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PROPAGATION OF PLANE ELASTIC-TOUGH-PLASTIC WAVES IN THE MATERIAL WITH YIELD DELAY

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Abstract. The use of electromechanical model of ideal elastic-tough-plastic material with yield delay while investigating the propagation of plane one-dimensional waves is being considered. The materials being investigated possesses the property of toughness only in the plastic region and is elastic up to the plastic state. The solving of the problem is being carried out in the conditions of one-parameter loading when the permanent (steady) uniformly distributed stress is being suddenly applied to the boundary surface of non-deformed half-space. The value of applied stress exceeds the limit level of elastic state in the case of static deformation and remains unchanged for the whole region of disturbed half-space. In order to conduct the corresponding calculation the dependence between the components the tensor of normal stress and linear deformations along the load axis is being determined. The defining equations of elastic-tough-plastic medium behind the front of the wave of transition from elastic to plastic state are being derived and the simple relationship for determination of the time (duration) of yield delay is deduced. The basic solution of these equations is obtained for the stresses in elastic and plastic region of the half-space. The special representation of the basic pattern of changing of lateral (transverse) stresses and longitudinal deformations for the region of active loading of material is defined. The distribution of velocities of the half-space points in the plastic region is obtained. The difference between the characters of propagation of plane elastic-plastic wave and the wave processes in the material with yield delay is shown.

It is noted that two different solutions are obtained despite of the simple type of loading. The first one detects the strange character of medium's behaviour when the material of disturbed half-space in the plastic region is in the state of “trembling”, which causes the pulsations of lateral (transverse) stresses and the impulses of deformations along the axis that is normal to the boundary surface of the half-space.

Keywords: yield delay, plane elastic-tough-plastic waves, dynamic criterion of plasticity, plastic state, overstress, medium “trembling”.

Introduction

A lot of publications are devoted to the theoretical research of propagation of plane elastic-plastic waves. The earliest works in this field are presented in [1-4]. The further complication of problems related with the investigation of such type of dynamic processes took its place due to the changing of the character of external loading and taking into account more complete features of materials [5–8]. The works related with propagation of waves in materials that possess the feature of yield delay are not so widespread. The short overview of the specific features of investigations of dynamic processes in such mediums are presented in [9]. In theoretical works related with wave propagation in materials with yield delay, the great difficulties and awkwardness of research are noted [10]. The analytical solution may be obtained only for the simplest boundary conditions and idealized features of the medium [11]. This is caused by the fact that on the basis of Yu. N. Rabotnov model [12] after the exhaustion of delay possibility, the transition from the overstressed state to plastic one is carried out almost instantly by means of elastic unloading (relief) if the functional condition of Cottrell is satisfied. As a result of this the waves of large discontinuities, which cause very complicated motion pattern when interacting with elastic and plastic waves, arise. The great variety of discontinuities in certain problems makes it almost impossible to obtain even numerical solution with accurate accounting of all discontinuities [10].

Let us carry out the investigation of dynamic state of the material with yield delay, which fill the semi-infinite space, on the basis of electromechanical model of the medium [13]. The specific feature of the formulated problem consists in the fact that the transition from the elastic state to the plastic one (to the lower yield limit) is being carried out not suddenly but during the certain period of time. That is why the behaviour of the medium won't cause the occurrence of the wave front of large discontinuities. Hence, there occurs the possibility of considerable simplification of obtaining of general pattern of wave processes for such materials.

Derivation of relation between stress and deformation of ideal elastic-plastic material for the plane problem of propagation of compression waves

When investigating the propagation of elastic-tough-plastic waves in the rod on the basis of electromechanical model of the material with yield delay, the assumption related with neglect of inertial forces caused by lateral (transverse) deformations were made [14]. In the considered plane problem, such assumptions are not necessary because the lateral (transverse) deformations are missing. Let us formulate more detailed statement of the problem and determine the necessary features of the medium on the basis of single-axis diagram of tension-compression of ideal elastic-plastic material.

Let the motion of particles of the medium is carried out by the wave that propagates in non-movable and non-stressed half-space along the *x*-axis, which is perpendicular to the boundary surface (plane). The *y*-axis and *z*-axis of the Cartesian coordinate system *x*, *y*, *z* lie in this plane. The disturbance is conducted as a result of application of permanent uniformly distributes normal stress $s_{xx} = P_0$ to the half-space surface (Fig. 1). The other values of stresses and deformations are the functions of coordinate *x* and time *t* because the load on the half-plane boundary doesn't depend on *y* and *z*. So we may write that the components of the tensor of stresses and deformations are as follows:

$$
\begin{aligned}
&s_{xx} \neq 0, \quad s_{yy} = s_{zz} \neq 0, \quad t_{xy} = t_{yz} = t_{zx} = 0, \\
&e_{xx} \neq 0, \quad e_{yy} = e_{zz} = e_{xy} = e_{yz} = e_{zx} = 0.
\end{aligned} \tag{1}
$$

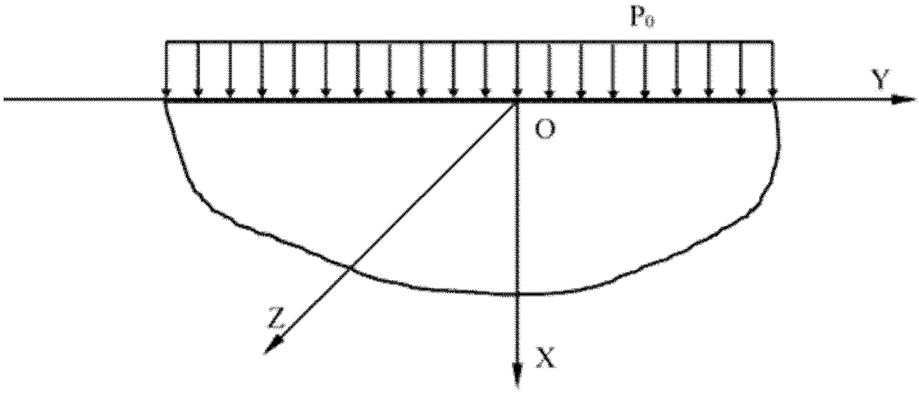

***Fig. 1.** The half-space, on the surface of which the uniform normal stress P_0 acts*

In order to obtain differential equations of dynamic state of material it is necessary to define the relationship between s_{xx} and e_{xx} taking into account the diagram of single-axis tension-compression of rods $s : e$, which is formed on the basis of experiments under static loading. The typical stress-deformation curve of compression of rods made of ideal elastic-plastic material is shown in Fig. 2, where s_S is the yield limit of the material, e_S is the maximal value of elastic deformation of compression.

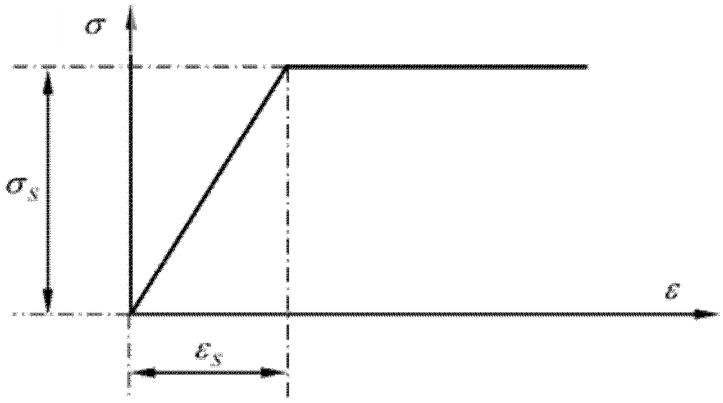

Fig. 2. *Static diagram of single-axis compression of ideal elastic-plastic material of the rod*

Taking into account (1), let us write the Hooke's law for the region of elastic deformations considering the plane problem. On the diagram (Fig. 2), this corresponds to the region $s \in [0, s_S]$:

$$e_{xx} = \frac{1}{E}\left(s_{xx} - 2n\, s_{yy}\right),$$
$$0 = \frac{1}{E}[(1-n)s_{yy} - n\, s_{xx}], \tag{2}$$

where E is Young's modulus; n is Poisson's ratio.

