ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

COMPORTAMIENTO ASINTÓTICO DE SOLUCIONES NO OSCILATORIAS DE CUARTO ORDEN DE ECUACIONES DIFERENCIALES LINEALES

ANíBAL CORONEL*, FERNANDO HUANCAS † , AND MANUEL PINTO**

Abstract

This article deals with the asymptotic behavior of nonoscillatory solutions of fourth order linear differential equation where the coefficients are perturbations of linear constant coefficient equation. We define a change of variable and deduce that the new variable satisfies a third order nonlinear differential equation. We assume three hypotheses. The first hypothesis is related to the constant coefficients and set up that the characteristic polynomial associated with the fourth order linear equation has simple and real roots. The other two hypotheses are related to the behavior of the perturbation functions and establish asymptotic integral smallness conditions of the perturbations. Under these general hypotheses, we obtain four main results. The first two results are related to the application of a fixed point argument to prove that the nonlinear third order equation has a unique solution. The next result concerns with the asymptotic behavior of the solutions of the nonlinear third order equation. The fourth main theorem is introduced to establish the existence of a fundamental system of solutions and to precise the formulas for the asymptotic behavior of the linear fourth order differential equation. In addition, we present an example to show that the results introduced in this paper can be applied in situations where the assumptions of some classical theorems are not satisfied.

Keywords. Poincaré-Perron problem; asymptotic behavior; Riccati type equations.

Resumen

Este artículo trata sobre el comportamiento asintótico de soluciones no oscilatorias de cuarto orden de ecuaciones diferenciales lineales donde los coeficientes son perturbaciones de la ecuación coeficiente constante lineal. Definimos un cambio de variable y deducimos que la nueva variable satisface una ecuación diferencial no lineal de tercer orden. Suponemos tres hipótesis. La primera hipótesis está relacionado con los coeficientes constantes y establece que la característica del polinomio asociado a la ecuación lineal de cuarto orden tiene raíces simples y reales. Las otras dos hipótesis están relacionadas con el comportamiento de las funciones de perturbación y establecen pequeñas condiciones de perturbación para las integrales asintóticas. Bajo estas hipótesis generales, se obtienen cuatro resultados principales. Los dos primeros resultados están relacionados con la aplicación de un argumento punto fijo para demostrar que el tercero no lineal ecuación de orden tiene una solución única. El siguiente resultado esta relacionado con el comportamiento asintótico de las soluciones no lineales de la ecuación de tercer orden. El cuarto principal teorema se introduce para establecer la existencia de un sistema fundamental de soluciones y precisa las fórmulas para el comportamiento asintótico de la cuarta ecuación diferencial de orden lineal. En adición, presentamos un ejemplo para mostrar que los resultados introducidos en este de artículo se pueden aplicar en situaciones en las suposiciones de algunos teoremas clásicas no están satisfechos.

Palabras clave. Problema de Poincaré-Perron, comportamiento asintótico, ecuaciones tipo Riccati.

_____

*GMA, Departamento de Ciencias Básicas, Universidad del Bío-Bío, Chile, (`acoronel@ubiobio.cl,fihuanca@gmail.com`).

**Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Chile (`pintoj@uchile.cl`).

1. Introduction. Linear fourth-order differential equations appear as the more basic mathematical models in several areas of science and engineering. These simplified equations arise from different linearization approaches used to give an ideal description of the physical phenomenon or to analyze (analytically solve or numerically simulate) the corresponding nonlinear governing equations. For example in the following cases, the one-dimensional model of Euler-Bernoulli in linear theory of elasticity [1, 29], the optimization of quadratic functionals in optimization theory [1], the mathematical model in viscoelastic flows [8, 19], and the biharmonic equations in radial coordinates in harmonic analysis [13, 18].

