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THE $GL(2,\mathbb{R})$-COMITANTS FOR THE HOMOGENEOUS BIDIMENSIONAL POLYNOMIAL SYSTEM OF DIFFERENTIAL EQUATIONS OF THE FOURTH DEGREE

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For the homogeneous bidimensional polynomial system of differential equations of the fourth degree, the types, subtypes and the number of irreducible $GL(2,\mathbb{R})$-comitants and $GL(2,\mathbb{R})$-invariants up to the eighteen degree including were determined. A minimal polynomial bases of $GL(2,\mathbb{R})$-comitants and of $GL(2,\mathbb{R})$-invariants up to eighteen degree including were constructed for the mentioned system.
Keywords: *polynomial systems of differential equations, comitants, invariants, transvectants, minimal polynomial basis.*

$GL(2,\mathbb{R})$-COMITANȚII SISTEMULUI OMOGEN BIDIMENSIONAL DE ECUAȚII DIFERENȚIALE DE GRADUL PATRU

Pentru sistemul omogen bidimensional de ecuații diferențiale de gradul patru au fost stabilite tipurile, subtipurile și numărul de $GL(2,\mathbb{R})$-comitanți și $GL(2,\mathbb{R})$-invarianți ireductibili până la gradul optsprezece inclusiv. Pentru sistemul menționat, au fost construite baze polinomiale minimale ale $GL(2,\mathbb{R})$-comitanților și ale $GL(2,\mathbb{R})$-invarianților până la gradul optsprezece inclusiv.
Cuvinte-cheie: *sistem polinomial de ecuații diferențiale, comitant, invariant, transvectant, bază polinomială minimală.*

1. Definitions and Notations

Let us consider the homogeneous polynomial system of differential equations of the fourth degree:

$$\frac{dx}{dt} = \mathbf{P}_4(x,y), \frac{dy}{dt} = \mathbf{Q}_4(x,y), \tag{1}$$

where $\mathbf{P}_4(x,y), \mathbf{Q}_4(x,y)$ are homogeneous polynomials of degree 4 in x and y with real coefficients.

The system (1) can be written in the following coefficient form:

$$\begin{aligned} \frac{dx}{dt} &= \mathrm{g}x^4 + 4\mathrm{h}x^3y + 6\mathrm{k}x^2y^2 + 4\mathrm{l}xy^3 + \mathrm{m}y^4, \\ \frac{dy}{dt} &= \mathrm{n}x^4 + 4\mathrm{p}x^3y + 6\mathrm{q}x^2y^2 + 4\mathrm{r}xy^3 + \mathrm{s}y^4. \end{aligned} \tag{2}$$

We denote by A the 10-dimensional coefficient space of the system (1), by $\mathbf{a} \in A$ the vector of coefficients $\mathbf{a} = (\mathrm{g},\mathrm{h},\mathrm{k},\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{p},\mathrm{q},\mathrm{r},\mathrm{s})$, by $\boldsymbol{q} \in Q \subseteq Aff(2,\mathbb{R})$ a non-degenerate linear transformation of the phase plane of system (1), by $\mathbf{q}$ the transformation matrix and by $r_{\boldsymbol{q}}(\mathbf{a})$ the linear representation of the coefficients of transformed system in the space A.

Definition 1. [1, 2] *A polynomial $\mathcal{K}(\mathbf{a},\mathbf{x})$ in coefficients of the system (1) and the coordinates of vector $\mathbf{x} = \binom{x}{y} \in \mathbb{R}^2$ is called a comitant of the system (1) with respect to the group Q if there exists a function $\lambda: Q \to \mathbb{R}$ such that*

$$\mathcal{K}(r_{\boldsymbol{q}}(\mathbf{a}), \mathbf{q}\mathbf{x}) \equiv \lambda(\mathbf{q}) \cdot \mathcal{K}(\mathbf{a},\mathbf{x})$$

for every $\boldsymbol{q} \in Q$, $\mathbf{a} \in A$ and $\mathbf{x} \in \mathbb{R}^2$.

If Q is the group $GL(2,\mathbb{R})$ of non-degenerate linear transformations

$$\mathbf{u} = \mathbf{q}\mathrm{x}, \Delta_{\mathbf{q}} = det\mathbf{q} \neq 0$$

of the phase plane of the system (1), where $\mathbf{u} = \binom{u}{v}$ is a vector of new phase variables and $\mathbf{q} = \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix}$ is the transformation matrix, then the comitant is called $GL(2,\mathbb{R})$-comitant or center-affine comitant. In what follows, only $GL(2,\mathbb{R})$-comitants are considered. If a comitant does not depend on the coordinates of the vector $\mathbf{x}$, then it is called invariant.

The function $\lambda(\boldsymbol{q})$ is called multiplicator. It is known [1, 2] that the function $\lambda(\mathbf{q})$ has the form $\lambda(\boldsymbol{q}) = \Delta_{\mathbf{q}}^{-\chi}$, where χ is an integer, which is called the weight of the comitants $\mathcal{K}(\mathbf{a},\mathbf{x})$. If $\chi = 0$, then the comitant is called absolute, otherwise it is called relative.

According to [1,2] if a $GL(2,\mathbb{R})$-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ is a non-homogeneous polynomial with respect to $\mathbf{x}$ and $\mathbf{a}$, then each its homogeneity is also a $GL(2,\mathbb{R})$-comitant. So, in what follows, we shall consider only homogeneous invariant polynomials.

We say that a comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ has character $(\rho;\chi;\delta)$ if it has degree ρ with respect to coordinates of the vector $\mathbf{x}$ (the order of comitant), weight χ and degree δ with respect to the coefficients of the system (1) (the degree of comitant).

We say that a comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ has type $(\rho;\chi;l_4)$ if it has degree ρ with respect to the coordinates of the vector $\mathbf{x}$, weight χ and degree l_4 with respect to the coefficients of the homogenity of degree four for the system (1). In the case of system (1) these two notions are similar.