After certain transformations we obtain:

$$s_{xx} = \frac{E(1-n)}{(1+n)(1-2n)} e_{xx}. \tag{3}$$

Using (2) and (3), let us determine the values of s_{xx} and e_{xx} which causes plastic deformation of the material. As a condition of transition to the plastic state, we take the Mize's criterion that matches the Trask's criterion in our case. In the considered problem, the yield condition may be written in the simple form:

$$s_{xx} - s_{yy} = s_S. \tag{4}$$

Substituting s_{yy} from (2) to (4), we obtain:

$$s_{xx}^{s} = \frac{1-n}{1-2n} s_S. \tag{5}$$

Obviously, one can see that $s_{xx}^{s} > s_S$. Similarly, for e_{xx}^{s} from (3) we obtain:

$$e_{xx}^{s} = \frac{1+n}{E} s_S = e_S(1+n). \tag{6}$$

Let us determine $s_{xx}(e_{xx})$ in plastic region. Taking into account the non-compressiability of plastic deformation and the condition (4), let us perform the following transformations for volumetric deformation:

$$e_{xx} + e_{yy} + e_{zz} = e_{xx}^{e} + 2e_{yy}^{e} = \frac{s_{xx} + 2s_{yy}}{3K} = \frac{3s_{xx} - 2s_S}{3K}, \tag{7}$$

where $K = \dfrac{E}{3(1-2n)}$ is volumetric modulus of elasticity.

Here we take into account the separation of deformation on elastic and plastic components:

$$e_{xx} = e_{xx}^{e} + e_{xx}^{p}, \qquad e_{yy} = e_{yy}^{e} + e_{yy}^{p}. \tag{8}$$

On the basis of (1) we may state that $e_{xx} = e_{yy} = 0$. That's why with a help of (7) we may defind the dependence $s_{xx}(e_{xx})$ for the material of the half-space, which is in the yield state:

$$s_{xx} = \frac{Ee_{xx}}{3(1-2n)} + \frac{2}{3}s_s. \tag{9}$$

Using the relationships (3) and (9), let us transform the diagram of single-axis compression of the material (Fig. 2) into the dependence $s_{xx} \sim e_{xx}$ for the plane problem (Fig. 3).

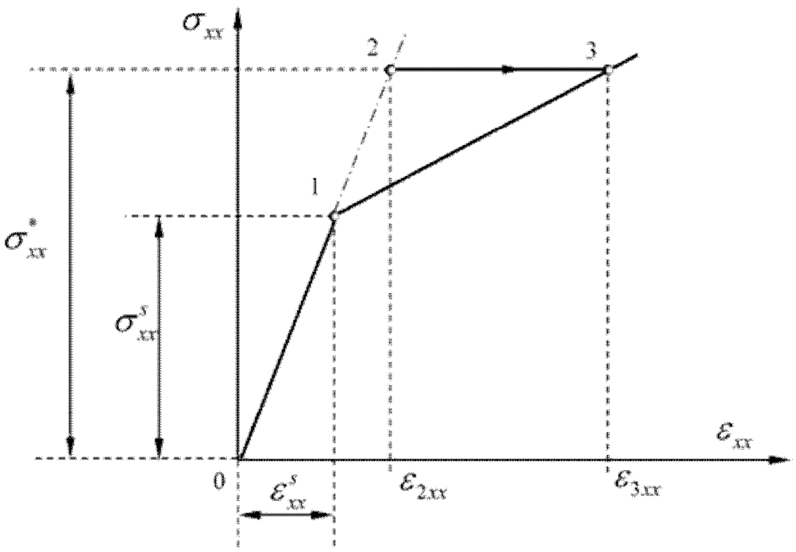

Fig. 3. *The dependence $s_{xx}(e_{xx})$ for the complete limitation of lateral (transverse) deformations*

In order to carry out the comparative analysis and to use the certain values of obtained data in the following investigations, let us consider the propagation of plane elastic-plastic waves when the normal stress $P_0 = s^*_{xx}$, which exceeds the plasticity limit s^s_{xx}, is suddenly applied to the boundary of the half-space. Let us write down the dynamic equations of motion for small deformations and missing mass forces:

$$\frac{\partial s_{xx}}{\partial x} = r\frac{\partial^2 u}{\partial t^2}, \tag{10}$$

where u is the displacement of the medium particles along the x-axis; r is the density of the material that fills the half-space.

Let us differentiate equation (10) with respect to x and relationships for elastic region (3) and for plastic region (9) twice with respect to time;

$$\frac{\partial^2 s_{xx}}{\partial x^2} = r\frac{\partial^3 u}{\partial t^2 \partial x}, \quad \frac{\partial^2 s_{xx}}{\partial t^2} = \frac{E(1-n)}{(1+n)(1-2n)}\frac{\partial^2 e_{xx}}{\partial t^2}, \quad \frac{\partial^2 s_{xx}}{\partial t^2} = \frac{E}{3(1-2n)}\frac{\partial^2 e_{xx}}{\partial t^2}. \tag{11}$$

Taking into account the fact that $e_{xx} = \frac{\partial u}{\partial x}$, let us exclude the deformation e_{xx} from the system of equations (11):

$$\frac{\partial^2 s_{xx}}{\partial t^2} = a_1^2\frac{\partial^2 s_{xx}}{\partial x^2}, \quad \frac{\partial^2 s_{xx}}{\partial t^2} = a_2^2\frac{\partial^2 s_{xx}}{\partial x^2},$$
$$a_1 = \sqrt{\frac{E(1-n)}{r(1+n)(1-2n)}}, \quad a_2 = \sqrt{\frac{E}{3(1-n)r}}. \tag{12}$$

The obtained equations (12) allows us to state that a_1 is the velocity of propagation of the front of elastic wave, on which the stress spasmodically reaches the value of yield limit s_{xx}^{s}. Similarly, on the plastic wave, which propagates with the velocity a_2, the new discontinuity (jump) of the disturbance to the value of applied normal stress s_{xx}^{*} takes place. Hence, the disturbance propagates in the medium in the form of step-type waves, which are schematically shown in Fig. 4

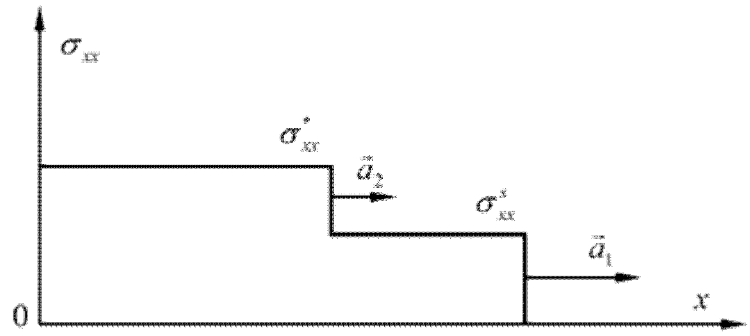

Fig. 4. *The propagation of elastic-plastic waves in the half-space*

Formation and solving of defining equations of elastic-tough-plastic state of the material behind the front of yield delay

