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A New Solution to the Heat Equation in One-Dimension

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Abstract Recall from [1] the Green's function is a solution to the following initial value problem

$$u_t(x,t) - ku_{xx}(x,t) = 0$$

$$u(x,0) = \delta(x)$$

where δ is the Dirac Delta function, $(x,t) \in \mathbb{R} \times (0,\infty)$. In the present paper, inspired by the $T-X$ family of distributions [2] and the following fundamental solution

$$\Psi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(- \frac{x^2}{4kt}\Big)$$

we present a new solution to the above intitial value problem.

Keywords: Green's function, fundamental solution, heat equation

1 Introduction

The $T-X(W)$ Family of Distributions family of distributions is a generalization of the the beta-generated family of distributions first proposed by Eugene et.al [3]. In particular, let $r(t)$ be the PDF of the random variable $T \in [a,b]$, $-\infty \leq a < b \leq \infty$, and let $W(F(x))$ be a monotonic and absolutely continuous function of the CDF $F(x)$ of any random variable X. The CDF of a new family of distributions defined by Alzaatreh et.al [2] is given by

$$G(x) = \int_a^{W(F(x))} r(t)dt = R\{W(F(x))\}$$

where $R(\cdot)$ is the CDF of the random variable T and $a \geq 0$

Remark 1 *The PDF of the T-X(W) family of distributions is obtained by differentiating the CDF above*

Remark 2 *When we set $W(F(x)) := -ln(1-F(x))$, then we use the term "T-X Family of Distributions" to describe all sub-classes of the T-X(W) family of distributions induced by the weight function $W(x) = -ln(1-x)$. A description of different weight functions that are appropirate given the support of the random variable T is discussed in [2]*

A plethora of results studying properties and application of the T-X(W) family of distributions have appeared in the literature, and the research papers, assuming open access, can be easily obtained on the web via common search engines, like Google, etc.

Generalizations of the normal distribution [4] have been shown to be solutions of partial differential equations of mathematical physics, see [5,7], for example, and some of these partial differential equations are derivable via partial difference equations arising from random walks problems, by going in the continuum limit, and for example, see [6].

This paper is organized as follows. In the next section we present the new solution and show it graphically[see, fig 1 in appendix]. The final section is devoted to the conclusions, in which we leave the reader with a conjecture.

2 Main Result

First define

$$\Phi(x,t) := \int_0^x \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{r^2}{4kt}\Big)dr$$

and notice that we have, the following explicit representation

$$\Phi(x,t) = \frac{1}{2}\text{Erfc}\left[-\frac{x}{2\sqrt{kt}}\right]$$

where $\text{Erfc}[z] := 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$. Further observe that

$$\frac{d\Phi(x,t)}{dx} = \Psi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big)$$

With $k, t > 0$, one can check that

$$\int_{-\infty}^{\infty} \Psi(x,t)dx = 1$$

Thus, with $k, t > 0$, $\Psi(x,t)$ is a probability density function. Now we introduce the following

Definition 1 *A random variable X will be called Green-X distributed if the CDF admits the following integral representation*

$$J(x,t) := \int_0^{-log(1-F(x))} \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{m^2}{4kt}\Big)dm$$

where with $k, t > 0$ and the random variable X has CDF $F(x)$

From the above definition the following is immediate

Theorem 1 *The CDF of the Green-X family of distributions is given by*

$$J(x,t) = \frac{1}{2}Erfc\left[-\frac{-log[1-F(x)]}{2\sqrt{kt}}\right]$$

where $Erfc[z] := 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$, with $k, t > 0$, and the random variable X has CDF $F(x)$

Theorem 2 *The PDF of the Green-Standard Exponential distribution is a solution to the initial value problem*

$$j_t(x,t) - kj_{xx}(x,t) = 0$$
$$j(x,0) = \delta(x)$$

Proof. One can show that the PDF of the Green-Standard Exponential distribution is given by

$$j(x,t) = \frac{e^{-\frac{\text{Log}\left[e^{-x}\right]^2}{4kt}}}{2\sqrt{\pi}\sqrt{kt}}$$

and thus it is easy to verify for any $k > 0$

$$j_t(x,t) - kj_{xx}(x,t) = 0$$

Moreover, since

$$\int_{-\infty}^{\infty} j(x,0)dx = 1$$

It is clear that $j(x,0)$ behaves like the Dirac Delta function, hence the initial condition.

3 Concluding Remarks

In this paper we have shown the following conjecture below is true when $n = 1$

Conjecture 1

$$Z(x_1, x_2, \cdots, x_{n-1}, x_n, t) = \prod_{i=1}^{n} j(x_i, t)$$

where

$$j(x_i, t) = \frac{e^{-\frac{Log\left[e^{-x_i}\right]^2}{4kt}}}{2\sqrt{\pi}\sqrt{kt}}$$

is a solution to the the following initial value problem

$$Z_t(x_1, x_2, \cdots, x_n, t) = k[Z_{x_1x_1}(x_1, x_2, \cdots, x_n, t) + \cdots + Z_{x_{n-1}x_{n-1}}(x_1, x_2, \cdots, x_n, t) + Z_{x_{n-1}x_{n-1}}(x_1, x_2, \cdots, x_n, t)]$$

$$Z(x_1, \cdots, x_n, 0) = \delta^*(x_1, x_2, \cdots, x_n)$$

where δ^* is the n-dimensional Dirac Delta function

References

1. Wikipedia contributors. "Heat equation." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 16 Apr. 2018. Web. 31 May. 2018
2. A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, Metron, 71(1), (2013), 63-79
3. Eugene, N, Lee, C, Famoye, F: The beta-normal distribution and its applications. Communications in Statistics-Theory and Methods 31(4), 497âĂŞ512 (2002)
4. Wikipedia contributors. "Normal distribution." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 5 Jul. 2018. Web. 17 Jul. 2018
5. Clement Boateng Ampadu, A Solution to a Partial Differential Equation Arising from a Random Walk Problem with Time Dependence II, International Journal of Mathematics and Statistics Invention, ISSN (Online): 2321-4767, ISSN (Print): 2321-4759, To Appear
6. Ampadu, Clement B. (2010), Random Walks and Partial Differential Equations (PhD Dissertation), Retrieved from ProQuest Dissertations and Theses database. (UMI No. 3406252)
7. R. Wojnar, Rayleigh's Distribution, Wigner's Surmise and Equation of the Diffusion, ACTA PHYSICA POLONICA A, Vol. 123(3) (2013), pp 624-628

Appendix

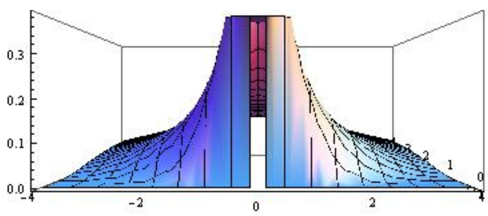

Figure 1. The Frontal View of the PDF of $j(x,t) = \frac{e^{-\frac{\text{Log}\left[e^{-x}\right]^2}{4kt}}}{2\sqrt{\pi}\sqrt{kt}}$ where $k = 1$, $-4 \le x \le 4$, and $0 \le t \le 4$