Every comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ of the system (1) of type $(\rho;\chi;l_4)$ can be represented in the form

$$\mathcal{K}(\mathbf{a},\mathbf{x}) = T_0(\mathbf{a})x^\rho + T_1(\mathbf{a})x^{\rho-1}y + \cdots + T_{\rho-1}(\mathbf{a})xy^{\rho-1} + T_\rho(\mathbf{a})y^\rho,$$

where $T_i(\mathbf{a})$ are homogeneous polynomials in coefficients of the system (1). Polynomial $T_0(\mathbf{a})$ is called semi-invariant of the comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ and is denoted by $\mathcal{SK}(\mathbf{a})$. Thus,

$$\mathcal{SK}(\mathbf{a}) = \frac{1}{\rho!}\cdot\frac{\partial^\rho \mathcal{K}(\mathbf{a},\mathbf{x})}{\partial x^\rho}.$$

$GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of the system (1) (or (2)) have the form

$$R_4 = \mathbf{P}_4(x,y)y - \mathbf{Q}_4(x,y)x =$$
$$= -\mathrm{n}x^5 + (g-4\mathrm{p})x^4y + (4h-6\mathrm{q})x^3y^2 + (6\mathrm{k}-4\mathrm{r})x^2y^3 + (4\mathrm{l}-\mathrm{s})xy^4 + \mathrm{m}y^5,$$
$$S_4 = \frac{1}{4}\left(\frac{\partial \mathbf{P}_4(x,y)}{\partial x} + \frac{\partial \mathbf{Q}_4(x,y)}{\partial y}\right) = (\mathrm{g}+\mathrm{p})x^3 + 3(\mathrm{h}+\mathrm{q})x^2y + 3(\mathrm{k}+\mathrm{r})xy^2 + (\mathrm{l}+\mathrm{s})y^3. \quad (3)$$

Definition 2. [3,4] *Let φ and ψ be homogeneous polynomials in the coordinates of the vector* $\mathbf{x} = \binom{x}{y} \in \mathbb{R}^2$ *of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1-j)!\,(\rho_2-j)!}{\rho_1!\,\rho_2!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}}$$

is called transvectant of the index j of the polynomials φ and ψ.

Remark 1. *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants of the system* (1) *with the characters $(\rho_\varphi;\chi_\varphi;\delta_\varphi)$ and $(\rho_\psi;\chi_\psi;\delta_\psi)$ (with the types $(\rho_\varphi;\chi_\varphi;l_{4\varphi})$ and $(\rho_\psi;\chi_\psi;l_{4\psi})$), respectively, then the transvectant of the index $j \le min\{\rho_\varphi,\rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of the system* (1) *with the character $(\rho_\varphi+\rho_\psi-2j;\chi_\varphi+\chi_\psi+j;\delta_\varphi+\delta_\psi)$* [5] *(with the type $(\rho_\varphi+\rho_\psi-2j;\chi_\varphi+\chi_\psi-j;l_{4\varphi}+l_{4\psi})$). If $j > min\{\rho_\varphi,\rho_\psi\}$, then $(\varphi,\psi)^{(j)} = 0$.*

We say that a comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ has subtype $(\rho;\chi;r_4;s_4)$ if it has degree ρ with respect to the coordinates of vector $\mathbf{x}$, weight χ, degree r_4 with respect to the coefficients of the comitant R_4 and degree s_4 with respect to the coefficients of comitant S_4.

Remark 2. *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants of the system* (1) *with the subtypes $(\rho_\varphi;\chi_\varphi;r_{4\varphi};s_{4\varphi})$ and $(\rho_\psi;\chi_\psi;r_{4\psi};s_{4\psi})$, respectively, then the transvectant of the index $j \le min\{\rho_\varphi,\rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of the system* (1) *with the subtype $(\rho_\varphi+\rho_\psi-2j;\chi_\varphi+\chi_\psi-j;r_{4\varphi}+r_{4\psi};s_{4\varphi}+s_{4\psi})$. If $j > min\{\rho_\varphi,\rho_\psi\}$, then $(\varphi,\psi)^{(j)} = 0$.*

A comitant of the system (1) with respect to the group Q is called reducible, if it can be represented as a polynomial of comitants of the lower degrees.

Definition 3. [1,2,6] *Set $\mathcal{S}$ of the comitants (respectively, invariants) is called a polynomial basis of comitants (respectively, invariants) for the system* (1) *with respect to the group Q, if any comitant (respectively, invariant) of the system* (1) *with respect to group Q is a polynomial of elements of the set $\mathcal{S}$.*

Definition 4. [6] *Set $\mathcal{S}$ of the comitants (respectively, invariants) of degrees less or equal to δ is called a polynomial bases of the comitants for the system* (1) *up to the δ degree inclusively with respect to the group Q if any comitant (respectively, invariant) of the degree less or equal to δ of the system* (1) *with respect to group Q is a polynomial of elements of this set.*

Definition 5. [1,2,6] *A polynomial basis of the comitants (respectively, invariants) for the system* (1) *with respect to the group* Q *is called minimal, if by removal from it of any comitant (respectively, invariant) it ceases to be a polynomial basis.*

Definition 6. [1,2] *Any relation of form* $\mathcal{P}(\mathcal{S}) = 0$, *where* $\mathcal{P}(\mathcal{S})$ *is a polynomial of the comitants from* $\mathcal{S}$, *which is an identity with respect to variables* **a** *and* **x,** *i.e. with respect to the coefficients and variables* x *and* y *of the system* (1), *and it is not an identity with respect to the comitants from* $\mathcal{S}$, *is called a syzygy among comitants of this system.*