Let us define the pattern of motion of the medium particles in the half-space on the basis of electromechanical model of material that possesses the property of yield delay [13]. At the same time, the normal stress $s_{xx}^{*} > s_{xx}^{s}$ is suddenly applied to the boundary surface. As a result of such loading the shock (impact) wave will propagates through the medium with the velocity a_1. On the front of this wave, $x = a_1 t$, the stress and deformation spasmodically increase to the values of $s_{xx} = s_{xx}^{*}$ and $e_{xx} = e_{2xx} = \frac{(1+n)(1-2n)}{E(1-n)} s_{xx}^{*}$. Unlike the elastic-plastic material, the transition to the stress s_{xx}^{*} is carried out not along the curve 0 – 1 – 3 zs shown on the diagram (Fig. 3), but along the straight line 0 – 1 – 2 described by the equation (3). The region of the half-space, for which staying in the elastic state will be exhausted by the time of yield delay t, will transit to the plastic state. The boundary of transition of the material fom elastic state to plastic one may be considered as the front of the wave of yield delay, which propagates with the velocity of elastic wave a_1 and is defined by the equation $x = a_1(t - t)$ (Fig. 5).

In the region between the waves fronts $a_1(t-t) \le x \le a_1 t$, the material of the half-space will be in the elastic state. Behind the front of the yield delay wave, the plasticity state takes place. Let us note that according to the problem statement the normal stress along the *x*-axis in the plastic region will remain permanent and equal s_{xx}^{*}. The stressed and deformated state in the investigated material doesn't cause the initiation of rotation motion of particles. That's why, on the basis of the theorem of Cauchi-Helmgolz about the velocities of particles of infinitely small part of the medium [15], the velocity of its arbitrary particle will consist of the velocity of translational motion of the part as absolutely rigid body and of the velocity of pure strain (deformation). On the basis of (10) one may state that the velocity of translational motion is permanent and equal to the velocity of particles on the front of yield delay wave. In the case when the elastic solution is known, this velocity may be defined as:

$$V_0 = a_1 e_{2xx} = \frac{(1+n)(1-2n)}{E(1-n)} a_1 s_{xx}^{*} = \sqrt{\frac{(1+n)(1-2n)}{E(1-n)r}}\, s_{xx}^{*}. \qquad (13)$$

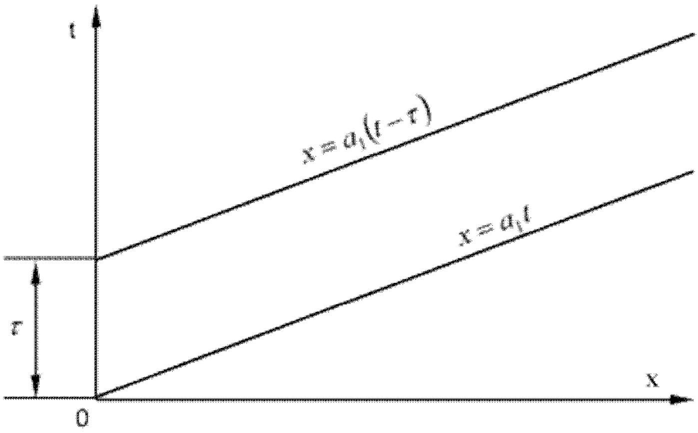

Fig. 5. *The distribution of the state regions of the half-space in the plane x, t*

The dependence of the velocity of the arbitrary particle of infinitely small part from its deformation in the yield state may be determined on the basis of defining equations of plasticity. The equations that define the dynamic condition of material plasticity with yield delay are as follows [13]:

$$\left(S_{ij}-\frac{2}{3}\mu\dot{\varepsilon}_{ij}^{p}\right)\left(S_{ij}-\frac{2}{3}\mu\dot{\varepsilon}_{ij}^{p}\right)=\frac{2}{3}\left[\sigma_s+k\left(\frac{3}{2}\dot{\varepsilon}_{ij}^{e}\dot{\varepsilon}_{ij}^{e}\right)^{1/n}\sqrt{\frac{\sqrt{\frac{3}{2}S_{ij}^{*}S_{ij}^{*}}}{\sigma_s}-1}\right]^2, \tag{14}$$

$$S_{ij}=\sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk},\quad \dot{\varepsilon}_{ij}^{e}=\dot{\varepsilon}_{ij}^{e}-\frac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}^{e},$$

where S_{ij}, S_{ij}^{*} are the components of stress deviator in the plastic region and on the boundary of transition from elastic region to plastic one; σ_{ij} are the components of stress tensor: $\dot{\varepsilon}_{ij}^{e}$, $\dot{\varepsilon}_{ij}^{e}$ are the components of the tensor and deviator of velocities of elastic deformations; μ is the coefficient of viscosity; k and n are the constants that characterize the material; δ_{ij} is the symbol of Kronecker.

Using the association law of plasticity and the equation (14), let us derive the relationship for determination of components of the tensor of velocity of plastic deformations $\dot{\varepsilon}_{ij}^{p}$ [14]:

$$\dot{\varepsilon}_{ij}^{p}=\frac{3}{2\mu}\left[1-\frac{\sigma_s+k\left(\frac{3}{2}\dot{\varepsilon}_{ij}^{e}\dot{\varepsilon}_{ij}^{e}\right)^{1/n}\sqrt{\frac{\sqrt{\frac{3}{2}S_{ij}^{*}S_{ij}^{*}}}{\sigma_s}-1}}{\sqrt{\frac{3}{2}S_{ij}S_{ij}}}\right]S_{ij}. \tag{15}$$

On the basis of equations (15), let us describe the process of material deformation in the region that corresponds to the section 2–3 of the diagram $\sigma_{xx}(e_{xx})$ (Fig. 3). Because of the fact that in the case of our plane problem the components of the tensor of stresses and defromations have the values defined in conditions (1), for the components S_{ij} we will obtain: $S_{xx}=\frac{2}{3}(\sigma_{xx}-\sigma_{yy})$, $S_{yy}=S_{zz}=\frac{1}{3}(\sigma_{yy}-\sigma_{xx})$. So for $\frac{3}{2}S_{ij}S_{ij}$ and $\frac{3}{2}S_{ij}^{*}S_{ij}^{*}$ we may define:

$$\frac{3}{2}S_{ij}S_{ij}=\frac{1}{3}(\sigma_{xx}-\sigma_{yy})^2,\quad \frac{3}{2}S_{ij}^{*}S_{ij}^{*}=(\sigma_{xx}^{*}-\sigma_{yy}^{*})^2. \tag{16}$$

Similarly, for the components $\dot{e}_{ij}^{e}$ we may obtain:

$$\dot{e}_{xx}^{e} = \frac{2}{3}\left(\dot{e}_{xx}^{e} - \dot{e}_{yy}^{e}\right), \quad \dot{e}_{yy}^{e} = e_{zz}^{e} = -\frac{1}{3}\left(\dot{e}_{xx}^{e} - \dot{e}_{yy}^{e}\right). \tag{17}$$

Let us note that according to (1) the components of deformation tensor $e_{yy} = e_{zz} = 0$, however the components of the tensor of elastic deformations in plastic region $e_{yy}^{e} = e_{zz}^{e} \neq 0$. So for $\frac{3}{2}\dot{e}_{ij}^{e}\dot{e}_{ij}^{e}$ we may wrtite:

$$\frac{3}{2}\dot{e}_{ij}^{e}\dot{e}_{ij}^{e} = \left(\dot{e}_{xx}^{e} - \dot{e}_{yy}^{e}\right)^{2}. \tag{18}$$

Taking into account the Hook's law for elastic components of deformation tensor, after corresponding transformations we obtain:

$$\frac{3}{2}\dot{e}_{ij}^{e}\dot{e}_{ij}^{e} = \frac{(1+n)^{2}}{E^{2}}\left(\dot{s}_{xx} - \dot{s}_{yy}\right)^{2}, \tag{19}$$

where $\dot{s}_{xx}, \dot{s}_{yy}$ are velocities of changing of normal stresses.

