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Chattering Elimination in Fuzzy Sliding Mode Control of Fractional Chaotic Systems Using a Fractional Adaptive Proportional Integral Controller

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Abstract: In this paper, a Fractional Adaptive Fuzzy Logic Control (FAFLC) strategy based on active fractional sliding mode (FSM) theory is considered to synchronize chaotic fractional-order systems. Takagi-Sugeno fuzzy systems are used to estimate the plant dynamics represented by unknown fractional order functions.
One of the main contributions in this work is to combine an adaptive fractional order PI^{λ} control law with the fractional-order adaptive sliding mode controller in order to eliminate the chattering action in the control signal. Based on Lyapunov theory, the stability analysis of the proposed control strategy is performed for an acceptable synchronization error level. Numerical simulations illustrate the efficiency of the proposed fractional fuzzy adaptive control scheme through the synchronization of two different fractional order chaotic Duffing systems. We show that the introduction of the additional fractional adaptive PI^{λ} control action is able to eliminate the chattering phenomena in the control signal.

Keywords: Nonlinear fractional systems, Fractional sliding mode control, Adaptive fuzzy control, Fractional adaptive PI^{λ} controller, Chattering elimination, Chaos synchronization.

1. Introduction

Since more than three centuries, a great number of researchers focused on the mathematical topic of Fractional calculus, dealing with derivatives and integrations of non-integer order. When compared to the classical theory, fractional differential equations describe more accurately many systems in interdisciplinary fields, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, the nonlinear oscillation of earthquakes, mechanics, some finance systems, and electromagnetic wave systems [1, 2].

Fractional order systems have shown very attractive performances and properties, and there for many Applications of such systems have been reported in different areas such as signal processing, image processing [3], automatic control [4], robotics [5], and renewable energy.

In the last decade, a great number of research works focused on fractional systems that display chaotic behavior like: Chua circuit [6], Duffing system [7], Chen dynamic [8], characterization [9], Rössler system and Newton-Leipnik formulation [10]. Synchronization or control of these systems is a difficult task because a main characteristic of chaotic systems is their high sensitivity to initial conditions, but it is gathering more and more research effort due to several potential applications especially in cryptography [11].

For the particular case of fractional order chaotic systems, many approaches have been proposed to achieve chaos synchronization, such as PC control [12], nonlinear state observer method [13], adaptive control [14, 15] and sliding mode control [16].

In this paper we are interested by the problem of uncertain fractional order chaotic systems synchronization by mean of adaptive fuzzy sliding mode control. Sliding mode control is a very suitable method for handling such nonlinear systems because of low sensitivity to disturbances and plant parameter variations and its order reduction property, which relaxes the burden of the necessity of exact modeling. In the proposed control configuration, a fuzzy logic approximation method is used to modelize the uncertain fractional order system [17] [18].

Based on the Lyapunov stability theorem, an efficient adaptive control algorithm by means of fuzzy logic models is proposed that guarantees the feedback control system stability and that is able to attenuate the effects of additive noises and estimation errors on the tracking performance to any prescribed error level via the sliding mode robust tracking design technique.

However, the important problem of sliding mode techniques from the control perspective is the discontinuity of the control signal required to obtain robustness. This destructive phenomenon, so-called chattering, may affect control accuracy or incur an unwanted wear of a mechanical component. Various solutions to reduce the chattering have been studied in the literature [19, 20]. Comparing with a similar previous work [21], an improved synchronization technique is proposed here for a robust sliding mode control of nonlinear systems with fractional order dynamics that is able to eliminate the chattering phenomena for uncertain systems with unknown parameters' variation.

The main contribution of this work consists in combining an adaptive fractional PI^{λ} control law with the sliding mode controller in order to improve the control signal quality by eliminating the undesirable chattering. The Grünwald-Letnikov numerical approximation method is used for fractional order differential equation resolution with improved performance result.

This paper is organized as follows. Section 2 presents an introduction to fractional calculus with some numerical approximation methods. A description of the Takagi-Sugeno is given in section 3. Section 4 presents the proposed adaptive sliding mode fuzzy synchronization technique of uncertain fractional order systems. The stability analysis is performed in section 5. In section 6, application of the obtained control scheme on a Duffing fractional order system is investigated. Finally, concluding remarks with future works are presented in section 7.

2. Basics of Fractional Order Systems

2.1 Fractional derivatives and integrals

There exists many formulations for the fractional order derivative definition; the most popular are those of Grünwald-Letnikov (GL), Riemann-Liouville (RL) and Caputo [22, 23].

Riemann-Liouville (RL) fractional order integral is expressed as:

$$ {}^{RL}_{a}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \quad (1)$$

The fractional order derivative is defined as:

$$ {}^{RL}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \right] \quad (2)$$

Where $\Gamma(x) = \int_t^a y^{\alpha-1} e^{-y} dy$

$\Gamma(\cdot)$ is the Gamma function, a, t and α are real numbers, and alpha verifies $0<\alpha<1$. In we assume that a=0 without loss of generality. Also we use ${}_{a}D_t^{\alpha} = D^{\alpha}$.

2.2. Numerical approximation method

The specialized literature proposes different ways and techniques for approaching non-integer order operators. They result in various algorithms for the numerical simulation of these systems. The most common approach used in the fractional order chaotic systems literature is a modified version of the Adams-Bashforth-Moulton method based on predictor-correctors [24]. However, we will use in this work a simpler approach consist on the fractional order operator discretization following the Grünwald-Letnikov definition [25].

The Grünwald-Letnikov fractional order derivative definition is expressed as [26]:

$$ {}^{GL}_{t_0}D_t^{\alpha} f(t) = \lim_{n\to 0} \frac{1}{h^n} \times \left(\sum_{j=0}^{\left[\frac{t-\alpha}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) \right) \quad (4)$$

where $\left[\frac{t-\alpha}{h}\right]$ indicates the integer part and $(-1)^j \binom{\alpha}{j}$ are binomial coefficients $C_j^{(\alpha)}$ (j=0,1,…).

