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# Simple and Multi-collision of an Ellipsoid with Planar Surfaces. Part I: Theory

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*This paper discusses the problem of simultaneous collisions between an ellipsoid and some planar surfaces. The approach is one based on the theory of screws and uses the notion of inertance. The authors consider that the coefficients of restitution are different for each planar surface and they obtain the velocities after the collision. An example concludes the theory.*

***Keywords***: *multi-collision, inertance, ellipsoid, planar surface*

## 1. Introduction

The problem of the collision of a rigid solid with an obstacle or between rigid solids was considered in many papers [1-24, 26-34]. A presentation of the problem is discussed in [19] and [22] and will not repeat here. The approach is a multibody type one and it is based on the theory of screws.

The multi-collision of a rigid body with an obstacle is presented in [25]. This paper is a generalization of the reference [25] by considering more than one obstacle, each one being characterized by a different coefficient of restitution. The great problem is to determine the collision point between the rigid solid and one particular obstacle. In this paper the rigid solid is an ellipsoid and the obstacles are some planar surface. The theory is full developed and it can be generalized to other types of rigid solids and obstacles.

## 2. General aspects

Let us consider an ellipsoid moving in the space acted by a general system of forces that reduces at the point $C$ at the resultant $\mathbf{F}$ and a resultant moment $\mathbf{M}$ (Fig. 1).

Let $OXZY$ be a fixed reference frame having the $Z$ - axis vertical ascendant and the $X$ -axis and $Y$ -axis defining a horizontal plane.

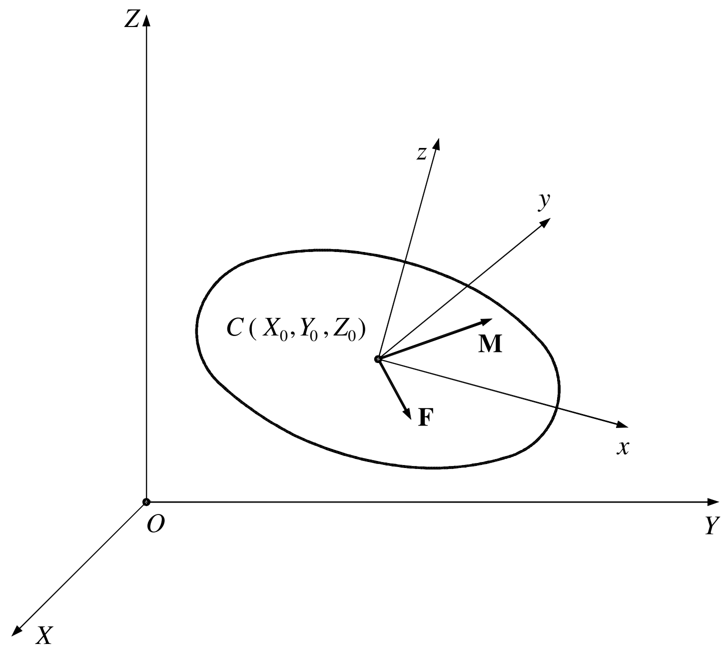


**Figure 1.** The motion of an ellipsoid

Let $Cxyz$ be a mobile reference frame, rigidly linked to the ellipsoid, the axes $Cx$, $Cy$ and $Cz$ being the principal central axes of inertia.

We denote by $X_0$, $Y_0$ and $Z_0$ the coordinates of the center of mass of the ellipsoid relative to the fixed reference frame, and by $J_x$, $J_y$ and $J_z$ its principal central moments of inertia and let $m$ be the mass of the ellipsoid. We may write

$$J_x = \frac{m\left(b^2 + c^2\right)}{5},\ J_y = \frac{m\left(c^2 + a^2\right)}{5},\ J_z = \frac{m\left(a^2 + b^2\right)}{5}, \tag{1}$$

where $a$, $b$ and $c$ are the semi-axes of the ellipsoid.