An important family of linear fourth order differential equations is given by the equations of the following type

$$(1.1) \qquad y^{(\mathrm{iv})} + [a_3 + r_3(t)]y''' + [a_2 + r_2(t)]y'' + [a_1 + r_1(t)]y' + [a_0 + r_0(t)]y = 0,$$

where a_i are constants and r_i are real-valued functions. Note that (1.1) is a perturbation of the following constant coefficient equation:

$$(1.2) \qquad\qquad y^{(\mathrm{iv})} + a_3y''' + a_2y'' + a_1y' + a_0y = 0.$$

We recall that the study of perturbed equations of the type (1.1), in the general case of n-order equations, was motivated by Poincaré [25]. Thus (1.1) is also known as the scalar linear differential equation of Poincaré type. Moreover, we recall that the classical analysis introduced in the seminal work [25] is mainly focused on two questions: the existence of a fundamental system of solutions for (1.1) and the characterization of the asymptotic behavior of its solutions. Later on, equations of the Ponicaré type (1.1) (of different orders) have been investigated by several authors with a long and rich history of results [4, 9, 15, 16]. Even though this is an old problem, it is still an issue which does not lose its topicality and importance in the research community. For instance, in the case of asymptotic behavior of third order equations, there are the following newer results [11, 12, 23, 27, 28].

In this contribution, we address the question of the asymptotic behavior of (1.1) under new general hypotheses for the perturbation functions. Historically, some landmarks in the analysis of the asymptotic behavior in linear ordinary differential equations are given by the works of Poincaré [25], Perron [22], Levinson [20], Hartman-Wintner [17] and Harris and Lutz [14, 15], see [7] for a short review.

Nowadays, there are three big approaches to study the problem of asymptotic behavior of solutions for scalar linear differential equations of Poincaré type: the analytic theory, the nonanalytic theory and the scalar method. In a broad sense, we recall that the essence of the analytic theory consists of the assumption of some representation of the coefficients and of the solution, for instance power series representation (see [5] for details). In relation to the nonanalytic theory, we know that the methods are procedures that consist of two main steps: first, a change of variable to transform the scalar perturbed linear differential equation in a system of first order of Poincaré type and then a diagonalization process (for further details, consult [6, 9, 10, 21]). Meanwhile, in the scalar method [2, 3, 4, 11, 12, 23, 27, 28] the asymptotic behavior of solutions for scalar linear differential equations of Poincaré type is obtained by a change of variable which reduces the order and transforms the perturbed linear differential equation in a nonlinear equation. Then, the results for the original problem are derived by analyzing the asymptotic behavior of this nonlinear equation.

In this paper we consider the scalar method. Moreover we note that this paper is a short version of the paper [7], recently published by the authors which can be consulted by further details and the extended proofs of the results.

2. General assumptions. For convenience of the presentation, we introduce some notation and summarize the main general hypotheses about the coefficients and perturbation functions in the following list

(H$_1$) $\left\{\lambda_i, i = \overline{1,4} \ : \ \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4\right\} \subset \mathbb{R}$ is the set of characteristic roots for (1.2).

(H$_2$) Consider $p(\lambda_i, s)$ defined by

$$p(\lambda_i, s) = \lambda_i^3 r_3(s) + \lambda_i^2 r_2(s) + \lambda_i r_1(s) + r_0(s).$$

The perturbation functions are selected such that $\mathcal{G}(p(\lambda_i, \cdot))(t) \to 0$ and $\mathcal{L}(r_j)(t) \to 0$,

$j = 0, 1, 2, 3$, when $t \to \infty$, where $\mathcal{G}$ and $\mathcal{L}$ are the functionals defined as follows

$$(2.1) \qquad \mathcal{G}(E)(t) = \left| \int_{t_0}^{\infty} g(t,s)E(s)ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s)E(s)ds \right|$$

$$+ \left| \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t,s)E(s)ds \right|,$$

$$(2.2) \qquad \mathcal{L}(E)(t) = \int_{t_0}^{\infty} \left[|g(t,s)| + \left| \frac{\partial g}{\partial t}(t,s) \right| + \left| \frac{\partial^2 g}{\partial t^2}(t,s) \right| \right] |E(s)| ds.$$