The theory of algebraic invariants and comitants for polynomial autonomous systems of differential equations has been developed by C.Sibirschi [1,2] and his disciples. An important problem concerning this theory is the construction of minimal polynomial bases of the invariants and of the comitants of the mentioned systems, with respect to different subgroups of the affine group of the transformations of their phase planes, in particular with respect to the subgroup $GL(2,\mathbb{R})$. Some important results in this direction were obtained by academician C.Sibirschi [1,2] and N.Vulpe [6-12]. We remark, that $GL(2,\mathbb{R})$-comitants ($GL(2,\mathbb{R})$-invariants) and the polynomial bases of $GL(2,\mathbb{R})$-comitants ($GL(2,\mathbb{R})$-invariants) for different combinations of homogeneous polynomials $\mathbf{P}_i(x,y)$, $\mathbf{Q}_i(x,y)$ $(i = 0,1,2,3)$ in the system were constructed by E.Gasinskaya-Kirnitskaya [13-15], D.D. Bik [16, 17], D.Boularas [18,19], M.Popa [18,20], V.Ciobanu [21,23], V.Danilyuk [22,23], Iu.Calin [24], E.Naidenova [25]. In [26] the author studied the problem of determination of the types and the number of irreducible comitants for the system (1). The problem of constructing of an integer algebraic base for the system (1) was investigated in [27]. In our work, we give the minimal polynomial bases of $GL(2,\mathbb{R})$-comitants and of $GL(2,\mathbb{R})$-invariants up to eighteen degree for the homogeneous bidimensional polynomial system of differential equations of the fourth degree.

2. The Algorithm of Construction a Polynomial Base of $GL(2,\mathbb{R})$-comitants

By using the following substitutions

$$\mathrm{g} = \frac{4\mathrm{P}+5\mathrm{H}}{5}, \mathrm{h} = \frac{5\mathrm{K}+3\mathrm{Q}}{5}, \mathrm{k} = \frac{5\mathrm{L}+2\mathrm{R}}{5}, \mathrm{l} = \frac{5\mathrm{M}+\mathrm{S}}{5}, \mathrm{m} = \mathrm{N}, \mathrm{n} = -\mathrm{G}$$
$$\mathrm{p} = \frac{\mathrm{P}-5\mathrm{H}}{5}, \mathrm{q} = \frac{2\mathrm{Q}-5\mathrm{K}}{5}, \mathrm{r} = \frac{3\mathrm{R}-5\mathrm{L}}{5}, \mathrm{s} = \frac{4\mathrm{S}-5\mathrm{M}}{5} \tag{4}$$

the system (2) is reduced to the form

$$\frac{dx}{dt} = \frac{4\mathrm{P}+\mathrm{H}}{5}x^4 + \frac{4\mathrm{Q}+2\mathrm{K}}{5}x^3y + \frac{4\mathrm{R}+3\mathrm{L}}{5}x^2y^2 + \frac{4\mathrm{S}+4\mathrm{M}}{5}xy^3 + \mathrm{N}y^4,$$
$$\frac{dy}{dt} = -\mathrm{G}x^4 + \frac{4\mathrm{P}-5\mathrm{H}}{5}x^3y + \frac{4\mathrm{Q}-3\mathrm{K}}{5}x^2y^2 + \frac{4\mathrm{R}-2\mathrm{L}}{5}xy^2 + \frac{4\mathrm{S}-\mathrm{M}}{5}y^4. \tag{5}$$

For the system (5) the comitants R_4, S_4 become of the form

$$R_4 = \mathrm{G}x^5 + 4\mathrm{H}x^4y + 6\mathrm{K}x^3y^2 + 6\mathrm{L}x^2y^3 + 4\mathrm{M}xy^4 + \mathrm{N}y^5,$$
$$S_4 = \mathrm{P}x^3 + 3\mathrm{Q}x^2y + 3\mathrm{R}xy^2 + \mathrm{S}y^3. \tag{6}$$

So, comitants R_4 and S_4 are binary forms of the fifth and the third order, respectively.

System (1) $\big($or $(2)\big)$ has the following tensorial form

$$\frac{dx^j}{dt} = a^j_{j_1j_2j_3j_4}x^{j_1}x^{j_2}x^{j_3}x^{j_4}, j, j_1, j_2, j_3, j_4 = 1,2 \tag{7}$$

where the tensor $a^j_{j_1j_2j_3j_4}$ is symmetrical with respect to the lower indices. System (7) has the following extended form

$$\frac{dx^1}{dt} = a^1_{1111}(x^1)^4 + 4a^1_{1112}(x^1)^3x^2 + 6a^1_{1122}(x^1)^2(x^2)^2 + 4a^1_{1222}x^1(x^2)^3 + a^1_{2222}(x^2)^4,$$
$$\frac{dx^2}{dt} = a^2_{1111}(x^1)^4 + 4a^2_{1112}(x^1)^3x^2 + 6a^2_{1122}(x^1)^2(x^2)^2 + 4a^2_{1222}x^1(x^2)^3 + a^2_{2222}(x^2)^4. \tag{8}$$

According to [3,6], weight χ of each coordinate of the tensor $a^j_{j_1j_2j_3j_4}$ (of each coefficient of system (8)), with respect to the coordinate x^2 of vector $\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^2$, is called the difference between the number of lower indices and the number of upper indices that have the value two. By weight χ of the product of the coordinates of tensor $a^j_{j_1j_2j_3j_4}$ (of the coefficients of the system (8)) we denote the sum of the weight of each factor from the product. So, each coefficient of system (8) (or (2)) and each coefficient of the binary forms (comitants) R_4 and S_4 (6) are assigned with weight: $\chi_{\mathrm{n}} = \chi_{\mathrm{G}} = -1$, $\chi_{\mathrm{g}} = \chi_{\mathrm{p}} = \chi_{\mathrm{H}} = \chi_{\mathrm{P}} = 0$, $\chi_{\mathrm{h}} = \chi_{\mathrm{q}} = \chi_{\mathrm{K}} = \chi_{\mathrm{Q}} = 1$, $\chi_{\mathrm{k}} = \chi_{\mathrm{r}} = \chi_{\mathrm{L}} = \chi_{\mathrm{R}} = 2$, $\chi_{\mathrm{l}} = \chi_{\mathrm{s}} = \chi_{\mathrm{M}} = \chi_{\mathrm{S}} = 3$, $\chi_{\mathrm{m}} = \chi_{\mathrm{N}} = 4$. To a product of the coefficients of the binary forms R_4 and S_4 (for the system (2)) the weight which is equal to the sum of weights of each factor from the product is assigned.