The moving equations of the ellipsoid are (Fig. 1)

$$m\begin{bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{bmatrix} = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix}, \tag{2}$$

$$\begin{gathered} J_x\dot{\omega}_x - \left(J_y - J_z\right)\omega_y\omega_z = M_x,\ J_y\dot{\omega}_y - \left(J_z - J_x\right)\omega_z\omega_x = M_y, \\ J_z\dot{\omega}_z - \left(J_x - J_y\right)\omega_x\omega_y = M_z, \end{gathered} \tag{3}$$

in which $\omega_x$, $\omega_y$ and $\omega_z$ are the angular velocities around the principal central axes of inertia, $F_X$, $F_Y$, $F_Z$ are the components of the resultant on the axes of the fixed frames, while $M_x$, $M_y$ and $M_z$ are the moments relative to the axes of the mobile reference system.

Let be $P$ a generic point situated on the surface of the ellipsoid, and let $X$, $Y$, $Z$, and $x$, $y$, $z$ respectively, be the coordinates of the point $P$ relative to the fixed and mobile systems of reference, respectively

If we make the notations

– $[\mathbf{A}]$ - the rotation matrix of the ellipsoid;

– $\{\mathbf{R}\}$, $\{\mathbf{R}_0\}$, $[\mathbf{r}]$, $\{\mathbf{r}\}$ - the matrices

$$\{\mathbf{R}\} = [X \quad Y \quad Z]^{\mathrm{T}}, \tag{4}$$

$$\{\mathbf{R}_0\} = [X_0 \quad Y_0 \quad Z_0]^{\mathrm{T}}, \tag{5}$$

$$\{\mathbf{r}\} = [x \quad y \quad z]^{\mathrm{T}}, \tag{6}$$

$$[\mathbf{r}] = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}; \tag{7}$$

– $\psi$, $\theta$, $\varphi$ - the Bryan rotation angles;

– $[\boldsymbol{\psi}]$, $[\boldsymbol{\theta}]$, $[\boldsymbol{\varphi}]$ - the matrices

$$[\boldsymbol{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}, \tag{8}$$

$$[\boldsymbol{\theta}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{9}$$

$$[\boldsymbol{\varphi}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{10}$$

Where from

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}]; \tag{11}$$

– $\{\boldsymbol{\beta}\}$ - the column matrix

$$\{\boldsymbol{\beta}\} = [\psi \quad \theta \quad \varphi]^{\mathrm{T}}; \tag{12}$$

– $\{\mathbf{u}_\psi\}$, $\{\mathbf{u}_\theta\}$, $\{\mathbf{u}_\varphi\}$ - the column matrices

$$\{\mathbf{u}_\psi\} = [1\ \ 0\ \ 0]^{\mathrm{T}}, \tag{13}$$

$$\{\mathbf{u}_\theta\} = [0\ \ 1\ \ 0]^{\mathrm{T}}, \tag{14}$$

$$\{\mathbf{u}_\varphi\} = [0\ \ 0\ \ 1]^{\mathrm{T}}; \tag{15}$$

– $[\mathbf{Q}]$ - the matrix

$$[\mathbf{Q}] = [\boldsymbol{\varphi}]^{\mathrm{T}}\left[[\boldsymbol{\theta}]^{\mathrm{T}}\{\mathbf{u}_\psi\}\ \{\mathbf{u}_\theta\}\ \{\mathbf{u}_\varphi\}\right]; \tag{16}$$

– $\{\boldsymbol{\omega}\}$, $[\boldsymbol{\omega}]$ - the matrices of angular velocity relative to the mobile reference system

$$\{\boldsymbol{\omega}\} = [\omega_x\ \ \omega_y\ \ \omega_z]^{\mathrm{T}}, \tag{17}$$

$$[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \tag{18}$$

Where from

$$\{\boldsymbol{\omega}\} = [\mathbf{Q}]\{\dot{\boldsymbol{\beta}}\}; \tag{19}$$

– $[\boldsymbol{\omega}^{(0)}]$ - the matrix of angular velocity relative to fixed reference system

$$[\boldsymbol{\omega}^{(0)}] = \begin{bmatrix} 0 & -\omega_z^{(0)} & \omega_y^{(0)} \\ \omega_z^{(0)} & 0 & -\omega_x^{(0)} \\ -\omega_y^{(0)} & \omega_x^{(0)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{20}$$