(H$_3$) Let us introduce some notations. Consider the operators $\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_4$ defined as follows

$$\mathbb{F}_1(E)(t) = \int_t^{\infty} e^{-(\lambda_2 - \lambda_1)(t-s)} |E(s)| ds,$$

$$\mathbb{F}_2(E)(t) = \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} |E(s)| ds + \int_t^{\infty} e^{-(\lambda_3 - \lambda_2)(t-s)} |E(s)| ds,$$

$$\mathbb{F}_3(E)(t) = \int_{t_0}^t e^{-(\lambda_2 - \lambda_3)(t-s)} |E(s)| ds + \int_t^{\infty} e^{-(\lambda_4 - \lambda_3)(t-s)} |E(s)| ds,$$

$$\mathbb{F}_4(E)(t) = \int_{t_0}^t e^{-(\lambda_3 - \lambda_4)(t-s)} |E(s)| ds;$$

and σ_i, A_i defined by

$$\sigma_i = 3|\lambda_i|^2 + 5|\lambda_i| + 3$$

$$+ \left(19 + 7|\lambda_i| + |12\lambda_i + 3a_3| + |6\lambda_i^2 + 3\lambda_i a_3 + a_2| \right)\eta, \quad \eta \in]0, 1/2[,$$

$$A_i = \frac{1}{|\Upsilon_i|} \sum_{(j,k,\ell) \in I_i} |\lambda_k - \lambda_\ell| \left(1 + |\lambda_j - \lambda_i| + |\lambda_j - \lambda_i|^2 \right),$$

with

$$\Upsilon_i = \prod_{k>j} (\lambda_k - \lambda_j), \quad k, j \in \{1,2,3,4\} - \{i\},$$

$$I_i = \left\{ (j,k,\ell) \in \{1,2,3,4\}^3 \quad : \quad (j,k,\ell) \neq (i,i,i), \ (k,\ell) \neq (j,j) \right\}.$$

Then, it is considered that the following inequality

$$\mathbb{F}_i(r_j)(t) \leq \rho_i := \text{mín}\{\mathbb{F}_i(1)(t), (A_i\sigma_i)^{-1}\}$$

holds. Thus, defining the sets

$$\mathcal{F}_i([t_0, \infty[) = \left\{ E : [t_0, \infty[\to \mathbb{R} \ : \ \mathbb{F}_i(E)(t) \leq \rho_i \right\},$$

we assume that the perturbation functions $r_0, r_1, r_2, r_3 \in \mathcal{F}_i([t_0, \infty[)$.

3. Revisited scalar method. Statement of the results. In this section we present the scalar method as a process of three steps. At each step we present the statement of the main results.

3.1. Change of variable and reduction of the order. We introduce a little bit different change of variable to those proposed by Bellman. Here, in this paper, the new variable z is of the following type

$$(3.1) \qquad z(t) = \frac{y'(t)}{y(t)} - \mu \quad \text{or equivalently} \quad y(t) = \exp \left(\int_{t_0}^t (z(s) + \mu) ds \right),$$

where y is a solution of (1.1) and μ is an arbitrary root of the characteristic polynomial associated to (1.2). Then, by differentiation of $y(t)$ and by replacing the results in (1.1), we deduce that z is

a solution of the following third order nonlinear equation

(3.2)
$$z''' + [4\mu + a_3]z'' + [6\mu^2 + 3a_3\mu + a_2]z' + [4\mu^3 + 3\mu^2 a_3 + 2\mu a_2 + a_1]z$$
$$= -\Big\{ r_3(t)z'' + [3\mu r_3(t) + r_2(t)]z' + [3\mu^2 r_3(t) + 2\mu r_2(t) + r_1(t)]z$$
$$+\mu^3 r_3(t) + \mu^2 r_2(t) + \mu r(t) + r_0(t) + 4zz'' + [12\mu + 3a_3 + 3r_3(t)]zz'$$
$$+6z^2 z' + 3[z']^2 + [6\mu^2 + 3\mu a_3 + a_2 + 3\mu r_3(t) + r_2(t)]z^2$$
$$+[4\mu + r_3(t)]z^3 + z^4 \Big\}.$$

Thus, the analysis of original linear perturbed equation of fourth order (1.1) is translated to the analysis of a nonlinear third order equation (3.2).

We note that the characteristic polynomial associated to (1.2) and the third constant coefficient equation defined by the left hand side of (3.2) are related in the sense Proposition 3.1. Thus, noticing that the change of variable (3.1) can be applied by each characteristic root λ_i and assuming that the equation (3.2) with $\mu = \lambda_i$ has a solution, we can prove that (1.1) has a fundamental system of solutions, see Lemma 3.2.