For the determination of a polynomial basis of $GL(2,\mathbb{R})$-comitants for the system (1), we use the Gordan method [3,4] for the construction of polynomial bases of $GL(2,\mathbb{R})$-comitants for a system of binary forms and the dimensions of linear spaces of $GL(2,\mathbb{R})$-comitants of the same type (subtype). Let $f_1, f_2, \dots, f_l$ be a system of binary forms of degrees $m_1, m_2, \dots, m_l$, respectively, and $h = \max(m_1, m_2, \dots, m_l)$. For the considered system of binary forms, a minimal polynomial basis of $GL(2,\mathbb{R})$-comitants up to the degree $m-1$, was constructed. According to Gordan method, any comitant K_m represents a liniar combination of the transvectants of the form $(K_{m-1}, f_i)^{(k)}$, $i = \overline{1,l}$, where K_{m-1} are comitants of the degree $m-1$ for this system of binary forms. For the determination of the irreducible comitants of degree m, it is necessary to construct all transvectants of the form $(K_{m-1}, f_i)^{(k)}$, $i = \overline{1,l}$, taking $k = 1, 2, 3, \dots, h$, for all liniar independent comitants K_{m-1} and to exclude all comitants that are expressed polynomially by comitants of degrees less than m, i.e. reducible comitants. According to Hilbert theory, starting with a certain degree with respect to the coefficients of the system of the forms, all comitants of the form $(K_{m-1}, f_i)^{(k)}$, $i = \overline{1,l}$, will be reducible and so a polynomial basis of $GL(2,\mathbb{R})$-comitants for the system of binary forms will be determined. In our case, the system of the forms is made up by the two comitants of the first degree R_4 and S_4, having respectively the weights -1 and 0.

The study of the linear dependence of the comitants of the same type (subtype) is simplified, if for the study of the linear dependence the semi-invariants are used.

Number N of the linear independent comitants of the type $(\rho; \chi; l_4)$ for the system (1) is determined from the relation $N = N_\chi - N_{\chi-1}$ [6], where N_χ is the number of all possible products of the weight χ and degree l_4 formed by the coefficients of the forms R_4 and S_4 (of the homogenity of degree four of the system).

Assume that a minimal polynomial basis of comitants for the system (1) up to degree $d-1$, inclusively is known. We denote by $\bar{N}$ the number of all reducible comitants of the type $(\rho; \chi; l_4)$ of degree d, obtained by the operation of multiplication of the comitants of the less degree, which are included in the minimal polynomial basis and by P we denote the maximum number of linear independent comitants of these $\bar{N}$ comitants. Then, in the minimal polynomial basis for the system (1) must be included exactly $N-P$ comitants of the considered type.

The number N of liniar independent comitants of the subtype $(\rho; \chi; r_4; s_4)$ for the system (1) is determined from the relation $N = N_\chi - N_{\chi-1}$, where N_χ is the number of all possible products of the weight χ and the degrees r_4 and s_4 with respect to the coefficients of the comitants R_4 and S_4, respectively.

Assume that a minimal polynomial basis of the comitants for the system (1) up to degree $d-1$, inclusively is known. We denote by $\bar{N}$ the number of all reducible comitants of the subtype $(\rho; \chi; r_4; s_4)$ of degree d, obtained by the operation of multiplication of the comitants of the less degree, which are included in the minimal polynomial basis and by P we denote the maximum number of linear independent comitants of these $\bar{N}$ comitants. Then, in the minimal polynomial basis for the system (1) must be included exactly $N-P$ comitants of the considered subtype.

3. A minimal Polynomial Basis of $GL(2,\mathbb{R})$-comitants and of $GL(2,\mathbb{R})$-invariants for the Bidimensional System of Differential Equations with Nonlinearities of the Fourth Degree

The problem of constructing a minimal polynomial basis of $GL(2,\mathbb{R})$ −comitants for the system (1) is reduced to the construction of a minimal polynomial basis for the binary forms R_4 and S_4.

For the construction of the elements of the minimal polynomial basis of $GL(2,\mathbb{R})$ −comitants for the system (1) (or (2)), we use the transvectants in differential form, the Gordan method for construction of polynomial bases of $GL(2,\mathbb{R})$-comitants for a system of binary forms, the dimensions of linear spaces of comitants of the same type (subtype) and the system of programs Mathematica. By using the $GL(2,\mathbb{R})$-

comitants of the first degree with respect to the coefficients of the system (1) and the transvectans, the following $GL(2,\mathbb{R})$ −comitants for homogeneous polynomial system of differential equations of the fourth degree were constructed:

$$J_1=(Q_{17},R_4)^{(5)};\ J_2=(Q_{11},S_4)^{(3)};\ J_3=(Q_{12},S_4)^{(3)};\ J_4=(Q_{13},S_4)^{(3)};$$
$$J_5=(Q_{14},S_4)^{(3)};\ J_6=(Q_{16},S_4)^{(3)};\ J_7=(Q_{47},S_4)^{(3)};\ J_8=(Q_{49},S_4)^{(3)};$$
$$J_9=(Q_{50},S_4)^{(3)};\ J_{10}=(Q_{52},S_4)^{(3)};\ J_{11}=(Q_9,Q_{10})^{(1)};\ J_{12}=(Q_{53},S_4)^{(3)};$$
$$J_{13}=(Q_{55},S_4)^{(3)};\ J_{14}=(Q_{58},Q_3)^{(2)};\ J_{15}=(Q_{36},Q_9)^{(1)};\ J_{16}=(Q_{36},Q_{10})^{(1)};$$
$$J_{17}=(Q_{37},Q_9)^{(1)};\ J_{18}=(Q_{38},Q_9)^{(1)};\ J_{19}=(Q_{37},Q_{10})^{(1)};\ J_{20}=(Q_{38},Q_{10})^{(1)};$$
$$J_{21}=(Q_{39},Q_9)^{(1)};\ J_{22}=(Q_{39},Q_{10})^{(1)};\ J_{23}=(Q_{43},Q_9)^{(1)};\ J_{24}=(Q_{44},Q_9)^{(1)};$$
$$J_{25}=(Q_{43},Q_{10})^{(1)};\ J_{26}=(Q_{44},Q_{10})^{(1)};\ J_{27}=(Q_{46},Q_9)^{(1)};\ J_{28}=(Q_{46},Q_{10})^{(1)};$$
$$J_{29}=(Q_{103},S_4)^{(3)};\ J_{30}=(Q_{72},Q_9)^{(1)};\ J_{31}=(Q_{72},Q_{10})^{(1)};\ J_{32}=(Q_{73},Q_9)^{(1)};$$
$$J_{33}=(Q_{73},Q_{10})^{(1)};\ J_{34}=(Q_{75},Q_9)^{(1)};\ J_{35}=(Q_{78},Q_9)^{(1)};\ J_{36}=(Q_{75},Q_{10})^{(1)};$$
$$J_{37}=(Q_{80},Q_9)^{(1)};\ J_{38}=(Q_{82},Q_9)^{(1)};\ J_{39}=(Q_{82},Q_{10})^{(1)};\ J_{40}=(Q_{87},Q_9)^{(1)};$$
$$J_{41}=(Q_{90},Q_9)^{(1)};\ J_{42}=(Q_{90},Q_{10})^{(1)};\ J_{43}=(Q_{72},Q_{36})^{(1)};\ J_{44}=(Q_{72},Q_{37})^{(1)};$$
$$J_{45}=(Q_{99},Q_9)^{(1)};\ J_{46}=(Q_{100},Q_9)^{(1)};\ J_{47}=(Q_{104},Q_9)^{(1)};\ J_{48}=(Q_{105},Q_9)^{(1)};$$
$$J_{49}=(Q_{106},Q_9)^{(1)};\ J_{50}=(Q_{106},Q_{36})^{(1)};$$
$$Q_1=R_4;\ Q_2=S_4;\ Q_3=(R_4,R_4)^{(4)};\ Q_4=(R_4,S_4)^{(3)};$$
$$Q_5=(S_4,S_4)^{(2)};\ Q_6=(R_4,S_4)^{(2)};\ Q_7=(R_4,R_4)^{(2)};\ Q_8=(R_4,S_4)^{(1)};$$
$$Q_9=(Q_3,S_4)^{(2)};\ Q_{10}=(Q_4,S_4)^{(2)};\ Q_{11}=(Q_3,R_4)^{(2)};\ Q_{12}=(Q_3,S_4)^{(1)};$$
$$Q_{13}=(Q_7,S_4)^{(3)};\ Q_{14}=(Q_4,S_4)^{(1)};\ Q_{15}=(Q_6,S_4)^{(2)};\ Q_{16}=(Q_5,S_4)^{(2)};$$
$$Q_{17}=(Q_3,R_4)^{(1)};\ Q_{18}=(Q_7,S_4)^{(2)};\ Q_{19}=(Q_6,S_4)^{(1)};\ Q_{20}=(Q_7,S_4)^{(1)};$$
$$Q_{21}=(Q_7,R_4)^{(1)};\ Q_{22}=(Q_{11},S_4)^{(2)};\ Q_{23}=(Q_{17},S_4)^{(3)};\ Q_{24}=(Q_9,S_4)^{(1)};$$
$$Q_{25}=(Q_{13},S_4)^{(2)};\ Q_{26}=(Q_{10},S_4)^{(1)};\ Q_{27}=(Q_{15},S_4)^{(2)};\ Q_{28}=(Q_{11},R_4)^{(2)};$$
$$Q_{29}=(Q_{11},S_4)^{(1)};\ Q_{30}=(Q_{17},S_4)^{(2)};\ Q_{31}=(Q_{13},S_4)^{(1)};\ Q_{32}=(Q_{18},S_4)^{(2)};$$
$$Q_{33}=(Q_{15},S_4)^{(1)};\ Q_{34}=(Q_{11},R_4)^{(1)};\ Q_{35}=(Q_{21},S_4)^{(3)};\ Q_{36}=(Q_{28},R_4)^{(4)};$$
$$Q_{37}=(Q_{28},S_4)^{(3)};\ Q_{38}=(Q_3,Q_9)^{(1)};\ Q_{39}=(Q_{22},S_4)^{(2)};\ Q_{40}=(Q_{23},S_4)^{(2)};$$
$$Q_{41}=(Q_4,Q_9)^{(1)};\ Q_{42}=(Q_{24},S_4)^{(2)};\ Q_{43}=(Q_{25},S_4)^{(2)};\ Q_{44}=(Q_4,Q_{10})^{(1)};$$
$$Q_{45}=(Q_{26},S_4)^{(2)};\ Q_{46}=(Q_{27},S_4)^{(2)};\ Q_{47}=(Q_{28},R_4)^{(3)};\ Q_{48}=(Q_{28},S_4)^{(2)};$$
$$Q_{49}=(Q_{34},S_4)^{(3)};\ Q_{50}=(Q_{22},S_4)^{(1)};\ Q_{51}=(Q_{30},S_4)^{(2)};\ Q_{52}=(Q_{35},S_4)^{(3)};$$
$$Q_{53}=(Q_{25},S_4)^{(1)};\ Q_{54}=(Q_{32},S_4)^{(2)};\ Q_{55}=(Q_{27},S_4)^{(1)};\ Q_{56}=(Q_{28},S_4)^{(1)};$$
$$Q_{57}=(Q_{28},R_4)^{(1)};\ Q_{58}=(Q_{47},R_4)^{(3)};\ Q_{59}=(Q_{36},S_4)^{(1)};\ Q_{60}=(Q_{47},S_4)^{(2)};$$
$$Q_{61}=(Q_{37},S_4)^{(1)};\ Q_{62}=(Q_{48},S_4)^{(2)};\ Q_{63}=(Q_{49},S_4)^{(2)};\ Q_{64}=(Q_{39},S_4)^{(1)};$$
$$Q_{65}=(Q_{51},S_4)^{(2)};\ Q_{66}=(Q_{52},S_4)^{(2)};\ Q_{67}=(Q_{43},S_4)^{(1)};\ Q_{68}=(Q_{54},S_4)^{(2)};$$
$$Q_{69}=(Q_{46},S_4)^{(1)};\ Q_{70}=(Q_{36},R_4)^{(1)};\ Q_{71}=(Q_{57},S_4)^{(3)};\ Q_{72}=(Q_{70},R_4)^{(4)};$$
$$Q_{73}=(Q_{58},S_4)^{(2)};\ Q_{74}=(Q_{70},S_4)^{(3)};\ Q_{75}=(Q_{59},S_4)^{(2)};\ Q_{76}=(Q_{60},S_4)^{(2)};$$
$$Q_{77}=(Q_{71},S_4)^{(3)};\ Q_{78}=(Q_{23},Q_9)^{(1)};\ Q_{79}=(Q_{61},S_4)^{(2)};\ Q_{80}=(Q_{62},S_4)^{(2)};$$
$$Q_{81}=(Q_{63},S_4)^{(2)};\ Q_{82}=(Q_{23},Q_{10})^{(1)};\ Q_{83}=(Q_{24},Q_9)^{(1)};\ Q_{84}=(Q_{64},S_4)^{(2)};$$
$$Q_{85}=(Q_{65},S_4)^{(2)};\ Q_{86}=(Q_{66},S_4)^{(2)};\ Q_{87}=(Q_{24},Q_{10})^{(1)};\ Q_{88}=(Q_{67},S_4)^{(2)};$$
$$Q_{89}=(Q_{68},S_4)^{(2)};\ Q_{90}=(Q_{26},Q_{10})^{(1)};\ Q_{91}=(Q_{69},S_4)^{(2)};\ Q_{92}=(Q_{58},S_4)^{(1)};$$
$$Q_{93}=(Q_{70},R_4)^{(2)};\ Q_{94}=(Q_{93},R_4)^{(3)};\ Q_{95}=(Q_{72},S_4)^{(1)};\ Q_{96}=(Q_{93},S_4)^{(3)};$$
$$Q_{97}=(Q_{73},S_4)^{(1)};\ Q_{98}=(Q_{94},S_4)^{(2)};\ Q_{99}=(Q_{58},Q_9)^{(1)};\ Q_{100}=(Q_{95},S_4)^{(2)};$$
$$Q_{101}=(Q_{96},S_4)^{(2)};\ Q_{102}=(Q_{97},S_4)^{(2)};\ Q_{103}=(Q_{94},R_4)^{(2)};\ Q_{104}=(Q_{58},Q_{36})^{(1)};$$
$$Q_{105}=(Q_{94},Q_9)^{(1)};\ Q_{106}=(Q_{94},Q_{36})^{(1)}.$$