Then the following relations way be written [21]

$$\{\dot{\mathbf{R}}\} = \{\dot{\mathbf{R}}_{\mathbf{0}}\} + [\mathbf{A}] + [\mathbf{r}]^{\mathrm{T}}[\mathbf{Q}]\{\dot{\boldsymbol{\beta}}\}, \tag{21}$$

$$[\boldsymbol{\omega}^{(0)}][\mathbf{A}][\boldsymbol{\omega}][\mathbf{A}]^{\mathrm{T}}. \tag{22}$$

Assuming now that the rigid solid collides simultaneously at the points $P_i$, $i = \overline{1, n}$, with some fixed obstacles, the coefficients of restitution being $k_i$, $i = \overline{1, n}$, then we have [25]

$$\{\mathbf{v}\} = \{\mathbf{v}^0\} + \sum_{i=1}^{n} \frac{(1 + k_i)v_{in}^0}{g_i}[\mathbf{M}_c]^{-1}\{\mathbf{N}_i\}, \tag{23}$$

where $\{\mathbf{v}\}$ and $\{\mathbf{v}^0\}$ are the column matrices of the velocities after and before de collision

$$\{\mathbf{v}\} = [\omega_x\ \ \omega_y\ \ \omega_z\ \ v_x\ \ v_y\ \ v_z]^{\mathrm{T}}, \tag{24}$$

$$\{\mathbf{v}^0\} = [\omega_x^0\ \ \omega_y^0\ \ \omega_z^0\ \ v_x^0\ \ v_y^0\ \ v_z^0]^{\mathrm{T}}, \tag{25}$$

$$\mathbf{n}_i = -\frac{\nabla f|_{P_i}}{\left\|\nabla f|_{P_i}\right\|} = a_i\mathbf{i} + b_i\mathbf{j} + c_i\mathbf{k}\,, \tag{26}$$

$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ being the unit vectors of the axes $OX$, $OY$ and $OZ$,

$$\mathbf{CP}_i \times \mathbf{n}_i = d_i\mathbf{i} + e_i\mathbf{j} + f_i\mathbf{k}\,, \tag{27}$$

$$\{\mathbf{N}_i\} = [a_i \;\; b_i \;\; c_i \;\; d_i \;\; e_i \;\; f_i]^{\mathrm{T}} \tag{28}$$

$$[\mathbf{M}_c] = \begin{bmatrix} 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \\ J_x & 0 & 0 & 0 & 0 & 0 \\ 0 & J_y & 0 & 0 & 0 & 0 \\ 0 & 0 & J_z & 0 & 0 & 0 \end{bmatrix}, \tag{29}$$

$$[\mathbf{M}_c]^{-1} = \begin{bmatrix} 0 & 0 & 0 & J_x^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_y^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_z^{-1} \\ m^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & m^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & m^{-1} & 0 & 0 & 0 \end{bmatrix}, \tag{30}$$

$$\{\mathbf{n}_i\} = [a_i \;\; b_i \;\; c_i], \tag{31}$$

$$v_{in} = \{\dot{\mathbf{R}}\}^{\mathrm{T}}\{\mathbf{n}_i\}, \tag{32}$$

$$[\boldsymbol{\eta}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

$$g_i = \{\mathbf{N}_i\}^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_c]^{-1}\{\mathbf{N}_i\} = \frac{1}{m} + \frac{d_i^2}{J_x} + \frac{e_i^2}{J_y} + \frac{f_i^2}{J_z}\,. \tag{34}$$