In order to carry out the further transformations, let us note that Hook's law is valid on the front of the yield delay wave. That's why on the basis of equations (2) we may write:

$$(1-n)s_{yy}^{*} - n\, s_{xx}^{*} = 0. \tag{20}$$

Hence, for the relationship (16) we may define:

$$\frac{3}{2}S_{ij}^{*}S_{ij}^{*} = \left(\frac{1-2n}{1-n}s_{xx}^{*}\right)^{2}. \tag{21}$$

Making the substitution of (16), (19), (21) into (15), let us determine the velocity of plastic deformation $\dot{e}_{xx}^{p}$:

$$\dot{e}_{xx}^{p} = \frac{1}{m}\left[s_{xx} - s_{yy} - s_{s} - k\frac{(1+n)^{2/n}}{E^{2/n}}\left(\dot{s}_{xx} - \dot{s}_{yy}\right)^{2/n}\sqrt{\frac{(1-2n)\,s_{xx}^{*}}{(1-n)\,s_{s}} - 1}\right]. \tag{22}$$

Let us carry out the additional transformations of the equation (22). In order to simplify its presentation let us consider the applied stress s_{xx}^{*} as:

$$s_{xx}^{*} = s_{xx}^{s} + s_{s} = \frac{1-n}{1-2n}s_{s} + s_{s}. \tag{23}$$

Taking into account the plastic non-compressibility of the material, let us write:

$$e_{xx}^{p} + 2e_{yy}^{p} = 0. \tag{24}$$

If we represent the deformation in the view of (8) and taking into account the fact that it is missing in transverse direction $e_{yy} = 0$, we may write $e_{yy}^{p} = -e_{yy}^{e}$. So, on the basis of (24), after differentiation with respect ro time we obtain:

$$\dot{e}_{xx}^{p} = 2\dot{e}_{yy}^{e}. \tag{25}$$

Let us write down the Hook's law for the elastic lateral (transverse) deformations in the plastic region:

$$e_{yy}^{e} = \frac{1}{E}\left[(1-n)s_{yy} - n\, s_{xx}\right]. \tag{26}$$

Differentiating the equating (26) with respect to time and taking into account the fact that $s_{xx} = s^{*}_{xx} = const$, let us define:

$$\dot{e}^{e}_{yy} = \frac{1-n}{E}\dot{s}_{yy}. \tag{27}$$

As a result of substitution of relationships (23), (25), (27) into (22), after carrying out certain transformations we obtain the following form of equations (22):

$$\frac{2(1-n)}{E} m\dot{s}_{yy} + k\frac{(1+n)^{2/n}}{E^{2/n}}\left(-\dot{s}_{yy}\right)^{2/n}\sqrt{\frac{1-2n}{1-n}} + s_{yy} - \frac{1-n}{1-2n}s_{s} = 0. \tag{28}$$

In order to perform the corresponding calculations, let us adopt the following numerical values of material constants, which were used in the work [13]:

$$E = 2{,}1\cdot 10^{11}\,Pa,\quad s_{s} = 2{,}15\cdot 10^{8}\,Pa,\quad m = 1{,}2\cdot 10^{6}\,Pa\cdot s,\quad k = 1{,}5\cdot 10^{8}\,Pa\cdot s^{1/12},$$

$$r = 7800\frac{kg}{m^{3}},\quad n = 24,\quad n = 0{,}25.$$

If we substitute the values of corresponding constants of material into the equation (28), we obtain:

$$8{,}57\cdot 10^{-6}\dot{s}_{yy} + 1{,}42\cdot 10^{7}\left|\dot{s}_{yy}\right|^{1/12} + s_{yy} - 3{,}225\cdot 10^{8} = 0. \tag{29}$$

The obtained equation (29) may be characterized as defining one, on the basis of which solution one may describe the stressed and deformed state of material of the half-space in the plastic region. However, as a variable of this equation it is expedient to choose the internal time for each particle of the medium, which may be defined as a time of staying of the certain particle in the yield state. Let us consider this time as $z = t - t - \frac{x}{a_1}$.

In order to obtain the solution of the equation (29), it is necessary to determine the value of stress s_{yy} at the moment of time when z = 0, which will be equal to the stress on the front of the yield delay wave. On the basis of (2), let us write:

$$s^{*}_{yy} = s_{yy}(0) = \frac{n}{1-n}s^{*}_{xx}. \tag{30}$$

Taking into account the adopted applied stress s^{*}_{xx} equal to (23), after the certain transformations we obtain:

$$s_{yy}(0) = \frac{n(2-3n)}{(1-n)(1-2n)}s_{s}. \tag{31}$$

Substituting the numerical values, we obtain $s_{yy}(0) = 1{,}79\cdot 10^{8}\,Pa$.

The solution of the equation (29) is presented in Fig. 6 in the form of plot $s_{yy}(z)$. While determining the solution of the equation (29), the following condition has been adopted: $\dot{s}_{yy}(z)$ may accepts only positive values.

In order to obtain the special presentation of the general pattern of changing of the stress s_{yy} behind the front of the yield delay wave, in the solution of the equation (29) let us make the following change $z \to t - t - \frac{x}{a_1}$. In order to carry out the corresponding calculations, it is necessary to determine the numarical value of the time (duration) od yield delay t .

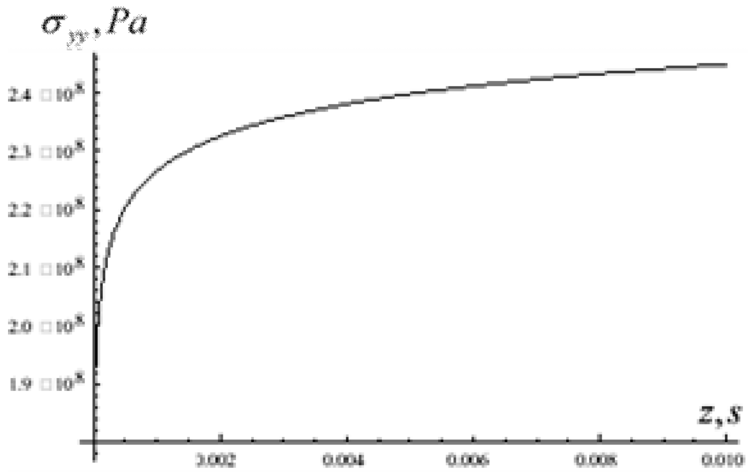

Fig. 6. *The graphical presentation of the solution of the state equation in the plastic region*

The investigation [16] shows that the yield delay time, when suddenly "overstressed" material $\left(\sqrt{\frac{3}{2}S^*_{ij}S^*_{ij}} > s_s\right)$ stays in the elastic state, equals the time, during which the yield starting stress may be achieved when loading with a certain permanent velocity. That's why the yield delay time for the arbitrary stressed state may be determined as follows:

$$t = \frac{\sqrt{\frac{3}{2}S^*_{ij}S^*_{ij}} - s_s}{\sqrt{\frac{3}{2}\dot{S}_{ij}\dot{S}_{ij}}}, \tag{32}$$

where $\dot{S}_{ij}$ are the components of deviator of velocities of stresses in the elastic state.