For the comitants and invariants of the system (1), listed above the following theorem takes place:

Theorem. *A minimal polynomial basis of $GL(2,\mathbb{R})$-comitants (respectively, of $GL(2,\mathbb{R})$-invariants) of the system* (1) *up to eighteen degree consists from* 156 *elements (respectively,* 50 *elements), which must be of the following* 35 *(respectively,* 8*) types or 99 (respectively, 32) subtypes:*

No	Type of comitant	Subtype of comitant	The number of comitants included in the basis	The comitants included in the basis
1.	(5,-1,1)	(5,-1,1,0)	1	Q_1
2.	(3,0,1)	(3,0,0,1)	1	Q_2
3.	(2,2,2)	(2,2,2,0)	1	Q_3
		(2,2,1,1)	1	Q_4
		(2,2,0,2)	1	Q_5
4.	(4,1,2)	(4,1,1,1)	1	Q_6
5.	(6,0,2)	(6,0,2,0)	1	Q_7
		(6,0,1,1)	1	Q_8
6.	(1,4,3)	(1,4,2,1)	1	Q_9
		(1,4,1,2)	1	Q_{10}
7.	(3,3,3)	(3,3,3,0)	1	Q_{11}
		(3,3,2,1)	2	Q_{12}, Q_{13}
		(3,3,1,2)	2	Q_{14}, Q_{15}
		(3,3,0,3)	1	Q_{16}
8.	(5,2,3)	(5,2,3,0)	1	Q_{17}
		(5,2,2,1)	1	Q_{18}
		(5,2,1,2)	1	Q_{19}
9.	(7,1,3)	(7,1,2,1)	1	Q_{20}
10.	(9,0,3)	(9,0,3,0)	1	Q_{21}
11.	(0,6,4)	(0,6,4,0)	1	J_1
		(0,6,3,1)	1	J_2
		(0,6,2,2)	2	J_3, J_4
		(0,6,1,3)	1	J_5
		(0,6,0,4)	1	J_6
12.	(2,5,4)	(2,5,3,1)	2	Q_{22}, Q_{23}
		(2,5,2,2)	2	Q_{24}, Q_{25}
		(2,5,1,3)	2	Q_{26}, Q_{27}
13.	(4,4,4)	(4,4,4,0)	1	Q_{28}
		(4,4,3,1)	2	Q_{29}, Q_{30}
		(4,4,2,2)	2	Q_{31}, Q_{32}
		(4,4,1,3)	1	Q_{33}
14.	(6,3,4)	(6,3,4,0)	1	Q_{34}
		(6,3,3,1)	1	Q_{35}
15.	(7,1,5)	(1,7,5,0)	1	Q_{36}
		(1,7,4,1)	2	Q_{37}, Q_{38}
		(1,7,3,2)	3	Q_{39}, Q_{40}, Q_{41}
		(1,7,2,3)	3	Q_{42}, Q_{43}, Q_{44}
		(1,7,1,4)	2	Q_{45}, Q_{46}
16.	(3,6,5)	(3,6,5,0)	1	Q_{47}
		(3,6,4,1)	2	Q_{48}, Q_{49}
		(3,6,3,2)	3	Q_{50}, Q_{51}, Q_{52}
		(3,6,2,3)	2	Q_{53}, Q_{54}
		(3,6,1,4)	1	Q_{55}
17.	(5,5,5)	(5,5,4,1)	1	Q_{56}
18.	(7,4,5)	(7,4,5,0)	1	Q_{57}
19.	(0,9,6)	(0,9,5,1)	1	J_7
		(0,9,4,2)	1	J_8
		(0,9,3,3)	3	J_9, J_{10}, J_{11}
		(0,9,2,4)	1	J_{12}