The calculation of these coefficients is done by the formula of following recurrence:

$$c_0^{(\alpha)}=1, \quad c_j^{(\alpha)}=(1-\frac{1+\alpha}{j})c_{j-1}^{(\alpha)}$$

now, if we consider the fractional order differential equation : ${}_{t_0}^{GL}D_t^{\mu}y(t)=f(y(t),t)$

then, the numerical solution is expressed as:

$$y(t_k)=f(y(t_k),t_k)h^{\mu}-\sum_{j=v}^{k}c_j^{(\mu)}y(t_{k-j}) \qquad (5)$$

This approximation of the fractional derivative within the meaning of Grünwald-Letnikov is on the one hand equivalent to the definition of Riemman-Liouville for a broad class of functions [27], on the other hand, it is well adapted to the definition of Caputo (Adams method) because it requires only the initial conditions and has a physical direction clearly.

3. T-S fuzzy systems

The uncertain fractional order chaotic system may be directly addressed by fuzzy logic systems by using the linguistic models (e.g., small, medium and large) [17, 28]. The Takagi-Sugeno (T-S) configuration of the system includes a fuzzy rule base, represented by a number of fuzzy IFTHEN rules in the form:

$$R^{(l)}: \text{IF} x_1 \text{ is } F_1^l \text{, and ..., } x_n \text{ is } F_n^l \text{ THEN}$$
$$y_l=\alpha_0^l+\alpha_1^l x_1+...+\alpha_n^l x_n=\theta_l^T[1\ x^T]^T \qquad (6)$$

where $(F_1^l,..., F_i^l,...,F_n^l)$ are input fuzzy sets and $\theta_l^T=[\alpha_0^l+\alpha_1^l+...+\alpha_n^l]$ represents the adjustable factors of the consequence part. y_l is a crisp value, and a fuzzy inference engine to combine the fuzzy IF-THEN rules in the fuzzy rule base into a mapping from an input linguistic vector $\mathbf{x}=[x_1,x_2,...,x_n]^T\in\Re^n$ to an output variable $y\in\Re$. The output of the fuzzy logic systems with central average defuzzifier, product inference and singleton fuzzifier can be expressed as

$$y(\mathbf{x})=\frac{\sum_{l=1}^{M}v^l y_l}{\sum_{l=1}^{M}v^l}=\frac{\sum_{l=1}^{M}v^l\theta_l^T[1\ \ x^T]^T}{\sum_{l=1}^{M}v^l} \qquad (7)$$

where M is the number of rules, $v^l=\prod_{i=1}^{n}\mu_{F_i^l(x_i)}$ is the true value of the l^{th} implication and $\mu_{F_i^l(x_i)}$ is the membership function value of the x_i [29]. Equation (7) can be rewritten as

$$y(\mathbf{x})=\theta_l^T\,\xi(\mathbf{x}) \qquad (8)$$

Where $\theta_l^T=[\theta_1^T,\theta_2^T,...,\theta_M^T]$ is the parameter vector and $\xi^l(\mathbf{x})=[\xi^1(x),\xi^2(x),...,\xi^M(x)]$ a fuzzy basis function vector defined as:

$$\xi^l(\mathbf{x})=\frac{v^l[1\ \ x^T]}{\sum_{l=1}^{M}v^l}$$

The output (7) is pumped out by the mean of the common defuzzification strategy.

The above fuzzy logic system is able to provide a uniform approximation of any well-defined nonlinear function over a compact set U_c to any degree of accuracy, as proved in the universal approximation theorem [18, 20].

4. Fractional fuzzy adaptive sliding mode algorithm

Let us now present the proposed adaptive fuzzy control strategy that will allow the control of nonlinear fractional order systems.

Consider the fractional order nonlinear system given as follows [29],

$$\begin{cases} x_1^{(q_1)}=x_2 \\ \vdots \\ x_{n-1}^{(q_{n-1})}=x_n \\ x_n^{(q_n)}=f(\mathbf{x},t)+g(\mathbf{x},t)u+d(t) \\ y=x_1 \end{cases} \qquad (9)$$

This system is called commensurate if $q_1=q_2=...=q_n=q$ and can be rewritten as,

$$\begin{aligned} x^{(nq)}&=f(\mathbf{x},t)+g(\mathbf{x},t)u+d(t) \\ y&=x_1 \end{aligned} \qquad (10)$$

where $\mathbf{x}=[x_1,x_2,...,x_n]^T=[x,x^{(q)},...,x^{((n-1)q)}]^T$. $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ are unknown but bounded nonlinear functions, $d(t)$ is the external bounded disturbance, assuming that the upper bound of the disturbance $d(t)$ is D, that is $|d(t)| \le D$, and $u(t)$ is the control input.

The nonlinear system (10) is assumed to be controllable and the input gain $g(\mathbf{x},t) \neq 0$ has to be non-zero. Consequently, without loss of generality, we assume that $g(\mathbf{x},t) > 0$.

The control objective is to force the system output $\mathbf{y}$ to follow a given bounded reference signal $\mathbf{y}_d$, under the constraint that all signals involved must be bounded.

Let us now define the reference signal vector $\mathbf{y}_d$ and the tracking error vector $\mathbf{e}$ as follows:

$$\mathbf{y}_d = \left[y_d, y_d^{(q)}, \dots, y_d^{((n-1)q)} \right]^T \in \Re^n \tag{11}$$

$$\mathbf{e} = \mathbf{y}_d - \mathbf{y} = \left[e, e^{(q)}, \dots, e^{((n-1)q)} \right]^T \in \Re^n, \quad \mathbf{e}^{(iq)} = \mathbf{y}_d^{(iq)} - \mathbf{y}^{(iq)} \tag{12}$$

where $0 < q < 1$. Let $\mathbf{k} = [k_1, k_2, \dots, k_n]^T \in \Re^n$ to be chosen in a way that $h(p) = \sum_{i=1}^{n} k_i p^{(i-1)q}, k_n = 1$ is Hurwitz polynomial.