The impulse at the point $P_i$ is [25]

$$P_i = -\frac{(1 + k_i)v_{in}^0}{g_i}\,. \tag{35}$$

The equation of the ellipsoid reads

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{36}$$

Taking into account the relation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{37}$$

it results

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\mathbf{A}]^{\mathrm{T}} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}.. \tag{38}$$

Writing the matrix of rotation $[\mathbf{A}]$ in the form

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{39}$$

the expression (38) becomes

$$\begin{aligned} x &= a_{11}(X - X_0) + a_{21}(Y - Y_0) + a_{31}(Z - Z_0), \\ y &= a_{12}(X - X_0) + a_{22}(Y - Y_0) + a_{32}(Z - Z_0), \\ z &= a_{13}(X - X_0) + a_{23}(Y - Y_0) + a_{33}(Z - Z_0). \end{aligned} \tag{40}$$

## 3. The collision between the ellipsoid and the horizontal plane

Let us consider now the Fig. 2.

The distance between the point $P(X, Y, Z)$ and the plane $Z = 0$ is given by $|Z|$.

Let $F(x, y, z)$ be the Lagrange function

$$F(x, y, z) = Z + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right). \tag{41}$$

We get

$$\frac{\partial F}{\partial x} = \frac{\partial Z}{\partial x} + \frac{2\lambda x}{a^2} = a_{31} + \frac{2\lambda x}{a^2}, \tag{42}$$

$$\frac{\partial F}{\partial y} = \frac{\partial Z}{\partial y} + \frac{2\lambda y}{b^2} = a_{32} + \frac{2\lambda y}{b^2}, \tag{43}$$

$$\frac{\partial F}{\partial z} = \frac{\partial Z}{\partial z} + \frac{2\lambda z}{c^2} = a_{33} + \frac{2\lambda z}{c^2}, \tag{44}$$

$$\frac{\partial F}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1. \tag{45}$$

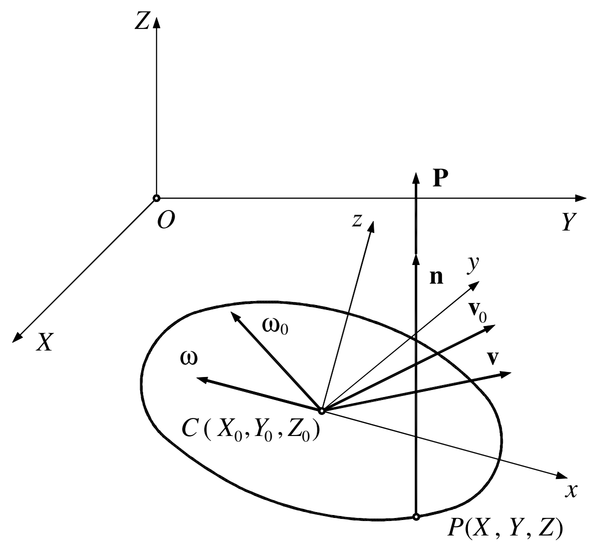


**Figure 2.** The collision between the ellipsoid and the horizontal plane.

Equating to zero the relations (42) – (45), one obtains

$$x = -\frac{a^2 a_{31}}{2\lambda},\ y = -\frac{b^2 a_{32}}{2\lambda},\ z = -\frac{c^2 a_{33}}{2\lambda}, \tag{46}$$

$$\lambda = \pm \frac{\sqrt{a^2 a_{31}^2 + b^2 a_{32}^2 + c^2 a_{33}^2}}{2}, \tag{47}$$

$$Z = Z_0 \mp \frac{a^2 a_{31} + b^2 a_{32} + c^2 a_{33}}{\sqrt{a^2 a_{31}^2 + b^2 a_{32}^2 + c^2 a_{33}^2}}. \tag{48}$$

If $Z > 0$, then no collision is present.