		(0,9,1,5)	1	J_{13}
20.	(2,8,6)	(2,8,6,0)	1	Q_{58}
		(2,8,5,1)	2	Q_{59}, Q_{60}
		(2,8,4,2)	3	Q_{61}, Q_{62}, Q_{63}
		(2,8,3,3)	3	Q_{64}, Q_{65}, Q_{66}
		(2,8,2,4)	2	Q_{67}, Q_{68}
		(2,8,1,5)	1	Q_{69}
21.	(4,7,6)	(4,7,6,0)	1	Q_{70}
		(4,7,5,1)	1	Q_{71}
22.	(1,10,7)	(1,10,7,0)	1	Q_{72}
		(1,10,6,1)	2	Q_{73}, Q_{74}
		(1,10,5,2)	4	$Q_{75}, Q_{76}, Q_{77}, Q_{78}$
		(1,10,4,3)	5	$Q_{79}, Q_{80}, Q_{81}, Q_{82}, Q_{83}$
		(1,10,3,4)	4	$Q_{84}, Q_{85}, Q_{86}, Q_{87}$
		(1,10,2,5)	3	Q_{88}, Q_{89}, Q_{90}
		(1,10,1,6)	1	Q_{91}
23.	(3,9,7)	(3,9,6,1)	1	Q_{92}
24.	(5,8,7)	(5,8,7,0)	1	Q_{93}
25.	(0,12,8)	(0,12,8,0)	1	J_{14}
		(0,12,7,1)	1	J_{15}
		(0,12,6,2)	3	J_{16}, J_{17}, J_{18}
		(0,12,5,3)	3	J_{19}, J_{20}, J_{21}
		(0,12,4,4)	3	J_{22}, J_{23}, J_{24}
		(0,12,3,5)	3	J_{25}, J_{26}, J_{27}
		(0,12,2,6)	1	J_{28}
26.	(2,11,8)	(2,11,8,0)	1	Q_{94}
		(2,11,7,1)	2	Q_{95}, Q_{96}
		(2,11,6,2)	1	Q_{97}
27.	(1,13,9)	(1,13,8,1)	2	Q_{98}, Q_{99}
		(1,13,7,2)	2	Q_{100}, Q_{101}
		(1,13,6,3)	1	Q_{102}
28.	(3,12,9)	(3,12,9,0)	1	Q_{103}
29.	(0,15,10)	(0,15,9,1)	2	J_{29}, J_{30}
		(0,15,8,2)	2	J_{31}, J_{32}
		(0,15,7,3)	3	J_{33}, J_{34}, J_{35}
		(0,15,6,4)	3	J_{36}, J_{37}, J_{38}
		(0,15,5,5)	2	J_{39}, J_{40}
		(0,15,4,6)	1	J_{41}
		(0,15,3,7)	1	J_{42}
30.	(1,16,11)	(1,16,11,0)	1	Q_{104}
		(1,16,10,1)	1	Q_{105}
31.	(0,18,12)	(0,18,12,0)	1	J_{43}
		(0,18,11,1)	1	J_{44}
		(0,18,10,2)	1	J_{45}
		(0,18,9,3)	1	J_{46}
32.	(1,19,13)	(1,19,13,0)	1	Q_{106}
33.	(0,21,14)	(0,21,13,1)	1	J_{47}
		(0,21,12,2)	1	J_{48}
34.	(0,24,16)	(0,24,15,1)	1	J_{49}
35.	(0,27,18)	(0,27,18,0)	1	J_{50}

Conclusion

It was established that a minimal polynomial basis of $GL(2,\mathbb{R})$-comitants for the system (1) up to eighteen degree consists of 156 comitants and invariants, but a minimal polynomial basis of $GL(2,\mathbb{R})$-invariatns for the system (1) up to eighteen degree consists of 50 invariants.

We establish a conjecture that the $GL(2,\mathbb{R})$-comitants $J_1 - J_{50}$, $Q_1 - Q_{106}$ form a minimal polynomial basis of $GL(2,\mathbb{R})$-comitants for the system (1), but the $GL(2,\mathbb{R})$-invariatns $J_1 - J_{50}$ form a minimal polynomial basis of $GL(2,\mathbb{R})$-invariants for the system (1).

For the system (1) it was established that if a linear space basis of $GL(2,\mathbb{R})$-comitants ($GL(2,\mathbb{R})$-invariants) contains at least one irreducible comitant (irreducible invariant), then there doesn't exist any syzygy between comitants (invariants) of the corresponding subtype.