According to the dynamic condition of plasticity (14), the following criterion of material yield limit may be obtained [13]:

$$\sqrt{\frac{3}{2}S^*_{ij}S^*_{ij}} = s_s + \frac{k^2}{s_s}\left(\frac{3}{2}\dot{e}^e_{ij}\dot{e}^e_{ij}\right)^{2/n}. \tag{33}$$

Let us write down the Hook's law in the form of the dependence between the components of the deviator of stresses and deformations:

$$e^e_{ij} = \frac{S_{ij}}{2G}, \text{ where the shearing modulus equals } G = \frac{E}{2(1+n)}. \tag{34}$$

After the differentiation of the equation (34) with respect to time and after substitution of the result into (33), we may write down:

$$\sqrt{\frac{3}{2}S^*_{ij}S^*_{ij}} = s_s + \frac{k^2}{s_s}\left[\frac{(1+n)^2}{E^2}\left(\frac{3}{2}\dot{S}_{ij}\dot{S}_{ij}\right)\right]^{2/n}. \tag{35}$$

Making the transformation of (35), we may define:

$$\sqrt{\frac{3}{2}\dot{S}_{ij}\dot{S}_{ij}} = \frac{E s_s^{n/4}}{(1+n)k^{n/2}}\left(\sqrt{\frac{3}{2}S^*_{ij}S^*_{ij}} - s_s\right)^{n/4}. \tag{36}$$

Substituting (36) into (32), we obtain the expression for determination of t in the following form:

$$t = \frac{k^{n/2}(1+n)}{E s_s^{n/4}\left(\sqrt{\frac{3}{2}S_{ij}^{*}S_{ij}^{*}} - s_s\right)^{n/4-1}}. \tag{37}$$

Taking into account (21), (23) and the value $n = 24$, after certain transformations we may write:

$$t = \frac{k^{12}(1+n)}{E s_s^{11}}\left(\frac{1-n}{1-2n}\right)^{5}. \tag{38}$$

Using in (38) the adopted material constants, we obtain $t = 1{,}29247 \cdot 10^{-4}\,s$.

The calculated value t allows to obtain the general pattern of the stress field $s_{yy}(x,t)$ behind the front of the yield delay wave. The result defined on the basis of solution of equation (29) and of corresponding changes is presented in Fig. 7.

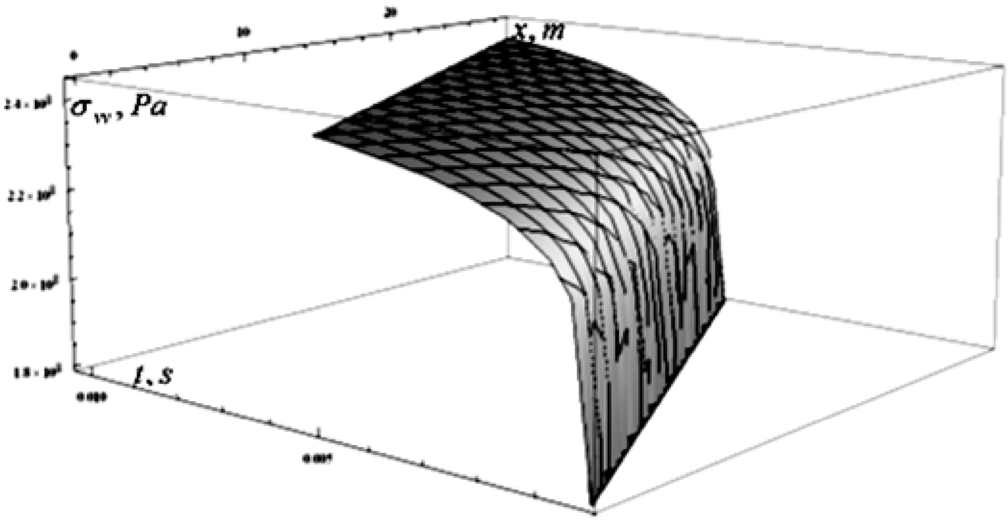

Fig. 7. *The stress field* $s_{yy}(x,t)$ *behind the front of the yield delat wave*

Taking into account the value of stress $s_{yy}^{*} = s_{yy}(0)$ obtained from (31), let uf form the following general pattern for the stress field $s_{yy}(x,t)$ (Fig. 8) between the front of elastic wave and the front of yield delay wave (Fig. 5).

On the basis of obtained solution of equation (29), let us define the law of changing of $e_{xx}(z)$ during transition from point 2 to point 3 on the diagram (Fig. 3). Using (24) and Hook's law (26) written for the plastic region, we obtain:

$$e_{xx}^{p} = 2e_{yy}^{e} = \frac{2}{E}\left[(1-n)s_{yy} - n\,s_{xx}^{*}\right]. \tag{39}$$

For the elastic deformation e_{xx}^{e} on the basis of (2), we may write:

$$e_{xx}^{e} = \frac{1}{E}\left(s_{xx}^{*} - 2n\,s_{yy}\right). \tag{40}$$

Taking into account the separation of deformation (8) for e_{xx}, let us define:

$$e_{xx} = e_{xx}^{e} + e_{xx}^{p} = \frac{1-2n}{E}\left(2s_{yy} + s_{xx}^{*}\right). \tag{41}$$

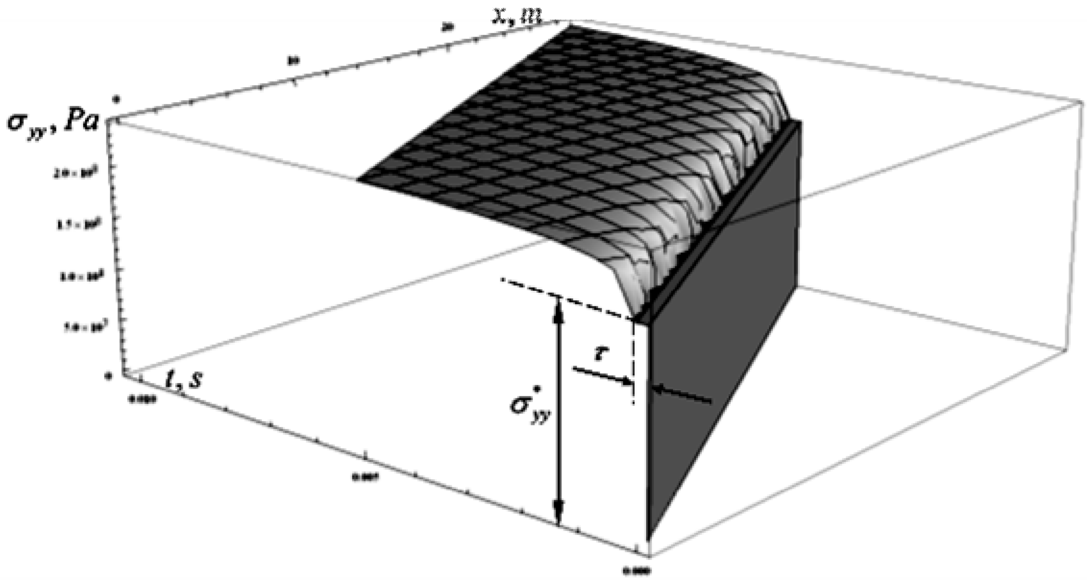

***Fig. 8.** The stress field $s_{yy}(x,t)$ for all regions of the disturbed half-space*

The equation (41) allows to define the law of deformation e_{xx} changing in the plastic region of “overstressed” material. In the point 2 of the diagram (Fig. 3) the deformation e_{xx} must have the value of $e_{2xx} = \dfrac{(1+n)(1-2n)}{E(1-n)} s^*_{xx}$. It is obvious that this information may be obtained if the initial value $s_{yy}(0)$ from the equality (30) is substituted into the equation (41). Similarly, in the point 3 we may obtain the deformation e_{3xx}. Making the change $s_{xx} \to s^*_{xx}$ in the equation (9), after simple transformations we obtain:

$$e_{3xx} = \frac{4-5n}{E} s_s. \tag{42}$$

This deformation is obtained on the final stage of the deformation process from the point 2 to the point 3 of the diagram (Fig. 3) if the value s_{yy} from (28) is to be substituted into the equation (41) and if we adopt that $\dot{s}_{yy}(t \to \infty) = 0$. As the final result after all remarks and explanations, let us substitute the solution of equation (29) into (41). We obtain the law of deformation e_{xx} changing with respect to the internal time z (Fig. 9).

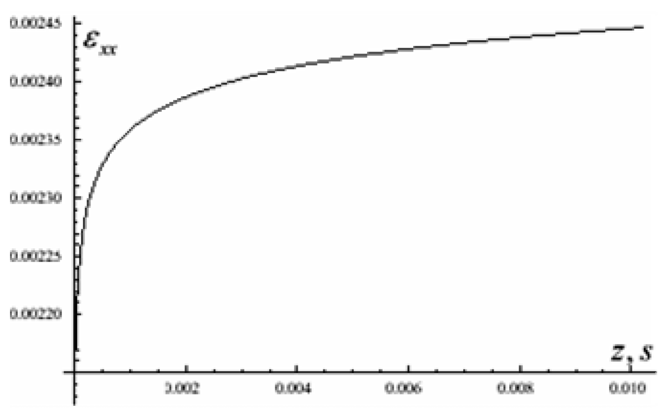

***Fig. 9.** The law of deformation e_{xx} changing for the points of the half-space which are in the yield state*

If we make the following change $z \to t - t - \frac{x}{a_1}$, we obtain the general pattern of the deformation field $e_{xx}(x,t)$ in the plastic region (Fig. 10).

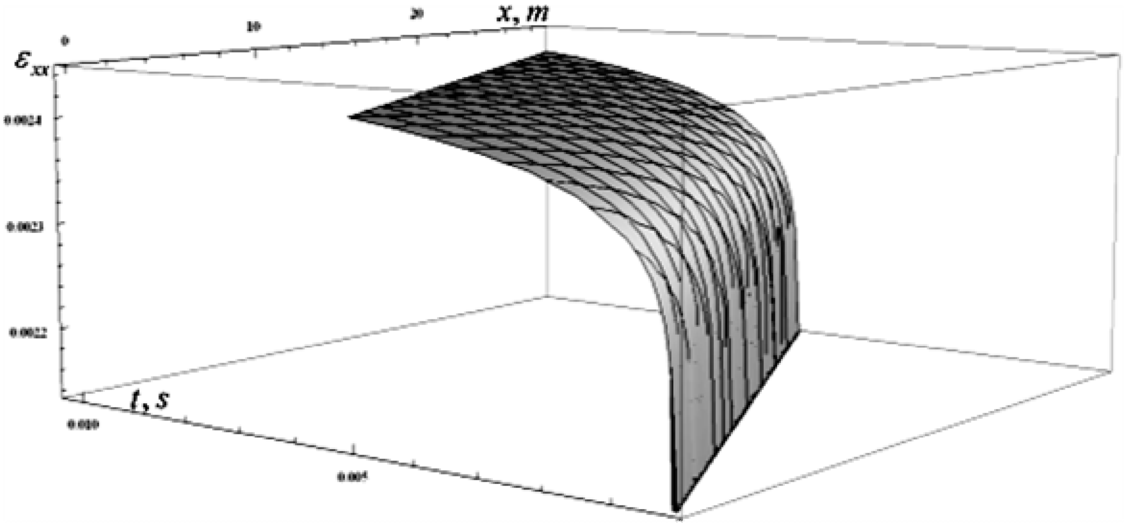

Fig. 10. *The field of deformations e_{xx} in the plastic region of the half-space*

Taking into account the value of deformation e_{2xx} for the narrow zone that is estimated by the time t (Fig. 5), we obtain the complete pattern of the deformation field $e_{xx}(x,t)$ for all regions of the disturbed half-space (Fig. 11).

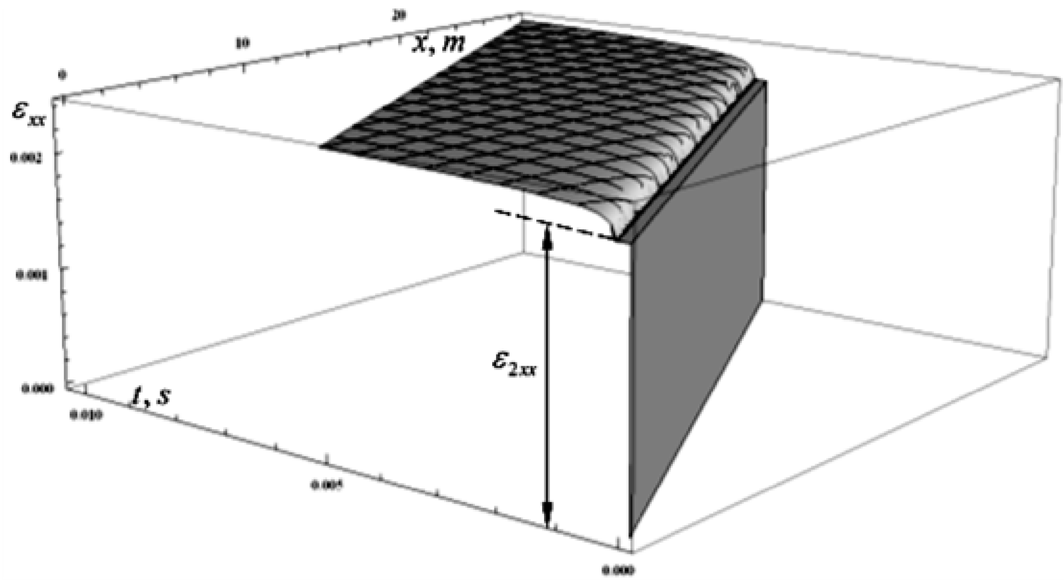

Fig. 11. *The field of deformations $e_{xx}(x,t)$ for all regions of the disturbed half-space*

In order to define the distribushion of velocities $v_x(x,t)$ of the material particles in the plastic region of the half-space, let us use the following relation ship at small deformations:

$$\frac{\partial v_x}{\partial x} = \frac{\partial e_{xx}}{\partial t}. \tag{43}$$

Differentiating (41) with respect to internal time z, we obtain:

$$\dot{\varepsilon}_{xx}(z) = \frac{2(1-2\nu)}{E}\dot{\sigma}_{yy}(z). \tag{44}$$

Adopting for convenience the compressing deformations e_{xx} as positive ones and taking into account (43), (44) and the expression for the time z, we may write down:

$$v_x(x,t) = V_0 + 2a_1\frac{1-2\nu}{E}\int_0^{t-\tau-\frac{x}{a_1}} \dot{\sigma}_{yy}(z)\,dz, \tag{45}$$

where V_0 is the velocity of particles on the boundary of elastic and plastic regions, which may be determined with a help of (13).