The sliding surface is defined as:

$$s(\mathbf{x},t) = -(\mathbf{ke}) = -(k_1 e + k_2 e^{(q)} + \dots + k_{n-1} e^{((n-2)q)} + e^{((n-1)q)}) \tag{13}$$

when $e(0) = 0$, the tracking problem $x = y_d$ implies that the sliding surface $s(\mathbf{e}) = 0, \forall t \ge 0$. Correspondingly, the sliding mode control will be designed in two phases:

1. The reaching phase when $s(\mathbf{x},t) \neq 0$, and
2. The sliding phase by $s(\mathbf{x},t) = 0$, for initial error $e(0) = 0$.

with the following sliding condition :

$$s(x,t)\dot{s}(x,t) \le -\eta |s(x,t)|, \quad \eta > 0 \tag{14}$$

must be satisfied.

In absence of uncertainty and external disturbance, the corresponding equivalent control force $u_{eq}(t)$, can be obtained by $\dot{s}(\mathbf{x},t) = 0$. This later classic derivative can be decomposed into a fractional type,

$$\dot{s}(\mathbf{x},t) = D^{(1-q)}(D^q(s(\mathbf{x},t))) = 0 \quad \text{then} \quad D^q(s(\mathbf{x},t)) = 0 \tag{15}$$

If the functions $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ are known and the system is free of external disturbance i.e., $d(t)$=0. The control signal in the following equation drives the dynamic to reach to the sliding surface:

$$\begin{aligned} s^{(q)} &= -\left(\sum_{i=1}^{n-1} k_i e^{(iq)} + e^{(nq)} \right) \\ &= -\left(\sum_{i=1}^{n-1} k_i e^{(iq)} + y_d^{(nq)} - y^{(nq)} \right) \\ &= -\left(\sum_{i=1}^{n-1} k_i e^{(iq)} - f(x,t) - g(\mathbf{x},t) u_{eq} \right) - \mathbf{y}_d^{(nq)} \\ &= -\sum_{i=1}^{n-1} k_i e^{(iq)} + f(\mathbf{x},t) + g(\mathbf{x},t) u_{eq} - \mathbf{y}_d^{(nq)} \\ &= 0 \end{aligned} \tag{16}$$

Therefore, the equivalent control law is given by :

$$u_{eq} = \frac{1}{g(\mathbf{x},t)} \left(\sum_{i=1}^{n-1} k_i e^{(iq)} - f(\mathbf{x},t) + \mathbf{y}_d^{(nq)} \right) \tag{17}$$

Substituting Eq. (17) into Eq. (10), we have

$$e^{(nq)} + k_n e^{(n-1)q} + \dots + k_1 e = 0 \tag{18}$$

which is the main objective of control $\lim_{t \to \infty} e(t) = 0$.

In the reaching phase we get $s(\mathbf{x},t) \neq 0$, and a switching-type control u_{sw} must be added in order satisfy the sufficient condition (14) which implies that the global control will be written as:

$$u_i = u_{eq} - u_{sw} \tag{19}$$

with

$$u_{sw} = \frac{1}{g(\mathbf{x},t)} \left(\psi_p \operatorname{sgn}(s) \right) \tag{20}$$

where $\psi_p \ge \eta > 0$ and,

$$\operatorname{sgn}(s) = \begin{cases} 1 & \text{for} \quad s > 0 \\ 0 & \text{for} \quad s = 0 \\ -1 & \text{for} \quad s < 0 \end{cases} \tag{21}$$

Therefore the global sliding mode control law is given by:

$$u^* = \frac{1}{g(\mathbf{x},t)}\Big(\sum_{i=1}^{n-1} k_i e^{(iq)} - f(\mathbf{x},t) + y_d^{(nq)} - \psi_p \operatorname{sgn}(s)\Big) \tag{22}$$

We can show by taking a Lyapunov function candidate defined as

$$V = \frac{1}{2} s^2(\mathbf{e}) \tag{23}$$

And differentiating (23) with respect to time to the fractional order q, $V^{(q)}(t)$ along the system trajectory, we obtain

$$\begin{aligned} V^{(q)} &= ss^{(q)} = s\left(\sum_{i=1}^{n-1} k_i e^{(iq)} + e^{(nq)}\right) \\ &= -s\left(\sum_{i=1}^{n-1} k_i e^{(iq)} - y_d^{(nq)} + y^{(nq)}\right) \\ &= -s\left(\sum_{i=1}^{n-1} k_i e^{(iq)} + f(\mathbf{x},t) + g(\mathbf{x},t)u_{eq} - y_d^{(nq)}\right) \\ &\le -\eta \left|s(\mathbf{x},t)\right| \end{aligned} \tag{24}$$

Hence, the sliding mode control u^* guarantees the sliding condition of Eq. (14).

However, as mentioned in [30], the functions f and g are usually unknown in practice and it is difficult to apply the control law (22) for an unknown nonlinear plant. Moreover, the chattering problem appears when adding the switching control term u_{sw}.

To deal with these problems, we consider the adaptive sliding mode control scheme using a fuzzy logic system and the fractional order PI^{λ} control law to avoid chattering problem.

The input and output of the continuous time fractional order PI^{λ} controller, where $\lambda = q$, are in the form:

$$u_{PI} = p\left(\mathbf{s}\middle|\theta_p\right) = \theta_{p_1} z_1 + \theta_{p_2} z_2 \tag{25}$$

where $z_1 = s$, $z_2 = s^{(q)}$, θ_{p_1} and θ_{p_2} are control gains to be designed. Equation (25) can be rewritten as :

$$u_{PI} = p\left(\mathbf{s}\middle|\theta_p\right) = \xi^T(\mathbf{s})\theta_p \tag{26}$$

where $\xi^T(\mathbf{s}) = [s, s^{(q)}] \in \Re^2$ and $\theta_p = [\theta_{p1}, \theta_{p2}]^T \in \Re^2$ is an adjustable parameter vector.