References:

1. SIBIRSKY, K.S. *Introduction to the Algebraic Theory of Invariants of Differential Equations*. Manchester University Press, 1988.
2. SIBIRSKY, K.S. *Algebraic invariants of differential equations and matrices.* Kishinev, Shtiintsa, 1976 (in Russian).
3. GUREVICH, G.B. *Foundations of the Theory of Algebraic Invariants*. Noordhoff, Groningen, 1964.
4. GRACE, J.H., YOUNG, A. *The Algebra of Invariants*. Cambridge, 1903. 384 p.; N.Y.: Chelsea, 1965. 384 p.
5. BOULARAS, D., CALIN, IU., TIMOCHOUK, L., VULPE, N. *T-comitants of quadratic systems: A study via the translation invariants.* Report 96-90, Delf University of Technology, Faculty of Technical Mathematics and Informatics, 1996, p.1-36.
6. VULPE, N.I. *Polynomial bases of comitants of differential systems and their applications in qualitative theory*. Chișinău: Știința, 1986 (in Russian).
7. VULPE, N.I. A minimal polynomial base of the affine invariants of a quadratic system. In: *Differentsial'nye Uravneniya*, 11, 1975, no.5, p.918-920 (in Russian).
8. VULPE, N.I. Construction of a polynomial basis of comitants of a differential system. In: *Differentsial'nye Uravneniya*, 15, 1979, no.8, p.1399-1410 (in Russian).
9. VULPE, N.I. The number of linearly independent comitants of a system of differential equations. In: *Differentsial'nye Uravneniya*, 15, 1979, no.6, p.963-973 (in Russian).
10. VULPE, N.I. A polynomial basis of center-affine comitants of a differential system. In: *Dokl. Akad. Nauk SSSR*, 250, 01980, no.5, p.1033-1037 (in Russian).
11. VULPE, N.I. The minimal basis of comitants of a differential system with quadratic nonlinearities. In: *Differentsial'nye Uravneniya*, 17, 1981, no.11, p.1955-1963 (in Russian).
12. VULPE, N.I. Polynomial basis of centroaffine comitants of a homogeneous cubic differential system. In: *Differentsial'nye Uravneniya*, 17, 1981, no.9, p.1682-1684 (in Russian).
13. GASINSKAYA, È.F., SIBIRSCHI, K.S. Affine invariants of a system with quadratic nonlinearities. In: *Differentsial'nye Uravneniya*, 9, 1973, p.1371-1382 (in Russian).
14. KIRNITSKAYA, È.F. Application of a computer for finding comitants of differential systems. Algebraic invariants of dynamic systems. In: *Mat. Issled.*, 1980, no.55, p.64-70 (in Russian).
15. VULPE, N.I., KIRNITSKAYA, È.F. Centroaffine invariants of a differential system with cubic nonlinearities. Algebraic invariants of dynamic systems. In: *Mat. Issled.*, 1980, no.55, p.46-57 (in Russian).
16. DANG DINI BIK. Affine invariants of an autonomous second order differential system with cubic homogeneous nonlinearities. In: *Mat. Issled.*, **9,** 1974, no.3(33), p.64-81 (in Russian).
17. DANG DINI BIK. Comitants of a cubic system. In: *Mat. Issled.*, **10,** 1975, no.3(37), p.44-55 (in Russian).
18. BOULARAS, D., POPA, M.N. Concomitants of a system with quadratic nonlinearities. In: *Differentsial'nye Uravneniya* 14, 1978, no.5, p.835-842 (in Russian).
19. BOULARAS, D., SIBIRSCHI, KS. Centroaffine invariants of a quadratic differential system. In: *Bul. Akad. Shtiintse RSS Moldoven.*, 1979, no.1, p.8-17 (in Russian).
20. POPA, M.N. The shortest minimal polynomial basis of affine invariants of a differential system with quadratic nonlinearities. Studies in differential equations. In: *Mat. Issled.*, 1983, no.73, p.55-62 (in Russian).
21. CIOBANU, V.M. Comitants of a differential system with cubic nonlinearities. Qualitative methods of the theory of differential equations. In: *Mat. Issled.*, 1984, no.77, p.146-161 (in Russian).
22. DANILYUK, V.I. Centro-affine concomitants of a quadratic-cubic two-dimensional differential system. Differential equations and dynamical systems. In: *Mat. Issled.*, 1985, no.80, p.57-66 (in Russian).
23. DANILYUK, V.I., CIOBANU, V.M. Centro-affine-invariant canonical forms of a two-dimensional differential system with cubic nonlinearities. In: *Mat. Issled.; No.124; In: Funktsional. Metod. Teorii Differ. Uravn.*, 1992, p.43-55 (in Russian).

24. CALIN, IU. Minimal polynomial basis of GL(2, R)-comitants of the generic cubic system of differential equations. In: International Conference *Algebraic System and their Applications in Differential Equations and other domains of mathematics*, Chisinau, August, 21-23, 2007, p.38-41.
25. NAIDENOVA, E.V. *Generators for the algebras* $S_{0,3}$ *and* $SI_{0,3}$. Abstracts of "First Conference of the Mathematical Society of the Republic of Moldova", August 16-18, 2001, Chisinau, p.100.
26. MAKARI, P.M. On the number and types of basis comitants of a differential system with fourth-order homogeneous right-hand sides. In: *Izv. Akad. Nauk Respub. Moldova Mat.*, 1995, no.2-3, p.55-65 (in Russian).
27. VULPE, N.I., MAKARI, P.M., POPA, M.N. A complete algebraic basis of centro-affine invariants of a differential system with fourth-order homogeneous right-hand sides. In: *Izv. Akad. Nauk Respub. Moldova Mat.*, 1996, no.1, p.48-55 (in Russian).

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