If $Z = 0$, then we have two situations:

a) the component $v_Z$ of the velocity of the point $P$ is a positive one. In this case there is no collision;

b) the component $v_Z$ of the velocity of the point $P$ is negative. In this case we have a collision, which can be discussed with the aid of the formulae presented in paragraph 2 for $i = 1$.

## 4. The collision between the ellipsoid and an oblique plane

We now refer to the Fig. 3.

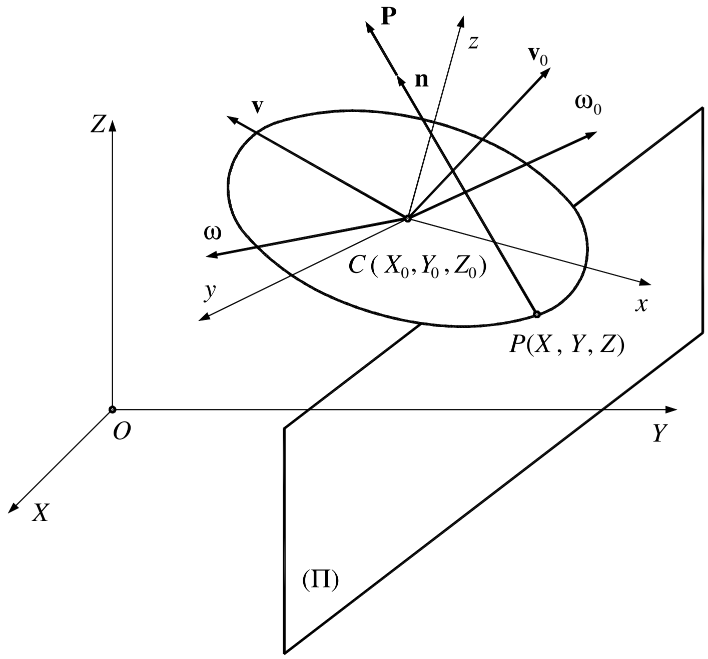


**Figure 3.** The collision between the ellipsoid and the oblique plane.

The equation of the plane reads

$$\alpha X + \beta Y + \gamma Z + \delta = 0 . \tag{49}$$

The distance between a point $P(X, Y, Z)$ and the plane is

$$dist = \frac{|\alpha X + \beta Y + \gamma Z + \delta|}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}, \tag{50}$$

so we can use the Lagrange function

$$F(x, y, z) = \frac{(\alpha X + \beta Y + \gamma Z + \delta)^2}{\alpha^2 + \beta^2 + \gamma^2} + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right). \tag{51}$$

The following system is obtained

$$\frac{\partial F}{\partial x} = \frac{2(\alpha X + \beta Y + \gamma Z + \delta)}{\alpha^2 + \beta^2 + \gamma^2}(\alpha a_{11} + \beta a_{12} + \gamma a_{13}) + \frac{2\lambda x}{a^2} = 0, \tag{52}$$

$$\frac{\partial F}{\partial y} = \frac{2(\alpha X + \beta Y + \gamma Z + \delta)}{\alpha^2 + \beta^2 + \gamma^2}(\alpha a_{21} + \beta a_{22} + \gamma a_{23}) + \frac{2\lambda y}{b^2} = 0, \tag{53}$$

$$\frac{\partial F}{\partial z}=\frac{2(\alpha X+\beta Y+\gamma Z+\delta)}{\alpha^2+\beta^2+\gamma^2}(\alpha a_{31}+\beta a_{32}+\gamma a_{33})+\frac{2\lambda z}{c^2}=0\,, \tag{54}$$

$$\frac{\partial F}{\partial \lambda}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0\,. \tag{55}$$