The resulting control law, which includes a fuzzy system to approximate the unknown functions $f(\mathbf{x})$ and $g(\mathbf{x})$ and a fractional adaptive PI^q controller that attenuates the chattering and improve performance, is as follows:

$$u_i = \frac{1}{g(\mathbf{x}|\theta_g)}\Big(\sum_{i=1}^{n-1} k_i e^{(iq)} - f(\mathbf{x}|\theta_f) + y_d^{(nq)} - p(\mathbf{s}|\theta_p)\Big) \tag{27}$$

The switching control u_{sw} is replaced by the action of the fractional adaptive PI^q controller to avoid the problem of chattering when the state is within a limited layer $\left|s(x,t)\right| < \phi$; the control action is maintained in the saturated state when the value is outside the boundary layer.

Hence, we set $\left|p(\mathbf{s}|\theta_p)\right| = D + \psi_p + \omega_{\max}$ when $s(\mathbf{x},t)$ is outside of the boundary layer, i.e., $\left|s(x,t)\right| \ge \phi$, where ϕ is the thickness of the boundary layer.

Note that the control law (22) is realizable only while $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ are well known.

However, $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ are unknown and external disturbance $d(t) \ne 0$, the ideal control effort (22) cannot be implemented. We replace $f(\mathbf{x},t), g(\mathbf{x},t)$ and u_{PI} by the fuzzy logic system $f(\mathbf{x}|\theta_f)$, $g(\mathbf{x}|\theta_g)$ and $p(\mathbf{s}|\theta_p)$ in a specified form as Eq. (9), i.e.,

$$\begin{aligned} f(x|\theta_f) &= \xi^T(\mathbf{x})\theta_f, \\ g(x|\theta_g) &= \xi^T(\mathbf{x})\theta_g, \\ p(s|\theta_p) &= \xi^T(\mathbf{s})\theta_p \end{aligned} \tag{28}$$

Here the fuzzy basis function $\xi^T(\mathbf{x})$ and $\xi^T(\mathbf{s})$ depends on the fuzzy membership functions and is supposed to be fixed, while θ_f, θ_g and θ_p are adjusted by adaptive laws based on a Lyapunov stability criterion [31, 32].

The optimal parameter estimations θ_f^*, θ_g^* and θ_p^* are defined as

$$\theta_f^* = \arg \min_{\theta_f \in \Omega_f} \left[\sup_{x \in \Omega_x} \left| f(x|\theta_f) - f(x,t) \right| \right]$$
$$\theta_g^* = \arg \min_{\theta_g \in \Omega_g} \left[\sup_{x \in \Omega_x} \left| g(x|\theta_g) - g(x,t) \right| \right]$$
$$\theta_p^* = \arg \min_{\theta_p \in \Omega_p} \left[\sup_{x \in \Omega_x} \left| p(s|\theta_p) - u_{sw} \right| \right] \quad (29)$$

where $\Omega_f, \Omega_g, \Omega_p$ and Ω_x are constraint sets of suitable bounds on θ_f, θ_g, θ_p and x respectively and they are defined as $\Omega_f = \{\theta_f \,|\, |\theta_f| \le M_f\}$, $\Omega_g = \{\theta_g \,|\, |\theta_g| \le M_g\}$, $\Omega_p = \{\theta_p \,|\, |\theta_p| \le M_p\}$ and $\Omega_x = \{x \,|\, |x| \le M_x\}$ where M_f, M_g, M_p and M_x are positive constants.

Assuming that the fuzzy parameters θ_f, θ_g and θ_p never reach the boundaries.
Let us define the minimum approximation error,

$$\omega = \left[f(x,t) - f(x|\theta_f^*) \right] + \left[g(x,t) - g(x|\theta_g^*) \right] u_i \quad (30)$$

and define the errors: $\tilde{\theta}_f = \theta_f - \theta_f^*$, $\tilde{\theta}_g = \theta_g - \theta_g^*$ and $\tilde{\theta}_p = \theta_p - \theta_p^*$.

Then, the equation of the sliding surface (16) can be rewritten as

$$\begin{aligned} s^{(q)} &= \omega + \left[f(\mathbf{x}|\theta_f^*) - f(\mathbf{x}|\theta_f) \right] + \left[g(\mathbf{x}|\theta_g^*) - g(\mathbf{x}|\theta_g) \right] u_i \\ &\quad - p(\mathbf{s}|\theta_p) + p(\mathbf{s}|\theta_p^*) - p(\mathbf{s}|\theta_p^*) + d(t) \\ &= \omega - \tilde{\theta}_p^T \xi(\mathbf{s}) - \tilde{\theta}_f^T \xi(\mathbf{x}) - \tilde{\theta}_g^T \xi(\mathbf{x}) u_i \\ &\quad - p(\mathbf{s}|\theta_p^*) + d(t) \end{aligned} \quad (31)$$

5. Stability analysis

The following theorem establishes the asymptotic stability of the proposed control system.

Theorem 1. *Consider the fractional order SISO nonlinear system (10) with the control input (27), if the fuzzy-based adaptive laws are chosen as*

$$\begin{aligned} \theta_f^{(q)} &= r_1 s \xi(\mathbf{x}) \\ \theta_g^{(q)} &= r_2 s \xi(\mathbf{x}) u_i \\ \theta_p^{(q)} &= r_3 s \xi(\mathbf{s}) \end{aligned} \quad (32)$$

Where r_1, r_2 and r_3 are positive constants, then, the overall adaptation scheme ensures the overall stability of the closed-loop system resulting in the sense that the tracking error converges to zero asymptotically and all the variables of the closed-loop system are bounded.

