(s^{2q}\right)^{\alpha_1+\frac{1}{2}}\left(x^{2q}-s^{2q}\right)^{2\beta_1-1}\tau'^{-}(s)\times$$

$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{x^{2q}-s^{2q}}{x^{2q}}\right)ds-\frac{\left(x^{2q}\right)^{\frac{1-\alpha_1+\beta_1}{2}}}{\gamma_2}h^{*}(x),\tag{26}$$

□

There holds the following preliminary assertion.

LEMMA. *If a function $\tau(x)$ has a positive maximum (respectively a negative minimum) at the point$x=x_0\in(0,h_1)$, then $\nu^{-}(x_0)>0$(respectively $\nu^{-}(x_0)<0$).*

Proof. Let's a function $\tau(x)$ has a positive maximum at the point $x=x_0\in(0,h_1)$ and $h^{*}(x)\equiv 0$, then from (26) we have:

$$\nu^{-}(x_0)=\frac{2q\gamma_1\left(\alpha_1+\beta_1\right)}{\gamma_2\Gamma(2\beta_1)}\left(x_0^{2q}\right)^{-\alpha_1-\beta_1}\int_0^{x_0}\tau'^{-}(t)dt\int_t^{x_0}\left(x_0^{2q}-s^{2q}\right)^{2\beta_1-1}\times$$

$$\times\left(s^{2q}\right)^{2\alpha_1}F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{x_0^{2q}-s^{2q}}{x_0^{2q}}\right)ds+$$

$$+\frac{2q\gamma_1}{\gamma_2\Gamma(2\beta_1)}\left(x_0^{2q}\right)^{\frac{1}{2}-\beta_1}\int_0^x\tau'^{-}(s)\left(s^{2q}\right)^{\alpha_1+\frac{1}{2}}\left(x_0^{2q}-s^{2q}\right)^{2\beta_1-1}\times$$

$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{x_0^{2q}-s^{2q}}{x_0^{2q}}\right)ds$$

Due to $\gamma_1>0$, $\gamma_2>0$, $\Gamma(2\beta_1)>0$, $0\leq\frac{x_0^{2q_1}-s^{2\,q_1}}{x_0^{2q_1}}\leq 1$,

$$\int_0^{x_0}\tilde{\tau}'(s)ds=\int_0^{x_0}\lim_{x_0\to s}\frac{\tilde{\tau}(x_0)-\tilde{\tau}(s)}{x_0-s}ds>0$$

and taking (12), (5) into account, from here we deduce that $\nu^{-}(x_0) > 0$.
Similarly, we can prove that on the point of negative minimum $\tilde{\nu}^{-}(x_0) < 0$. ***Lemma is proved.*** □

Based on the Lemma, we can conclude that, $\int_0^{h_1} \tau(x)\nu^{-}(x)dx \geq 0$,consequently from (24) we will get $\nu^{-}(x) \equiv \tau(x) \equiv 0$. Hence, based on the solution of the first boundary problem for the Eq.(10) [7],[13] owing to account (13) and (14) we will get $u(x,y) \equiv 0$in $\overline{\Omega}^{+}$, similarly, based on the solution (18) we obtain $u(x,y) \equiv 0$ in closed domain $\overline{\Omega}^{-}$.

The existence of solution of the Problem I

Theorem 2. *If satisfies all conditions of the* ***Theorem 1****. and*

$$\varphi_1(y),\, \varphi_2(y) \in C\left(\overline{I_2}\right) \cap C^1\left(I_2\right); h(x) \in C^1\left(\overline{I_1}\right) \cap C^2\left(I_1\right) \tag{27}$$

than the solution of the investigating problem is exist.

Proof.