After integrating (45), we obtain the field of velocities of the half-space particles in the plastic region (Fig. 12).

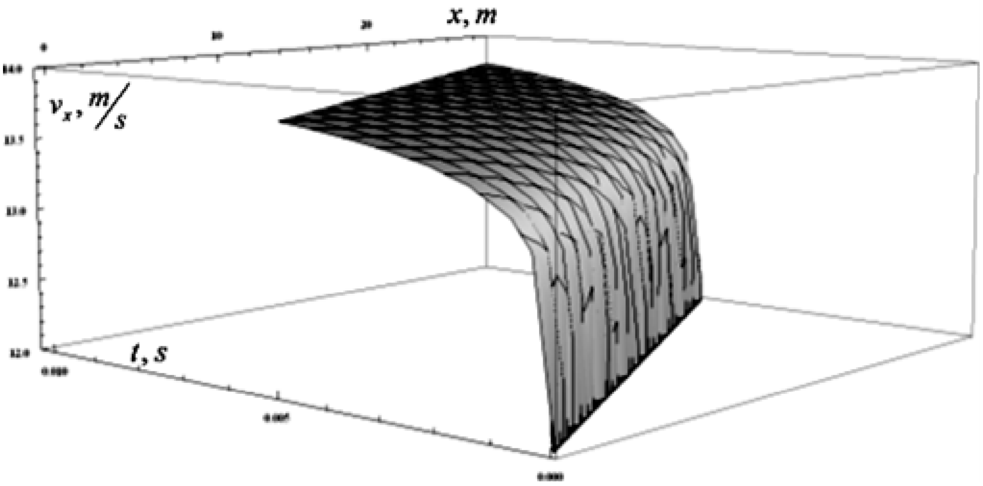

***Fig. 12.** The distribution of velocities of the half-space particles in the plastic region*

Derivation and investigation of singular solution of defining equations of elastic-tough-plastic state

All calculations presented above are based on the solution of equation (29) for derivation of which the condition that allow only increasing of the stress $\sigma_{yy}(z)$ has been imposed. This means that we neglected the sign of the absolute value of the $\dot{\sigma}_{yy}(z)$ in the second summand of the equation (29). This has been done consciously (knowingly) because this equation with the modulus has the strange (suprizing) solution with singularities. Taking into account the reccomendations [17], let us find this solution in the parametric form.

Let us make the change $\sigma_{yy} - 3,225 \cdot 10^8 = -u$ in the equation (29):

$$1,42 \cdot 10^7 |\dot{u}|^{1/12} - 8,57 \cdot 10^{-6}\dot{u} - u = 0. \tag{46}$$

Let us introduce the following parameter $\dot{u} = w = -\dot{\sigma}_{yy}$. Hence, from the equation (46) we obtain:

$$1,42 \cdot 10^7 |w|^{1/12} - 8,57 \cdot 10^{-6} w = u. \tag{47}$$

After differentiating (47) with respect to time z, we may write down:

$$w = \left(\frac{1,42 \cdot 10^7}{12} |w|^{-\frac{11}{12}} \operatorname{sign} w - 8,57 \cdot 10^{-6} \right) \dot{w}. \tag{48}$$

Let us rewrite this equation in the following form:

$$\frac{dz}{dw}=\frac{1,42\cdot 10^{7}}{12}\left|w\right|^{-\frac{23}{12}}-\frac{8,57\cdot 10^{-6}}{w}. \tag{49}$$

After integrating (49), let us find the general integral:

$$z=-\frac{1,42\cdot 10^{7}}{11}\left|w\right|^{-\frac{11}{12}} signw-8,57\cdot 10^{-6}\ln\left|w\right|+C. \tag{50}$$

Reverting to the equation (47), after making the recurrent change, we obtain:

$$s_{yy}=3,225\cdot 10^{8}+8,57\cdot 10^{-6}w-1,42\cdot 10^{7}\left|w\right|^{1/12}. \tag{51}$$

The relations (50) and (51) are the general solution of the differential equation (29) written in the parametric form. In order to define the constant of integration in the equation (50), let us find w_0 on the boundary of the front of yield delay wave. After substitution the value $s_{yy}(0)=1,79\cdot 10^{8}\,Pa$ into (51), we obtain the corresponding equation, after solving of which we obtain $w_0=-6,865\cdot 10^{11}\frac{Pa}{s}$. On the basis of equation (50), we obtain the following value of the constant of integration:

$$C=\frac{1,42\cdot 10^{7}}{11}\left|w_0\right|^{-\frac{11}{12}} signw_0+8,57\cdot 10^{-6}\ln\left|w_0\right|. \tag{52}$$

After substitution of (52) into (50), we may write:

$$z=8,57\cdot 10^{-6}\ln\left|\frac{w_0}{w}\right|+1,29\cdot 10^{6}\left(\left|w_0\right|^{-\frac{11}{12}} signw_0-\left|w\right|^{-\frac{11}{12}} signw\right). \tag{53}$$

In order to define the value of parameter w and of the corresponding moments of time when the solution of the equation (29) has the singularity in the form of discontinuity point of the first kind, let us equate the right part of (49) to zero:

$$1,18\cdot 10^{6}-\left|w\right|^{\frac{11}{12}}\cdot 8,57\cdot 10^{-6}=0. \tag{54}$$

Solving this equation and choosing the necessary root, we obtain:

$$w_2=1,4219\cdot 10^{12}\frac{Pa}{s}. \tag{55}$$

Now let us define the value of stress s_{yy} for the found w_2. Substituting w_2 into (51), we obtain:

$$s_{yy}(w_2)=3,225\cdot 10^{8}+8,57\cdot 10^{-6}w_2-1,42\cdot 10^{7}\left|w_2\right|^{1/12}=1,8846\cdot 10^{8}\,Pa. \tag{56}$$

Substituting the defined $s_{yy}(w_2)$ into the equation (51), let us determine the values of w which satisfy the obtained equation:

$$s_{yy}(w_2)=3,225\cdot 10^{8}+8,57\cdot 10^{-6}w-1,42\cdot 10^{7}\left|w\right|^{1/12}. \tag{57}$$

The results of calculation allow us to find the following solutions:

$$w_1=-3,743\cdot 10^{11}\frac{Pa}{s},\quad w_2=1,4219\cdot 10^{12}\frac{Pa}{s}. \tag{58}$$

Now we must define w_3 for which the transition from w_2 to w_3 is carried out at the constant time. This value may be calculated if we make the change of w_0 to w_2 and equate the time z to zero in the equation (53):

$$8,57\cdot 10^{-6}\ln\left|\frac{w_2}{w}\right|+1,29\cdot 10^{6}\left(\left|w_2\right|^{-\frac{11}{12}} signw_2-\left|w\right|^{=\frac{11}{12}} signw\right)=0. \tag{59}$$

Solving this equation, we obtain the following real root:

$$w_3 = -5{,}735 \cdot 10^{12} \frac{Pa}{s}. \tag{60}$$

In order to construct the plot of solution of the equation (29) on the basis of its parametric forms (50) and (51), it is expedient to find the initial time range Δz_0 when $w \in [w_0, w_1]$ and the iteration time intervals Δz_1 for $w \in [w_3, w_1]$. On the basis of equation (53) after making the following substitutions, we obtain:

$$\Delta z_0 = 1{,}875 \cdot 10^{-5}\, s, \quad \Delta z_1 = 5{,}257 \cdot 10^{-5}\, s. \tag{61}$$

On the basis of the conclusion about the state of the medium between the fronts of the waves (Fig. 5) and of the singular solution for the plastic region of the half-space material, let us construct the plot of dependence $s_{yy}(z)$ for various time intervals.