Proof – Let us choose the Lyapunov function candidate as:

$$V = \frac{1}{2}s^2 + \frac{1}{2r_1}\tilde{\theta}_f^T \tilde{\theta}_f + \frac{1}{2r_2}\tilde{\theta}_g^T \tilde{\theta}_g + \frac{1}{2r_3}\tilde{\theta}_p^T \tilde{\theta}_p \quad (33)$$

The derivative of (33) with respect to time using the Caputo derivative Lemma [15] [31-33], gives

$$\begin{aligned} V^{(q)} &\le s s^{(q)} + \frac{1}{r_1}\tilde{\theta}_f^T \tilde{\theta}_f^{(q)} + \frac{1}{r_2}\tilde{\theta}_g^T \tilde{\theta}_g^{(q)} \\ &\quad + \frac{1}{r_3}\tilde{\theta}_p^T \tilde{\theta}_p^{(q)} \end{aligned} \quad (34)$$

$$\begin{aligned} V^{(q)} &\le s\Big(\omega - \tilde{\theta}_p^T \xi(\mathbf{s}) - \tilde{\theta}_f^T \xi(\mathbf{x}) \\ &\quad - \tilde{\theta}_g^T \xi(\mathbf{x}) u_i - p(\mathbf{s}|\theta_p^*) + d(t)\Big) \\ &\quad + \frac{1}{r_1}\tilde{\theta}_f^T \tilde{\theta}_f^{(q)} + \frac{1}{r_2}\tilde{\theta}_g^T \tilde{\theta}_g^{(q)} + \frac{1}{r_3}\tilde{\theta}_p^T \tilde{\theta}_p^{(q)} \end{aligned} \quad (35)$$

$$\begin{aligned} V^{(q)} &\le s\omega - s\tilde{\theta}_p^T \xi(\mathbf{s}) - s\tilde{\theta}_f^T \xi(\mathbf{x}) \\ &\quad - s\tilde{\theta}_g^T \xi(\mathbf{x}) u_i - s p(\mathbf{s}|\theta_p^*) + s d(t) \\ &\quad + \frac{1}{r_1}\tilde{\theta}_f^T \tilde{\theta}_f^{(q)} + \frac{1}{r_2}\tilde{\theta}_g^T \tilde{\theta}_g^{(q)} + \frac{1}{r_3}\tilde{\theta}_p^T \tilde{\theta}_p^{(q)} \\ &\le \frac{1}{r_1}\tilde{\theta}_f^T \left(\tilde{\theta}_f^{(q)} - r_1 s \xi(\mathbf{x}) \right) \\ &\quad + \frac{1}{r_2}\tilde{\theta}_g^T \left(\tilde{\theta}_g^{(q)} - r_2 s \xi(\mathbf{x}) u_i \right) \\ &\quad + \frac{1}{r_3}\tilde{\theta}_p^T \left(\tilde{\theta}_p^{(q)} - r_3 s \xi(\mathbf{s}) \right) \\ &\quad - s(D+\eta)\operatorname{sgn}(s) + s d(t) + s\omega \end{aligned} \quad (36)$$

By considering the fractional robust compensator (27) and the fractional fuzzy adaptations laws (32), we get after a simple manipulation

$$V^{(q)} \le s w - s\psi_p \operatorname{sgn}(s) = s\omega - |s|\psi_p \le 0 \quad (37)$$

Since ω it the minimum approximation error, (37) is the best result that we can obtain. Therefore, all signals in the system are bounded. Obviously, if $e(0)$ is bounded, then $e(t)$ is also bounded for all t.

Since the reference signal $\mathbf{y}_d$ is bounded, then the system states $\mathbf{x}$ is bounded as well.
To complete the proof and establish asymptotic convergence of the tracking error, we need proving that $s \to 0$ as $t \to \infty$. Besides, assume that $\|s\| \le \psi_s$, then Eq. (37) can be rewritten as :

$$V^{(q)} \le \psi_s |\omega| - |s| \psi_p \tag{38}$$

The integral of Eq. (38) provides,

$$\int_0^T |s| d\tau \le \frac{1}{\psi_p}\left(|V(0)| + |V(T)|\right) + \frac{\psi_s}{\psi_p}\int_0^T |\omega| d\tau \tag{39}$$

then we have $s \in L_1$. Form Eq. (37), we know that s is bounded and every term in Eq. (37) is bounded.

The uniform continuity of the fractional order derivative (2), and its roundedness from Eq. (31) allow to apply Barbalat's Lemma [26] and the fractional extensions of Barbalat Lemma [34]. Hence, $(s, s^{(q)}) \in L_\infty$. We have $s(t) \to 0$ as $t \to \infty$, and from Eq. (13) the tracking error $e(t)$ will converge to zero, which proves the system stability.

6. Simulation results

Let us apply the proposed controller to synchronize two different fractional order chaotic Duffing systems [29].

Consider two fractional order chaotic Duffing systems (see similar examples in [11, 21, 23]):
The drive (master) system given by:

$$\begin{aligned} D^q y_1 &= y_2 \\ D^q y_2 &= y_1 - 0.25 y_2 - y_1^3 + 0.3\cos(t) \end{aligned} \tag{40}$$

The response (slave) system given by:

$$\begin{aligned} D^q x_1 &= x_2 \\ D^q x_2 &= x_1 - 0.3 x_2 - x_1^3 + 0.35\cos(t) \\ &\quad + u(t) + d(t) \end{aligned} \tag{41}$$

where $d(t) = 0.1\sin(t)$ is an external disturbance. In this study we consider the case $q = 0.98$. The main objective is to control the trajectories of the response system to track the reference trajectories obtained from the drive system.

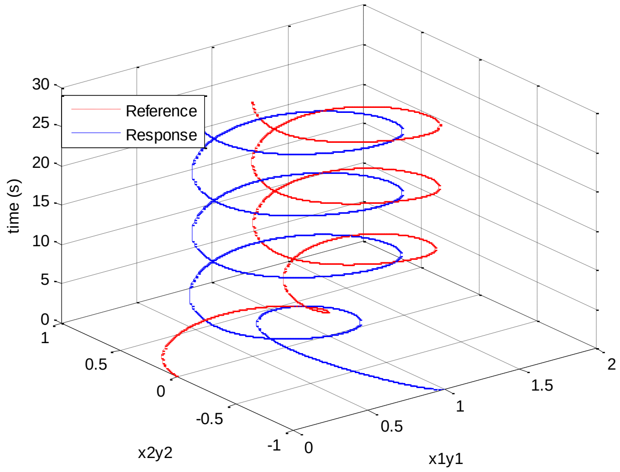

Figure.1 Phase portrait of Duffing master and slave systems (Without control action).