Taking (21) into account from Eq.(22) we will obtain

$$\tau''(x) = f(x) \tag{28}$$

where

$$f(x) = \Gamma(\alpha)\nu^{-}(x). \tag{29}$$

Solution of the equation (28) together with conditions $\tau(0) = \varphi_1(0)$, $\tau(h_1) = \varphi_2(0)$ has a form:

$$\tau(x) = \int_0^x (x-t)f(t)dt - x\int_0^1 (1-t)f(t)dt + \varphi_2(0)(1-x) + x\varphi_1(0),$$

consequently, we can find:

$$\tau'(x) = \int_0^x f(t)dt - \int_0^1 (1-t)f(t)dt + \varphi_1(0) - \varphi_2(0). \tag{30}$$

Further, considering (29) from (30), after some simplifications we will get

$$\tau'(x) = \lambda\Gamma(\alpha)\int_0^x \nu(t)dt - \lambda\Gamma(\alpha)\int_0^1 (1-t)\nu(t)dt + \varphi_1(0) - \varphi_2(0) \tag{31}$$

Substituting (26) into (31) we have:

$$\tau'(x) = \frac{2q\gamma_1(\alpha_1+\beta_1)\Gamma(\alpha)}{\gamma_2\Gamma(2\beta_1)}\int_0^x \left(t^{2q}\right)^{-\alpha_1-\beta_1} dt \int_0^t \left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1} \times$$

$$\times F\left(\alpha_1+\beta_1, \frac{2\beta_1-1}{2}, 2\beta_1, \frac{t^{2q}-s^{2q}}{t^{2q}}\right) ds \int_0^s \tau'(z)dz +$$

$$+\frac{2q\gamma_1\Gamma(\alpha)}{\gamma_2\Gamma(2\beta_1)}\int_0^x \left(t^{2q}\right)^{\frac{1-2\beta_1}{2}} dt \int_0^t \left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{\frac{2\alpha_1+1}{2}} \tau'(s) \times$$

$$\times F\left(\alpha_1+\beta_1, \frac{2\beta_1-1}{2}, 2\beta_1, \frac{t^{2q}-s^{2q}}{t^{2q}}\right) ds -$$

$$-\frac{2q\gamma_1(\alpha_1+\beta_1)\Gamma(\alpha)}{\gamma_2\Gamma(2\beta_1)}\int_0^1(1-t)\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\int_0^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$
$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds\int_0^s\tau'(z)dz-$$
$$-\frac{2q\gamma_1\Gamma(\alpha)}{\gamma_2\Gamma(2\beta_1)}\int_0^1(1-t)\left(t^{2q}\right)^{\frac{1-2\beta_1}{2}}dt\int_0^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{\frac{2\alpha_1+1}{2}}\tau'(s)\times$$
$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds+F(x),\qquad(32)$$

where

$$F(x)=\frac{2q\gamma_1(\alpha_1+\beta_1)\Gamma(\alpha)}{\gamma_2\Gamma(2\beta_1)}\int_0^x\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\int_0^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$
$$\times\varphi_2(0)F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds-$$
$$-\frac{2q(\alpha_1+\beta_1)\gamma_1\Gamma(\alpha)}{\gamma_2\Gamma(2\beta_1)}\int_0^1(1-t)\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\int_0^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$
$$\times\varphi_2(0)F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds-$$

$$-\frac{\Gamma(\alpha)}{\gamma_2}\int_0^x\left(t^{2q}\right)^{\frac{1-\alpha_1+\beta_1}{2}}h^*(t)dt+\frac{\Gamma(\alpha)}{\gamma_2}\int_0^1(1-t)\left(t^{2q}\right)^{\frac{1-\alpha_1+\beta_1}{2}}h^*(t)dt+\varphi_1(0)-\varphi_2(0).\qquad(33)$$

Changing the order of integration in (32), totally we have integral equation

$$\tau'(x)=\int_0^1K(x,z)\tau'(z)dz+F(x).\qquad(34)$$

Here

$$K(x,z)=\begin{cases}K_1(x,z);\ 0\le z\le x,\\ K_2(x,z);\ x\le z\le 1.\end{cases}\qquad(35)$$

$$K_1(x,z)=k_1\Gamma(\alpha)\int_z^x\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\int_z^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$
$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds-$$
$$-k_1\Gamma(\alpha)\int_z^1(1-t)\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\int_z^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$
$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds+$$
$$+k_2\Gamma(\alpha)\left(z^{2q}\right)^{\frac{2\alpha_1+1}{2}}\int_z^x\left(t^{2q}\right)^{\frac{1-2\beta_1}{2}}\left(t^{2q}-z^{2q}\right)^{2\beta_1-1}\times$$
$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-z^{2q}}{t^{2q}}\right)dt-$$

$$-k_2\Gamma(\alpha)\left(z^{2q}\right)^{\frac{2\alpha_1+1}{2}}\int_z^1(1-t)\left(t^{2q}\right)^{\frac{1-2\beta_1}{2}}\left(t^{2q}-z^{2q}\right)^{2\beta_1-1}\times$$