In the case when $z \in [0, t]$, the constant stress $s^*_{yy} = 1{,}79 \cdot 10^8\, Pa$ is to be transited behind the front of the elastic wave. For the time range $z \in [t, t + \Delta z_0]$, the law of changing of the stress $z \in [t, t + \Delta z_0]$ may be defined on the basis of solution of differential equation (29) written in the parametric form in the view of relations (51) and (53). Similarly, for the time range $z \in [t + \Delta z_0, t + \Delta z_0 + \Delta z_1]$ we use the same equations but after the changing of w_0 to w_3 in the equation (53). For the further time z course, the iteration of these solutions takes place, on the basis of which we may construct the plot of dependence $s_{yy}(z)$ (Fig. 13).

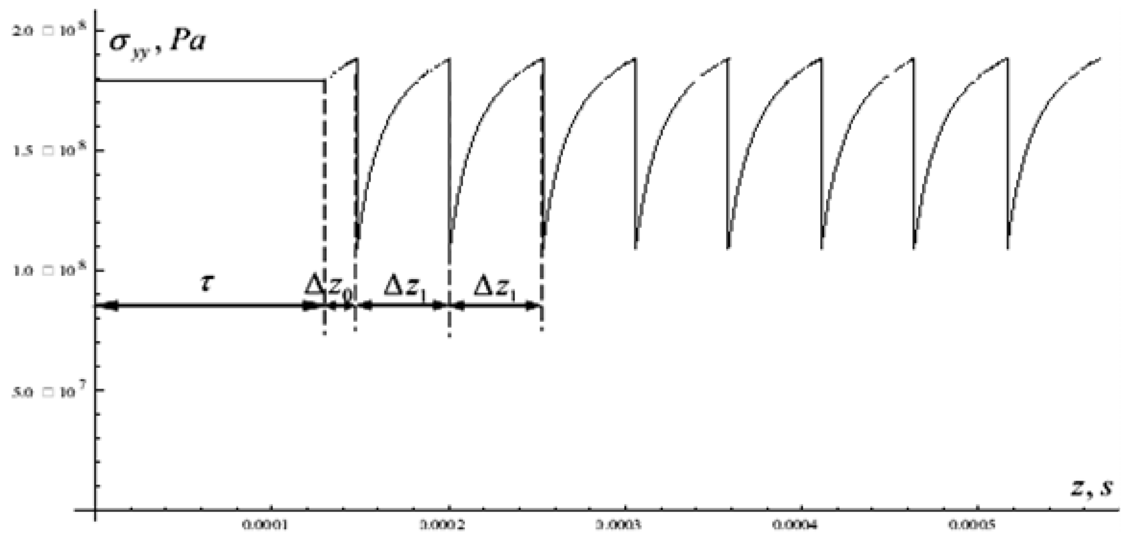

Fig. 13. *The plot of presentation of singular solution of the material state equation*

The spacial pattern of changing of the stress $s_{yy}(x,t)$ may be obtained by making the change $z \to t - t - \dfrac{x}{a_1}$. After making the corresponding numerical calculations we obtain the field of deformations shown in Fig. 14.

The obtained result shows the strange behavior of the material. The strangeness consists in the fact that at the constant stress s_{xx}, the pulsations of stresses in the transverse direction and the impulses of deformations along the *x*-axis take place. The whole material of the disturbed half-space in the plastic region becomes speckled caused by its “trembling”. On this stage of investigation it is not possible to define the criterion of truth (validity) of realization of regular or singular solution of the plastic state equation. However, the possibility of occurrence of the phenomenon of “trembling” of the material supports (confirms) the point of view that plastic deformation should be studied as the wave process.

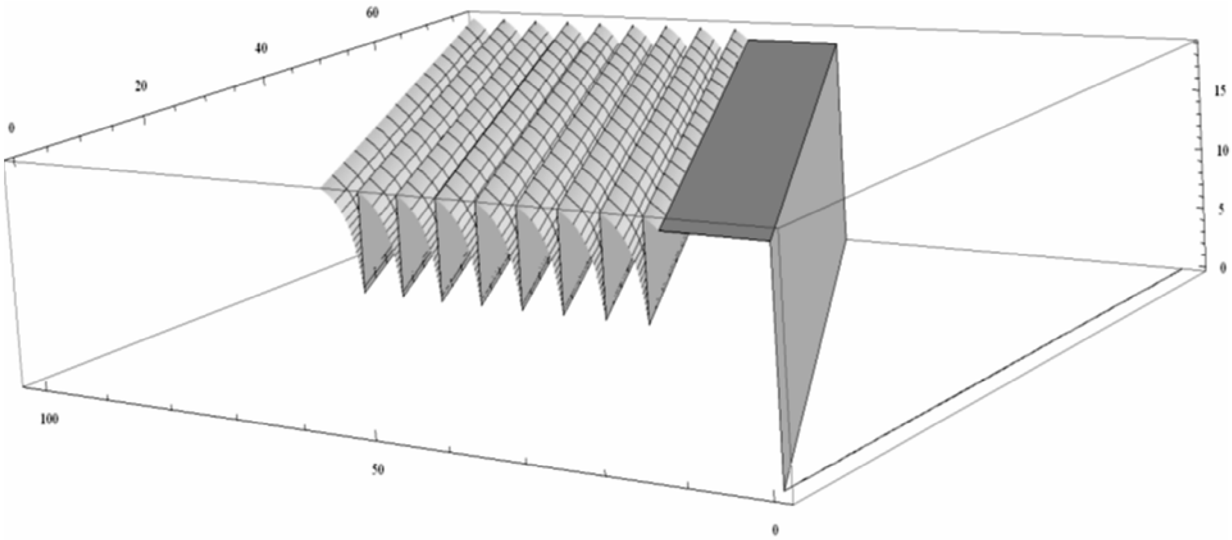

Fig. 14. *The field of stresses* $s_{yy}(x,t)$ *for all regions of the disturbed half-space when the singuliarity of the solution of the plastic state equations is taken into account*

Conclusions

The presentation of use of electromechanical model of ideal elastic-tough-plastic material with yield delay is carried out while investigating the propagation of plane one-dimensional waves. The difference of propagation of plane elastic-plastic waves in material without yield delay and of propagation of elastic-tough-plastic waves in the half-space filled by the material that possesses the property of yield delay is shown. On the basis of regular solution of the defining equations for ideal elastic-tough-plastic material, the field of stresses, deformations and velocities of the medium particles are obtained. The possibility of occurrence of singular solution with singularities in the form of discontinuity point of the first kind is substantiated. The character of behavior of the half-space material obtained on the basis of such solution looks like “trembling”. Such result supports (confirms) the point of view that plastic deformation should be studied as the wave process.

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