PROPOSITION 3.1. *If λ_i and λ_j are two distinct characteristic roots of the polynomial associated to (1.2), then $\lambda_j - \lambda_i$ is a root of the characteristic polynomial associated with the following differential equation*

$$z''' + [4\lambda_i + a_3]z'' + [6\lambda_i^2 + 3a_3\lambda_i + a_2]z' + [4\lambda_i^3 + 3\lambda_i^2 a_3 + 2\lambda_i a_2 + a_1]z = 0.$$

LEMMA 3.2. *Consider that (3.2) has a solution for each $\mu \in \{\lambda_1, \ldots, \lambda_4\}$. If the hypothesis (H_1) is satisfied, then the fundamental system of solutions of (1.1) is given by*

(3.3)
$$y_i(t) = \exp\Big(\int_{t_0}^t [\lambda_i + z_i(s)]ds \Big), \quad i \in \{1,2,3,4\},$$

where z_i is the solution of (3.2) with $\mu = \lambda_i$.

3.2. Well posedness and asymptotic behavior of (3.2). In this second step, we obtain three results. The first result is related to the conditions for the existence and uniqueness of a more general equation of that given in (3.2), see Theorem 3.3. Then, we introduce a second result concerning to the well posedness of (3.2), see Theorem 3.4. Finally, we present the result of asymptotic behavior for (3.2), see Theorem 3.5. Indeed, to be precise these three results are the following theorems:

THEOREM 3.3. *Given $t_0 \in \mathbb{R}$, let us introduce the notation $C_0^2([t_0, \infty[)$ for the following space of functions*

$$C_0^2([t_0, \infty[) = \Big\{ z \in C^2([t_0, \infty[, \mathbb{R}) \quad : \quad z(t), z'(t), z''(t) \to 0 \text{ when } t \to \infty \Big\},$$

and consider the equation

(3.4)
$$z''' + b_2 z'' + b_1 z' + b_0 z = \Omega(t) + F(t, z, z', z''),$$

where b_i are real constants, Ω and F are given functions such that the following restrictions
$(\mathcal{R}_1)$ *There are the functions $\hat{F}_1, \hat{F}_2, \Gamma : \mathbb{R}^4 \to \mathbb{R}$; $\Lambda_1, \Lambda_2 : \mathbb{R} \to \mathbb{R}^3$ and $\mathbf{C} \in \mathbb{R}^7$, such that*

$$F = \hat{F}_1 + \hat{F}_2 + \Gamma,$$
$$\hat{F}_1(t, x_1, x_2, x_3) = \Lambda_1(t) \cdot (x_1, x_2, x_3),$$
$$\hat{F}_2(t, x_1, x_2, x_3) = \Lambda_2(t) \cdot (x_1 x_2, x_1^2, x_1^3),$$
$$\Gamma(t, x_1, x_2, x_3) = \mathbf{C} \cdot (x_2^2, x_1 x_2, x_1 x_3, x_1^2, x_1^2 x_2, x_1^3, x_1^4),$$

where "$\cdot$" denotes the canonical inner product in $\mathbb{R}^n$.
$(\mathcal{R}_2)$ *The set of characteristic roots of (3.4) when $\Omega = F = 0$ is given by $\{\gamma_1 > \gamma_2 > \gamma_3\} \subset \mathbb{R}$.*
$(\mathcal{R}_3)$ *It is assumed that $\mathcal{G}(\Omega)(t) \to 0$, $\mathcal{L}(\|\Lambda_1\|_1)(t) \to 0$ and $\mathcal{L}\big(\|\Lambda_2\|_1\big)(t)$ is bounded, when $t \to \infty$. Here $\| \cdot \|_1$ denotes the norm of the sum in $\mathbb{R}^n$, $\mathcal{G}$ and $\mathcal{L}$ are the operators defined on (2.1) and (2.2), respectively.*

hold. Then, exists a unique $z \in C_0^2([t_0, \infty[)$ solution of (3.4).