The initial conditions of the drive and response systems are chosen as: $(y_1(0) \quad y_2(0))^T = (0 \quad 0)^T$ and $(x_1(0) \quad x_2(0))^T = (1 \quad -1)^T$ respectively.
The membership functions of x_i, for $f(\mathbf{x}|\theta_f)$ and $g(\mathbf{x}|\theta_g)$ are selected as [26]:

$$\mu_{F_i^l}(x_i) = \exp\left[-0.5\left(\frac{x_i - \bar{x}}{0.8}\right)^2\right] \tag{42}$$

where $i = 1:2$ and $l = 1,\ldots,7$, and $\bar{x}$ is selected from the interval $[-1, 2]$.

From Eq. (32) and Eq. (26), the control law (27) can be obtained as:

$$\begin{aligned} u_i(\mathbf{x}) = \frac{1}{g(\mathbf{x}|\theta_g)}\Big(&\sum_{i=1}^{n-1} k_i e^{(iq)} \\ &- f(\mathbf{x}|\theta_f) + y_d^{(nq)} - p(\mathbf{s}|\theta_p)\Big) \end{aligned} \tag{43}$$

Specifying the simulation parameters as : $\mathbf{k} = [1; 1]$, $r_1 = 200$, $r_2 = 40$ and $r_3 = 10$, the simulation time window $T = 30\text{sec}$ and the sampling period $\Delta = 0.001\text{sec}$; the simulations results are illustrated as follows:

Fig. 1 shows the 3-D phase portrait of the drive and response systems before the application of the proposed control scheme. The synchronization performance is very bad at this initial stage.

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6.1 Step 1: Fractional sliding mode control

Fig. 2 shows the synchronization trajectory after the introduction of the proposed control law: we can notice a fast and perfect synchronization of the fractional chaotic systems (drive and response).

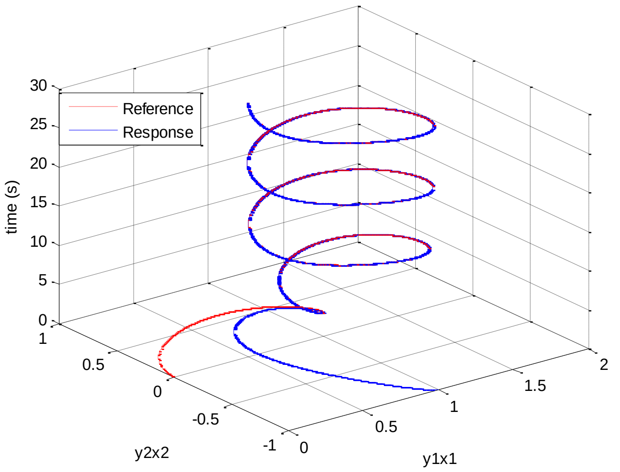
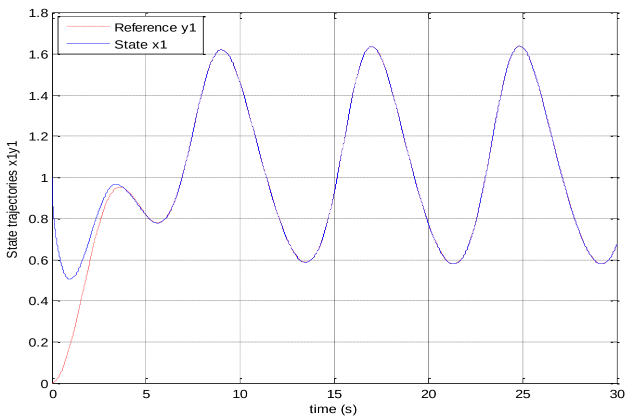
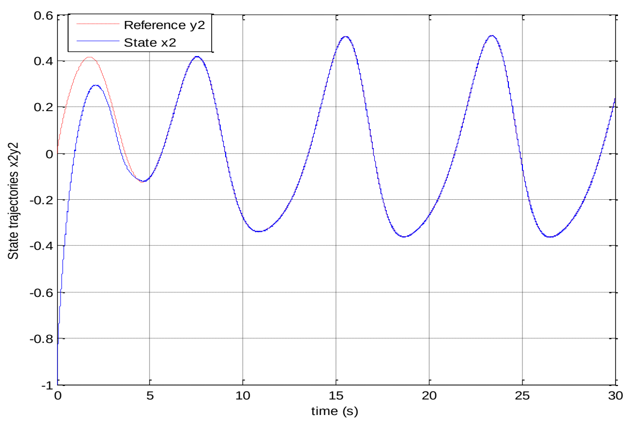


Figure.2 Synchronization performance of Duffing chaotic systems.



(a)



(b)

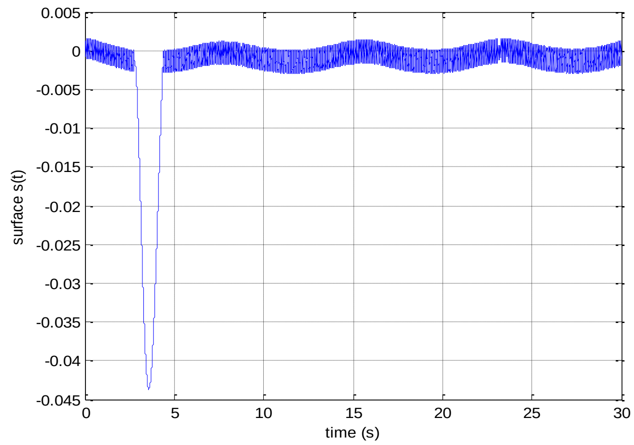
Figure.3 Synchronization of the state trajectories for the FSMC controller-(a):States x_1 and y_1 ,(b): States x_2 and y_2

Fig. 3 show the trajectories of the states x_1, y_1 and x_2, y_2. The sliding surface is shown in Fig. 4(a) and the control effort trajectory is given in Fig. 4(b).