Taking into account that

$$\begin{aligned}\alpha X+\beta Y+\gamma Z+\delta &= (\alpha a_{11}+\beta a_{21}+\gamma a_{31})x+(\alpha a_{12}+\beta a_{22}+\gamma a_{32})y\\ &+(\alpha a_{13}+\beta a_{23}+\gamma a_{33}z)+\alpha X_0+\beta Y_0+\gamma Z_0+\delta=B_1x+B_2y+B_3z+C\,,\end{aligned} \tag{56}$$

the equations (52) - (54) read

$$\begin{aligned}B_1x+B_2y+B_3z+C&=-\frac{\lambda x}{a^2}\frac{\alpha^2+\beta^2+\gamma^2}{\alpha a_{11}+\beta a_{12}+\gamma a_{13}}\,,\\ B_1x+B_2y+B_3z+C&=-\frac{\lambda y}{b^2}\frac{\alpha^2+\beta^2+\gamma^2}{\alpha a_{21}+\beta a_{22}+\gamma a_{23}}\,,\\ B_1x+B_2y+B_3z+C&=-\frac{\lambda x}{c^2}\frac{\alpha^2+\beta^2+\gamma^2}{\alpha a_{31}+\beta a_{32}+\gamma a_{33}}\end{aligned} \tag{57}$$

or, equivalently,

$$\begin{aligned}&\left[B_1+\frac{\lambda(\alpha^2+\beta^2+\gamma^2)}{a^2(\alpha a_{11}+\beta a_{12}+\gamma a_{13})}\right]x+B_2y+B_3z=-C\,,\\ &B_1x+\left[B_2+\frac{\lambda(\alpha^2+\beta^2+\gamma^2)}{b^2(\alpha a_{21}+\beta a_{22}+\gamma a_{23})}\right]y+B_3z=-C\,,\\ &B_1x+B_2y+\left[B_3+\frac{\lambda(\alpha^2+\beta^2+\gamma^2)}{c^2(\alpha a_{31}+\beta a_{32}+\gamma a_{33})}\right]z=-C\,,\end{aligned} \tag{58}$$

that is, a system of three linear equations with three unknowns $x$, $y$ and $z$.

Solving this system, we obtain the solution $x=x(\lambda)$, $y=y(\lambda)$ and $z=z(\lambda)$. Replacing this solution into (55) we get a second degree equation in $\lambda$ with the roots $\lambda_1$ and $\lambda_2$. For these roots we calculate the values $x_i$, $y_i$, $z_i$, $i=\overline{1,2}$, and then $X_i$, $Y_i$, $Z_i$, $i=\overline{1,2}$, with the aid of the formula (37). The next step consists in the calculation of

$$dist_i=\frac{|\alpha X_i+\beta Y_i+\gamma Z_i+\delta_i|}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\,,\ i=\overline{1,2}\,. \tag{59}$$

The following situations may occur:

a) both $dist_i>0$, $i=\overline{1,2}$. In this case there is no collision;

b) there exists a distance $d_i$ for which $dist_i=0$. The following sub cases have to be discussed (Fig. 3):

b.1) if $\mathbf{v}_P \cdot \mathbf{n} < 0$, then we have a collision at the point $P$ and we apply the formulae from section 2, for one point of collision,

b.2) if $\mathbf{v}_P \cdot \mathbf{n} \geq 0$, then there is no collision at the point $P$.

## 5. Simultaneous collision with two planes

We refer now to Fig. 4.

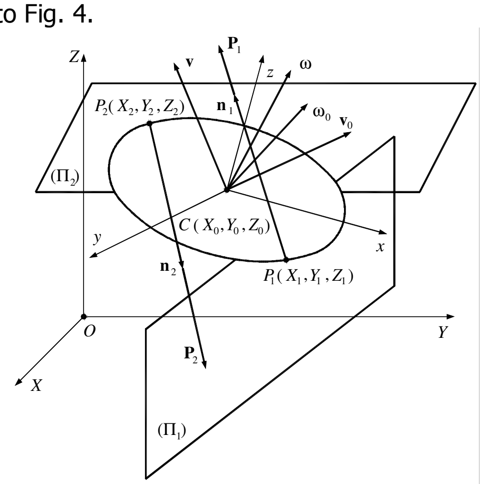


**Figure 4.** The collision between the ellipsoid and two planes.

The equations of the planes are

$$\alpha_i X + \beta_i Y + \gamma_i Z + \delta_i = 0\,,\ i = \overline{1,2}\,. \tag{60}$$

The procedure is similar to that described in paragraph 4, the only modification being given by the notations of $\alpha$, $\beta$ and $\gamma$ at which one added the index $i$.