THEOREM 3.4. *Let us consider that the hypotheses (H_1) and (H_2) are satisfied. Then, the equation (3.2) with $\mu = \lambda_i$ has a unique solution z_i such that $z_i \in C_0^2([t_0, \infty[)$.*

THEOREM 3.5. *Consider that the hypotheses (H_1),(H_2) and (H_3) are satisfied. Then z_i the solution of (3.2) with $\mu = \lambda_i$, has the following asymptotic behavior*

$$(3.5) \qquad z_i(t), z_i'(t), z_i''(t) = \begin{cases} O\left(\displaystyle\int_t^\infty e^{-\beta(t-s)} |p(\lambda_1, s)| ds \right), & i = 1, \beta \in [\lambda_2 - \lambda_1, 0[, \\ O\left(\displaystyle\int_{t_0}^\infty e^{-\beta(t-s)} |p(\lambda_2, s)| ds \right), & i = 2, \beta \in [\lambda_3 - \lambda_2, 0[, \\ O\left(\displaystyle\int_{t_0}^\infty e^{-\beta(t-s)} |p(\lambda_3, s)| ds \right), & i = 3, \beta \in [\lambda_4 - \lambda_3, 0[, \\ O\left(\displaystyle\int_{t_0}^t e^{-\beta(t-s)} |p(\lambda_4, s)| ds \right), & i = 4, \beta \in]0, \lambda_3 - \lambda_4], \end{cases}$$

where $p(\lambda_i, s) = \lambda_i^3 r_3(s) + \lambda_i^2 r_2(s) + \lambda_i r_1(s) + r_0(s)$.

3.3. Existence of a fundamental system of solutions for (1.1) **and its asymptotic behavior.** Here we translate the results for the behavior of z (see Theorem 3.4) to the variable y via the relation (3.1).

THEOREM 3.6. *Let us assume that the hypothesis (H_1) and (H_2) are satisfied. Denote by $W[y_1, \ldots, y_4]$ the Wronskian of $\{y_1, \ldots, y_4\}$, by π_i the number defined as follows*

$$\pi_i = \prod_{k \in N_i} (\lambda_k - \lambda_i), \quad N_i = \{1, 2, 3, 4\} - \{i\}, \quad i = 1, \ldots, 4,$$

by $p(\lambda_i, s)$ the function defined in Theorem 3.5 and by F the function defined in Theorem 3.3 with Λ_1, Λ_2 and $\mathbf{C}$ adequately defined. Then, the equation (1.1) has a fundamental system of solutions given by (3.3). Moreover the following properties about the asymptotic behavior

$$\frac{y_i'(t)}{y_i(t)} = \lambda_i, \quad \frac{y_i''(t)}{y_i(t)} = \lambda_i^2, \quad \frac{y_i'''(t)}{y_i(t)} = \lambda_i^3, \quad \frac{y_i^{(iv)}(t)}{y_i(t)} = \lambda_i^4,$$

$$W[y_1, \ldots, y_4] = \prod_{1 \le k < \ell \le 4} (\lambda_\ell - \lambda_k) \, y_1 y_2 y_3 y_4 \, (1 + o(1)),$$

are satisfied when $t \to \infty$. Furthermore, if (H_3) is satisfied, then

$$y_i(t) = e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t \Big[p(\lambda_i, s) + F(s, z_i(s), z_i'(s), z_i''(s)) \Big] ds \right),$$

$$y_i'(t) = \left(\lambda_i + o(1) \right) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t \Big[p(\lambda_i, s) + F(s, z_i(s), z_i'(s), z_i''(s)) \Big] ds \right),$$

$$y_i''(t) = \left(\lambda_i^2 + o(1) \right) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t \Big[p(\lambda_i, s) + F(s, z_i(s), z_i'(s), z_i''(s)) \Big] ds \right),$$

$$y_i'''(t) = \left(\lambda_i^3 + o(1) \right) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t \Big[p(\lambda_i, s) + F(s, z_i(s), z_i'(s), z_i''(s)) \Big] ds \right),$$

$$y_i^{(iv)}(t) = \left(\lambda_i^4 + o(1) \right) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t \Big[p(\lambda_i, s) + F(s, z_i(s), z_i'(s), z_i''(s)) \Big] ds \right),$$

holds, when $t \to \infty$ with z_i, z_i' and z_i'' given asymptotically by (3.5).