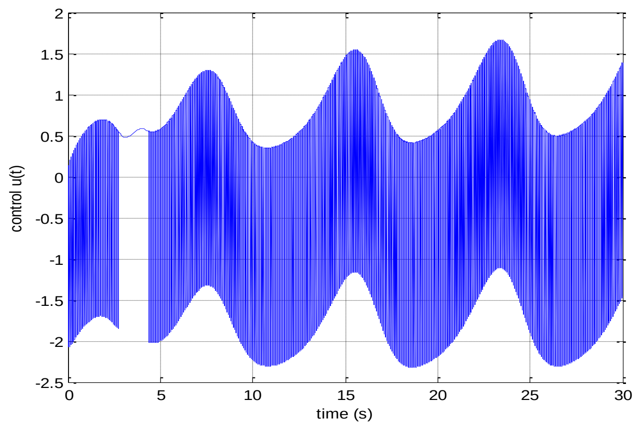
Fig. 5 illustrate the fast convergence of response system output to that of the drive one. However, the chattering phenomena appears like a big inconvenient for this control strategy, as illustrated in the sliding surface and the control effort trajectories in Fig. 4.

6.2 Step 2: Introduction of the fractional adaptive PI^{λ} controller

Let us consider now the problem of eliminating the chattering that appeared in the above results, and introduce the complementary fractional adaptive PI^{λ} controller. The simulation results are given in Fig. 6 for the synchronization performance of the Duffing chaotic master and slave systems, Fig. 7 for the states trajectories.



(a)



(b)

Figure.4 FSMC controller - (a): Sliding surface $s(t)$, (b): Control signal $u(t)$

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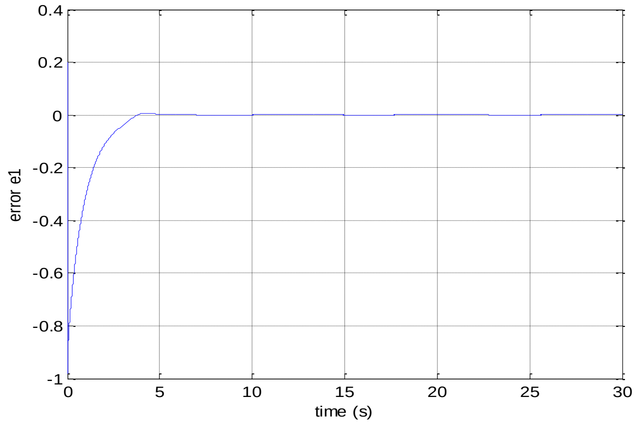
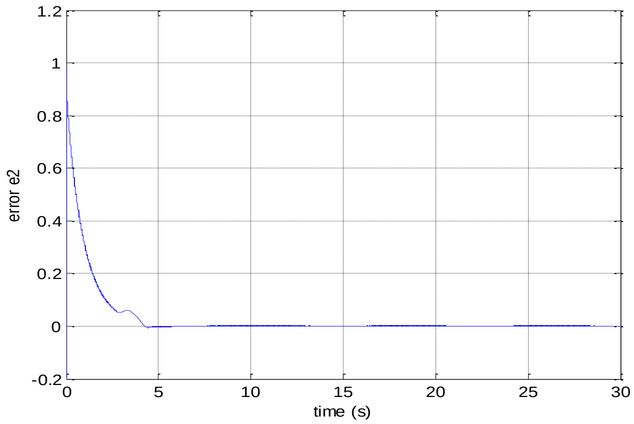


(a)



(b)

Figure.5 Synchronization errors - (a) : Error signal $e_1(t)$, (b): Error signal $e_2(t)$

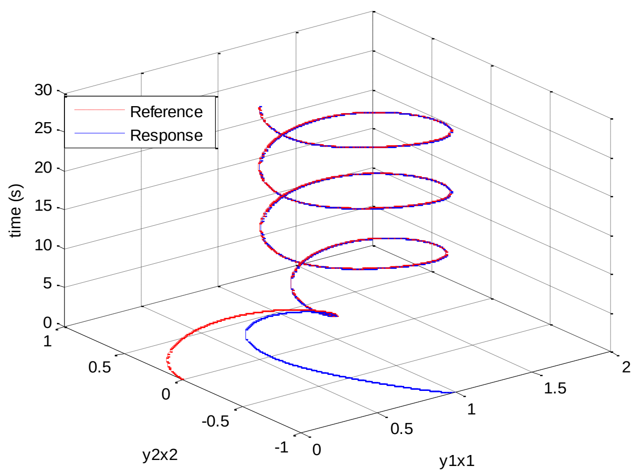


Figure.6 Synchronization performance

We see that the synchronization performance is enhanced even in presence of disturbance $d(t)$ and the chattering phenomena is eliminated in the sliding surface trajectory and the control signal $u(t)$ as illustrated from Fig. 8.

Remarks

The main objective of this work is to improve the adaptive fuzzy sliding mode control performance by eliminating chattering and steady-state error with the use of a fractional adaptive PI^{λ} control. As a result, the closed-loop system performance is obviously better.

We can remark a certain improvement in synchronization and tracking performance as shown from simulation results of Fig. 2-3 and Fig. 6-8, in comparison with similar results from literature [21, 29].

One reason for this improvement may be also the use of the Grünwald-Letnikov approximation method instead of Adams-Bashforth-Moulton method.

7. Conclusion

In this paper an improved fractional adaptive fuzzy sliding mode control strategy is proposed to deal with chaos synchronization of different uncertain fractional order chaotic systems. The main contribution of this work is introducing of an adaptive fractional PI^{λ} controller to eliminate the chattering phenomena in the fractional sliding mode controller. Thus, the well-known disadvantage of sliding mode techniques from the control perspective, i.e. the discontinuity of the control signal necessary to achieve robustness, is no longer pertinent in this context.

Based on the Lyapunov stability theorem, free parameters of the adaptive fuzzy controller can be tuned on line by the output feedback control law and adaptive laws to achieve fractional order chaotic systems synchronization.

B the asymptotic stability of the overall control system is established and an illustrative simulation example, chaos synchronization of two fractional order Duffing systems, is realized with the Grünwald-Letnikov numerical approximation approach to demonstrate the effectiveness of the proposed methodology.

Further researches will concern the application of the proposed methodology to discrete fractional order systems, and the investigation of other techniques for chattering elimination such as high-gain control laws and Type-2 fuzzy sets.