We obtain now two systems (58), corresponding to the two contact points $P_1$ and $P_2$.

Let us denote the distances by adding a superior index that corresponds to the points.

We have the following situations:

a) all distances $dist_i^{(j)} > 0$, $i = \overline{1,2}$, $j = \overline{1,2}$. In this case there is no collision;

b) there is only one distance for which $dist_i^{(j)} = 0$. We have to consider the following sub cases:

b.1) if $\mathbf{v}_{P_j} \cdot \mathbf{n}_j < 0$, then we have a collision at the point $P_j$ and we apply the formulae presented in the paragraph 2 for one point of collision, b.2) if $\mathbf{v}_{P_j} \cdot \mathbf{n}_j \geq 0$, then there is no collision;

c) there exist the distances $dist_i^{(j)} = 0$ and $dist_k^{(l)} = 0$, $j \neq l$. Now we have the following sub cases:

c.1) if $\mathbf{v}_{P_j} \cdot \mathbf{n}_j < 0$ and $\mathbf{v}_{P_l} \cdot \mathbf{n}_l < 0$, then we have two points of collision and we apply the formulae from section 2 for two points of collision,

c.2) if $\mathbf{v}_{P_j} \cdot \mathbf{n}_j < 0$ and $\mathbf{v}_{P_l} \cdot \mathbf{n}_l \geq 0$, then there is only one collision at point $P_j$ and again we apply the formulae from paragraph 2 for one point of collision,

c.3) if $\mathbf{v}_{P_j} \cdot \mathbf{n}_j \geq 0$ and $\mathbf{v}_{P_l} \cdot \mathbf{n}_l \geq 0$, then there is no collision.

## 6. Example

Let us consider that the two planes are the $OXZ$ and $OYZ$ planes for which $n_1 = \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i}$.

In addition, the ellipsoid's equation is

$$\frac{x^2}{1^2} + \frac{y^2}{1^2} + \frac{z^2}{4} = 1, \tag{i}$$

that is, $a = 1\,\mathrm{m}$, $b = 1\,\mathrm{m}$, $c = 2\,\mathrm{m}$. The mass of ellipsoid is $m = 500\,\mathrm{kg}$, wherefrom it results that the central principal moments of inertia read $J_x = 200\,\mathrm{kgm}^2$, $J_y = 200\,\mathrm{kgm}^2$, $J_z = 100\,\mathrm{kgm}^2$.

The first Lagrange function is

$$F_1(x, y, z, \lambda) = Y + \lambda_1\left(x^2 + y^2 + \frac{z^2}{4} - 1\right) \tag{ii}$$

and we obtain the system

$$\frac{\partial F_1}{\partial x} = a_{21} + 2\lambda_1 x = 0,\ \frac{\partial F_1}{\partial y} = a_{22} + 2\lambda_1 y = 0,\ \frac{\partial F_1}{\partial z} = a_{23} + \frac{2\lambda_1 z}{4} = 0,$$
$$x^2 + y^2 + \frac{z^2}{4} - 1 = 0. \tag{iii}$$

It successively results

$$x = -\frac{a_{21}}{2\lambda_1},\ y = -\frac{a_{22}}{2\lambda_1},\ z = -\frac{2a_{23}}{\lambda_1}, \tag{iv}$$

$$\lambda_1 = \pm\frac{1}{2}\sqrt{a_{21}^2 + a_{22}^2 + 4a_{23}^2}, \tag{v}$$

$$Y = Y_0 \mp \frac{a_{21} + a_{22} + 4a_{23}}{\sqrt{a_{21}^2 + a_{22}^2 + 4a_{23}^2}}. \tag{vi}$$