4. Sketch of the proof of the results. In this section we present the guidelines of the proofs of Theorems 3.3, 3.4, 3.5, and 3.6, for a complete proof of this results consult the paper [7], recently published by the authors.

4.1. Proof of Theorem 3.3. By the method of variation of parameters, the hypothesis ($\mathcal{R}_2$), implies that the equation (3.4) is equivalent to the following integral equation

$$(4.1) \qquad z(t) = \int_{t_0}^\infty g(t, s) \Big[\Omega(s) + F\big(s, z(s), z'(s), z''(s)\big) \Big] ds,$$

where g is the Green function adequately defined. Moreover, we recall that $C_0^2([t_0, \infty[)$ is a Banach space with the norm $\|z\|_0 = \sup_{t \geq t_0}[|z(t)| + |z'(t)| + |z''(t)|]$. Now, we define the operator T from $C_0^2([t_0, \infty[)$ to $C_0^2([t_0, \infty[)$ as follows

$$(4.2) \qquad Tz(t) = \int_{t_0}^{\infty} g(t,s)\Big[\Omega(s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big]ds.$$

Then, we note that (4.1) can be rewritten as the operator equation

$$(4.3) \qquad Tz = z \qquad \text{over} \qquad D_\eta := \Big\{z \in C_0^2([t_0, \infty[) \quad : \quad \|z\|_0 \leq \eta\Big\},$$

where $\eta \in \mathbb{R}^+$ will be selected in order to apply the Banach fixed point theorem. Indeed, the rest of the proof is reduced to prove that T satisfies the hypotheses of Banach fixed point theorem. Thus, we deduce that there are a unique $z \in D_\eta \subset C_0^2([t_0, \infty[)$ solution of (4.3).

4.2. Proof of Theorem 3.4. The proof of the Theorem 3.4 is followed by the application of Theorem 3.3 to equation (3.2). Indeed, we can verify the hypothesis $(\mathcal{R}_1)$-$(\mathcal{R}_3)$. First, the hypothesis $(\mathcal{R}_1)$ is satisfied since (3.3) can be rewritten as (3.4). Second by the application of Proposition 3.1, we deduce that the hypothesis $(\mathcal{R}_2)$ is satisfied. Meanwhile, we note that (H_2) implies $(\mathcal{R}_3)$. Thus, we deduce that conclusion of the Theorem 3.4 is valid.

4.3. Proof of Theorem 3.5. We prove the formula (3.5) by analyzing an iterative sequence and using the properties of the operator T defined in (4.2). For instance, in the first case we note that the operator T can be rewritten equivalently as follows

$$Tz(t) = \frac{1}{\Upsilon_1} \int_t^{\infty} g_1(t,s)\Big[p(\lambda_1, s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big]ds, \quad \text{for } t \geq t_0,$$

since $g_1(t,s) = 0$ for $s \in [t_0, t]$. Thus, the proof of (3.5) with $i = 1$ is reduced to prove that

$$(4.4) \qquad \exists\, \Phi_n \in \mathbb{R}_+ \ : \ |\omega_n(t)| + |\omega_n'(t)| + |\omega_n''(t)| \leq \Phi_n \int_t^{\infty} e^{-\beta(t-\tau)}|p(\lambda_1, \tau)|d\tau, \ \forall t \geq t_0,$$

$$(4.5) \qquad \exists\, \Phi \in \mathbb{R}_+ \ : \ \Phi_n \to \Phi, \text{ when } n \to \infty.$$

Hence, to complete the proof of (3.5) with $i = 1$, we proceed to prove (4.4) by mathematical induction on n and deduce that (4.5) is a consequence of the construction of the sequence $\{\Phi_n\}$.