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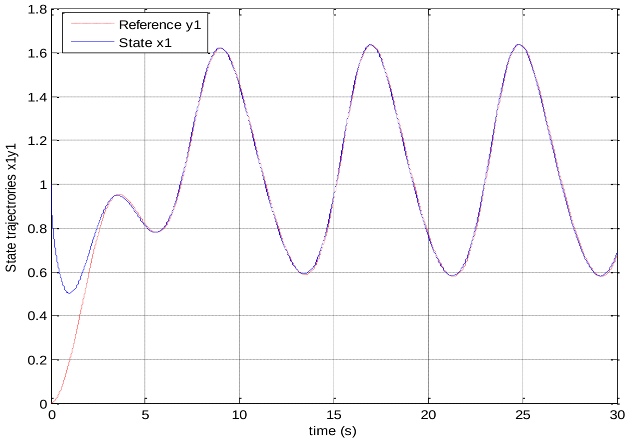
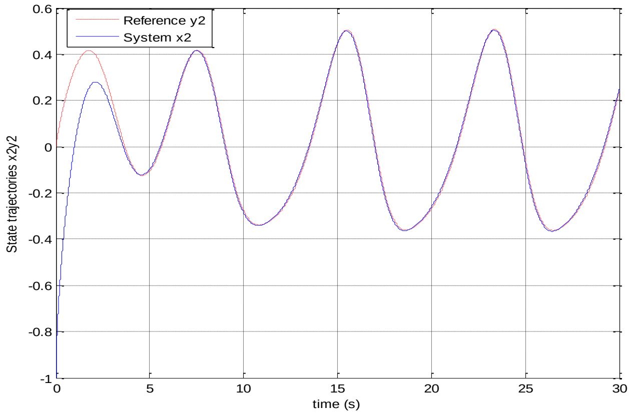
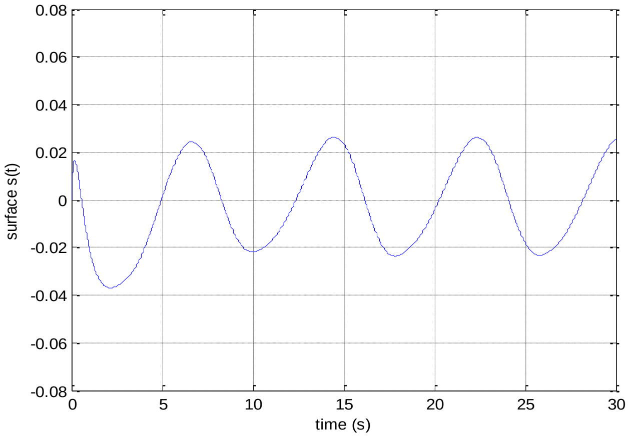


(a)



(b)

Figure.7 Synchronization of the state trajectories for the FSMC with the fractional order adaptive PI^{λ} controller (a): States x_1 and y_1, (b): States x_2 and y_2



(a)

Acknowledgments

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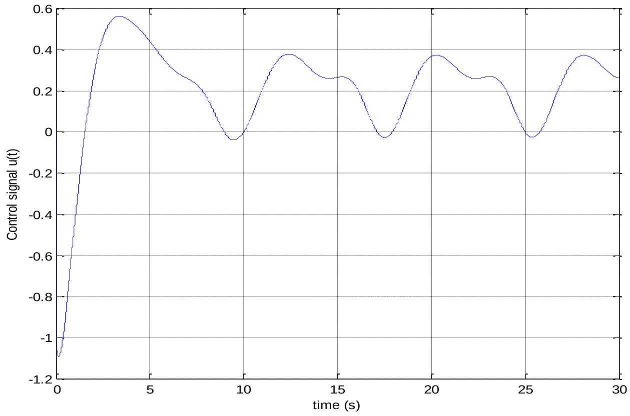
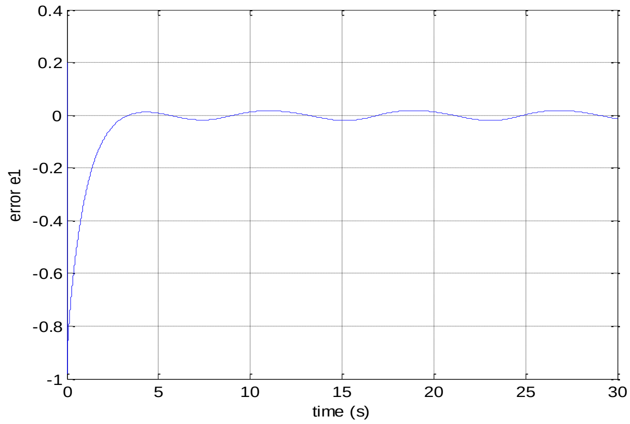
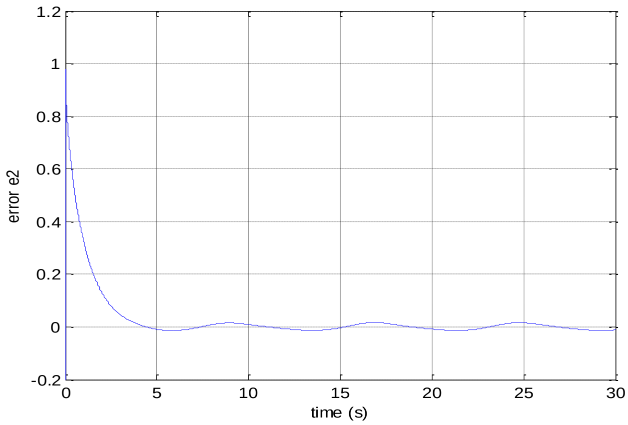


(b)

Figure.8 FSMC with the fractional order adaptive PI^{λ} controller-(a):Sliding surface $s(t)$,(b):Control signal $u(t)$



(a)



(b)

Figure.9 Synchronization errors with the fractional order adaptive PI^{λ} controller - (a): Error signal $e_1(t)$, (b): Error signal $e_2(t)$

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