$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-z^{2q}}{t^{2q}}\right)dt \tag{36}$$

$$K_2(x,z)=k_1\Gamma(\alpha)\int_z^1(1-t)\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\int_z^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}\times$$

$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-s^{2q}}{t^{2q}}\right)ds-$$

$$-k_2\Gamma(\alpha)\left(z^{2q}\right)^{\frac{2\alpha_1+1}{2}}\int_z^1(1-t)\left(t^{2q}\right)^{\frac{1-2\beta_1}{2}}\left(t^{2q}-z^{2q}\right)^{2\beta_1-1}\times$$

$$\times F\left(\alpha_1+\beta_1,\frac{2\beta_1-1}{2},2\beta_1,\frac{t^{2q}-z^{2q}}{t^{2q}}\right)dt \tag{37}$$

where

$$k_1=\frac{2q\gamma_1\left(\alpha_1+\beta_1\right)}{\gamma_2\Gamma(2\beta_1)},k_1=\frac{2q\gamma_1}{\gamma_2\Gamma(2\beta_1)}$$

Due to properties of hypergeometric function (5) from (37) we will get

$$|K_1(x,z)|\leq\left|\int_z^x\left(t^{2q}\right)^{-\alpha_1-\beta_1}dt\left|\int_z^t\left(t^{2q}-s^{2q}\right)^{2\beta_1-1}\left(s^{2q}\right)^{2\alpha_1}s^{1-2q}ds^{2q}\right|\right|. \tag{38}$$

Hence, due to class of given functions (38) considering (36) and (37)from (33) and (35) respectively we will receive $|K(x,z)|\leq const$ for all $0\leq x,\ z\leq 1$, $|F(x)|\leq const$, $0\leq x\leq 1$.

Since kernel $K(x,z)$ is continuous and function in right-side $F(x)$is continuously differentiable, solution of integral equation (34) we can write via resolvent-kernel:

$$\tau'(x)=F(x)-\int_0^1\Re(x,z)F(z)dz, \tag{39}$$

where $\Re(x,z)$ is the resolvent-kernel of $K(x,z)$.

Unknown functions $\nu^-(x)$ we will in accordingly from (26).

Solution of the Problem I in the domain Ω^+ we will write as follows [13], [7]:

$$u(x,y)=\int_0^y G_\xi(x,y,0,\eta)\psi(\eta)d\eta-\int_0^y G_\xi(x,y,1,\eta)\varphi(\eta)d\eta+\int_0^1 G_0(x-\xi,y)\tau(\xi)d\xi,$$

Here

$$G_0(x-\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y\eta^{-\alpha}G(x,y,\xi,\eta)d\eta,$$

$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{\infty}\left[e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\alpha/2}}\right)-e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\alpha/2}}\right)\right]$$

Is the Green's function of the first boundary problem Eq. (10) in the domain Ω^+ with the Riemanne-Liouville fractional differential operator instead of the Caputo ones [13],

$$e^{1,\delta}_{1,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\delta n)}$$

is the Wright type function [10].

Solution of the Problem I in the domain Ω^- will be found by the formulate (18). Hence, the **Theorem 2** is proved. □

References

[1] Kilbas A. A., Srivastava H. M., Trujillo J. J., *Theory and Applications of Fractional Differential Equations*. V. 204, North-Holland Mathematics Studies, Amsterdam, 2006.

[2] Miller K. S., Ross B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[3] Podlubny I., *Fractional Differential Equations*, Academic Press, New York, 1999.

[4] Samko S. G., Kilbas A. A., Marichev O. I., *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach, Longhorne, 1993.

[5] Marichev O. I., Kilbas A. A., Repin A. A., *Boundary value problems for partial differential equations with discounting coefficients. (In Russian)*, Izdat. Samar. Gos.Ekonom. Univ., Samara, 2008.

[6] Repin O. A., *Boundary value problems with shift for equations of hyperbolic and mixed type. (In Russian)*, Saratov Univ., Saratov, 1992.