5. Example. In this section we consider an example where some classical results cannot be applied. However, we can apply the Theorem 3.6. Indeed, consider the differential equation

$$(5.1) \qquad y^{(iv)} - 5y'' + [\sin(t^q) + 4]y = 0, \quad \text{with } q \in]2, \infty[,$$

which is of type (1.1) with $(a_0, a_1, a_2, a_3) = (4, 0, -5, 0)$ and $(r_0, r_1, r_2, r_3)(t) = (\sin(t^q), 0, 0, 0)$. Then, the classical generalizations of Poincaré type theorems [26], the Levinson Theorem [9, Theorem 1.3.1], the Hartman-Wintner [9, Theorem 1.5.1] or the Eastham Theorem [9, Theorem 1.6.1] can not be applied to obtain the asymptotic behavior of (5.1). However, the hypotheses of Theorem 3.6 are satisfied see [7]. Then, the asymptotic formulas are given by

$$y_1(t) = e^{2t} \exp\Big(\frac{1}{12} \int_{t_0}^t \Big\{\sin(s^q) - f_1(s)\Big\}ds\Big),$$

$$y_2(t) = e^t \exp\Big(-\frac{1}{6} \int_{t_0}^t \Big\{\sin(s^q) - f_2(s)\Big\}ds\Big),$$

$$y_3(t) = e^{-t} \exp\Big(\frac{1}{6} \int_{t_0}^t \Big\{\sin(s^q) - f_3(s)\Big\}ds\Big),$$

$$y_4(t) = e^{-2t} \exp\Big(-\frac{1}{12} \int_{t_0}^t \Big\{\sin(s^q) - f_4(s)\Big\}ds\Big),$$

where

$$f_1(t) = 3(z_1'(s))^2 + 24z_1''(s) + 4z_1(s)z_1''(s) + 6[z_1(s)]^2 z_1'(s) + 8[z_1(s)]^3 + [z_1(s)]^4$$
$$f_2(t) = 3(z_2'(s))^2 + 12z_2^2(s) + 4z_2(s)z_2''(s) + 12[z_2(s)]^2 z_2'(s) + 4[z_2(s)]^3 + [z_2(s)]^4$$
$$f_3(t) = 3(z_3'(s))^2 - 6z_3^2(s) + 4z_3(s)z_3''(s) + 6[z_3(s)]^2 z_3'(s) - 4[z_3(s)]^3 + [z_3(s)]^4$$
$$f_4(t) = 3(z_4'(s))^2 - 12z_4^2(s) + 4z_4(s)z_4''(s) + 6[z_4(s)]^2 z_4'(s) - 8[z_4(s)]^3 + [z_4(s)]^4$$

and $z_i(t)$ satisfies the following asymptotic behavior

$$z_i(t), z_i'(t), z_i''(t) = \begin{cases} O\Big(\displaystyle\int_t^\infty e^{-\beta(t-s)}|\sin(s^p)|ds\Big), & i = 1, \quad \beta \in [-1, 0[, \\ O\Big(\displaystyle\int_{t_0}^\infty e^{-\beta(t-s)}|\sin(s^p)|ds\Big), & i = 2, \quad \beta \in [-2, 0[, \\ O\Big(\displaystyle\int_{t_0}^\infty e^{-\beta(t-s)}|\sin(s^p)|ds\Big), & i = 3, \quad \beta \in [-1, 0[, \\ O\Big(\displaystyle\int_{t_0}^t e^{-\beta(t-s)}|\sin(s^p)|ds\Big), & i = 4, \quad \beta \in]0, 1]. \end{cases}$$

Referencias

[1]   A. R. AFTABIZADEH, Existence and uniqueness theorems for fourth-order boundary value problems, *J. Math. Anal. Appl.***116(2)** (1986), 415–426.

[2]   R. BELLMAN, *A Survey of the Theory of the Boundedness, Stability, and Asymptotic Behavior of Solutions of Linear and Non-linear Differential and Difference Equations*, Office of Naval Research of United States, Department of the Navy, NAVEXOS,P-596,1949

[3]   R. BELLMAN, On the asymptotic behavior of solutions of $u'' - (1 + f(t))u = 0$, *J. Math. Anal. Appl.*, **31(1)** (1950), 83–91.

[4]   R. BELLMAN, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[5]   E.A. CODDINGTON, N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[6]   W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*. D. C. Heath and Co., Boston, Mass., 1965.

[7]   A. Coronel, F. Huancas and M. Pinto; *Asymptotic integration of a linear fourth order differential equation of Poincaré type,* Electron. J. Qual. Theory Differ. Equ., **76** (2015) 1–24.