Similarly, one gets

$$X = X_0 \mp \frac{a_{11} + a_{22} + 4a_{23}}{\sqrt{a_{11}^2 + a_{12}^2 + 4a_{13}^2}}. \tag{vii}$$

The simultaneous collisions appear when (assuming $X \geq 0$, $Y \geq 0$)

$$Y_0 = \left|\frac{a_{21} + a_{22} + 4a_{23}}{\sqrt{a_{21}^2 + a_{22}^2 + 4a_{23}^2}}\right|, \; X_0 = \left|\frac{a_{11} + a_{12} + 4a_{13}}{\sqrt{a_{11}^2 + a_{12}^2 + 4a_{13}^2}}\right|. \tag{viii}$$

We assume that the Bryan rotation angles are $\psi = 45^0$, $\theta = 30^0$, and $\varphi = 90^0$; we may write

$$[\boldsymbol{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \; [\boldsymbol{\theta}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}, \; [\boldsymbol{\varphi}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ix}$$

and it results

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}] = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \end{bmatrix}, \tag{x}$$

$$\begin{gathered} a_{11} = 0, \; a_{12} = -\frac{\sqrt{3}}{2}, \; a_{13} = \frac{1}{2}, \; a_{21} = \frac{\sqrt{2}}{2}, \; a_{22} = -\frac{\sqrt{2}}{4}, \; a_{23} = -\frac{\sqrt{6}}{4}, \\ a_{31} = \frac{\sqrt{2}}{2}, \; a_{32} = \frac{\sqrt{2}}{4}, \; a_{33} = \frac{\sqrt{6}}{4}. \end{gathered} \tag{xi}$$

For the simultaneous collision one needs

$$X_O = \frac{4 - \sqrt{3}}{\sqrt{19}}, \; Y_O = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{26}}. \tag{xii}$$

If we consider that there is no rotation, that is $\psi = 0^0$, $\theta = 0^0$, $\varphi = 0^0$, which is equivalent to state that the ellipsoid has only translational motion, then

$$[\boldsymbol{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; [\boldsymbol{\theta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; [\boldsymbol{\varphi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{xiii}$$

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{xiv}$$

$$a_{11} = 1,\ a_{12} = 0,\ a_{13} = 0,\ a_{21} = 0,\ a_{22} = 1,\ a_{23} = 0,\ a_{31} = 0,\ a_{32} = 0, \\ a_{33} = 1, \tag{xv}$$

$$Y_0 = 1,\ X_0 = 1, \tag{xvi}$$

which is the expected answer, the problem being very simple in this case.

## 7. Conclusion

The paper presents the general approach for the study of the multi-collision of an ellipsoid with several planar surfaces. The general theory is discussed using the theory of inertances, the authors obtaining the velocities after the collision. The presentation starts with the simple case of the collision of the ellipsoid with the horizontal plane and is developed to the collision of the ellipsoid with an oblique plane and, finally, we discussed the collision of the ellipsoid with two planar surfaces. Obviously, the discussion may be generalized for the collision of the ellipsoid with several planar surfaces.

The coefficients of restitution may be different from one planar surface to another and no restriction is imposed for them in this paper. It is known [19] that in the case of the collision without friction the three coefficients of restitution: Newton, Poisson and energetic are equals. The formulae presented in this paper are deduced using the Poisson coefficient of restitution.

In the situations considered in this paper, the motion of the rigid solid after the collision is perfectly determined. A critique case appear at the simultaneous collision with two parallel planes when the distribution of velocities before the collision is a particular one (e.g. the component of the velocity parallel to the two planes does not vanish). These cases are eliminated from our discussion. Some considerations to this problem may be found in [20], where the authors considered the collision with friction.

The question: with how many planar surfaces may simultaneous collide the ellipsoid will be the subject of our next paper.

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