[7] Abdullaev O. Kh., "About a problem for loaded parabolic-hyperbolic type equation with fractional derivatives", *International journal of differential equations*, **2016**, 9815796.

[8] Kilbas A. A. and Repin O. A., "An analog of the Bitsadze-Samarskii problem for a mixed type equation with a fractional derivative", *Differential equations*, **39**:5 (2003), 674–680.

[9] Kilbas A. A., Repin O. A., "An analog of the Tricomi problem for a mixed type equation with a partial fractional derivative", *Fractional Calculus & Applied Analysis*, **13**:1 (2010), 69–84.

[10] Pskhu A. V., *Uravneniye v chasnykh proizvodnykh drobnogo poryadka. (Russian) [Partial differential equation of fractional order]*, Nauka, Moscow, 2005, 200 pp.

[11] Ochilova N. K., "Study the unique solvability of boundary value problem of Frankl for mixed-type equation degenerate on the boundary and within the region", *Vestnik KRAUNC. Fiz.-Mat. Nauki - Bulletin KRASEC. Phys. & Math. Sci.*, 2014, № 1(8), 20-32.

[12] Smirnov M. M., *Mixed type equations*, Nauka, Moscow, 2000.

[13] Pskhu A. V., "Solution of boundary value problems fractional diffusion equation by the Green function method", *Differential equation*, **39** (2003), 1509-1513.

References (GOST)

[1] Kilbas A. A., Srivastava H. M., Trujillo J. J. Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam. (2006).

[2] Miller K.S., Ross B. An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, (1993).

[3] Podlubny I. Fractional Differential Equations, Academic Press, New York, (1999).

[4] Samko S. G., Kilbas A. A., Marichev O. I. Fractional Integral and Derivatives: Theory and Applications, Gordon and Breach, Longhorne, PA, (1993).

[5] Marichev O. I., Kilbas A. A., Repin A. A. Boundary value problems for partial differential equations with discounting coefficients. (In Russian). Izdat.Samar.Gos.Ekonom. Univ., Samara (2008)

[6] Repin O. A. Boundary value problems with shift for equations of hyperbolic and mixed type. (In Russian). Saratov Univ., Saratov (1992)

[7] Abdullaev O.Kh. About a problem for loaded parabolic-hyperbolic type equation with fractional derivatives", International journal of differential equations., vol. 2016, Article ID 9815796, 12 pages.

[8] Kilbas A. A. and Repin O. A. An analog of the Bitsadze-Samarskii problem for a mixed type equation with a fractional derivative," Differential equations. (2003). vol. 39, no. 5, pp. 674–680.

[9] Kilbas A. A., Repin O. A. "An analog of the Tricomi problem for a mixed type equation with a partial fractional derivative," Fractional Calculus & Applied Analysis. (2010) vol. 13, no. 1, pp. 69–84.
[10] Pskhu A.V. Uravneniye v chasnykh proizvodnykh drobnogo poryadka. (Russian) [Partial differential equation of fractional order]. Nauka. Moscow. 2005. 200 p.
[11] Ochilova N. K. Study the unique solvability of boundary value problem of Frankl for mixed-type equation degenerate on the boundary and within the region. Vestnik KRAUNC. Fiz.-Mat. Nauki - Bulletin KRASEC. Phys. & Math. Sci. 2014. №1(8). pp. 20-32.
[12] Smirnov M. M. Mixed type equations. Moscow. Nauka. (2000).
[13] Pskhu A. V. Solution of boundary value problems fractional diffusion equation by the Green function method. Differential equation, 39. (2003), pp. 1509-1513.

Для цитирования: Islomov B. I., Ochilova N. K. About a problem for the degenerating mixed type equation fractional derivative // *Вестник КРАУНЦ. Физ.-мат. науки*. 2017. № 1(17). C. 22-32. DOI: 10.18454/2079-6641-2017-17-1-22-32

For citation: Islomov B. I., Ochilova N. K. About a problem for the degenerating mixed type equation fractional derivative, *Vestnik KRAUNC. Fiz.-mat. nauki*. 2017, **17**: 1, 22-32. DOI: 10.18454/2079-6641-2017-17-1-22-32

Поступила в редакцию / Original article submitted: 25.12.2016