[8]   A. R. DAVIES, A. KARAGEORGHIS, T. N. PHILLIPS, Spectral Galerkin methods for the primary two-point boundary value problem in modelling viscoelastic flows, *Int. J. Numer. Methods Engng.*, **26** (1988), 647–662

[9]   M.S.P. EASTHAM, *The Asymptotic Solution of Linear Differential Dystems, Applications of the Levinson theorem*. London Mathematical Society Monographs, volume 4, Oxford University Press, New York, 1989.

[10] M.V. FEDORYUK, *Asymptotic Analysis:Linear ordinary differential equations* (Translated from the Russian by Andrew Rodick). Springer-Verlag, Berlin, 1993.

[11] P. FIGUEROA, M. PINTO Riccati equations and nonoscillatory solutions of third order differential equations, *Dynam. Systems Appl.*, **17(3-4)** (2008), 459–475.

[12] P. FIGUEROA, M. PINTO, L^p-solutions of Riccati-type differential equations and asymptotics of third order linear differential equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **17(4)** (2010), 555–571.

[13] F. GAZZOLA, H.C. GRUNAU, Radial entire solutions for supercritical biharmonic equations, *Math. Ann.*, **334(4)** (2006), 905–936.

[14] W.A. HARRIS JR., D.A. LUTZ, On the asymptotic integration of linear differential systems, *J. Math. Anal. Appl.*, **48(1)** (1974), 1–16.

[15] W.A. HARRIS JR., D.A. LUTZ, A unified theory of asymptotic integration, *J. Math. Anal. Appl.*, **57(3)** (1977), 571–586.

[16] P. HARTMAN, Unrestricted solution fields of almost-separable differential equations, *Trans. Amer. Math. Soc.*, **63** (1948), 560–580.

[17] P. HARTMAN, A. WINTNER, Asymptotic integrations of linear differential equations, *American Journal of Mathematics*, **77(1)** (1955), 45–86.

[18] P. KARAGEORGIS, Asymptotic expansion of radial solutions for supercritical biharmonic equations, *Nonlinear Differential Equations and Applications,* **19(4)** (2012), 401–415.

[19] A. LAMNII, O. EL-KHAYYARI, J. DABOUNOU, Solving linear fourth order boundary value problem by using a hyperbolic splines of order 4, *Int. Electron. J. Pure Appl. Math.*, **7(2)** (2014), 85–98.

[20] N. LEVINSON, The asymptotic nature of solutions of linear systems of differential equations, *Duke Mathematical Journal,* **15(1)** (1948), 111–126.

[21] F.W.J. OLVER, *Asymptotics and Special Functions.* (Reprint of the 1974 original Academic Press, New York) AKP Classics, A.K. Peters, Ltd., Wellesley, MA, 1997.

[22] O. PERRON, Ber einen satz des henr Poincaré, *J. Reine Angew. Math.*, **136** (1909), 17–37.

[23] B. PIETRUCZUK, Resonance phenomenon for potentials of wignervon neumann type, *Geometric Methods in Physics* (P. Kielanowski, S.T. Ali, A. Odesskii, A. Odzijewicz, M. Schlichenmaier, and T. Voronov, editors). Trends in Mathematics, Springer Basel, pages 203–207, 2013.

[24] M. PINTO, Null solutions of difference systems under vanishing perturbation, *J. Difference Equ. Appl.*, **9(1)** (2003), 1–13.

[25] H. POINCARÉ, Sur les equations lineaires aux differentielles ordinaires et aux differences finies, *Amer. J. Math.*, **7(3)** (1885), 203–258.

[26] J. Šimša, An extension of a theorem of Perron, *SIAM J. Math. Anal.*, **19(2)** (1988), 460–472.
[27] S.A. Stepin, The wkb method and dichotomy for ordinary differential equations, *Doklady Mathematics*, **72(2)** (2005), 749–752.
[28] S.A. Stepin, Asymptotic integration of nonoscillatory second-order differential equations, *Doklady Mathematics*, **82(2)** (2010), 751–754.
[29] S.P. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill Book, New York, NY, USA, 2nd edition, 1961.