APPLICATION OF BAYES FACTOR FOR ONE WAY ANALYSIS OF VARIANCE USING RANDOM EFFECT MODELS

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Abstract. Bayesian analysis of variance (ANOVA) is gaining acceptance as an alternative to the hypothesis testing based on p-values in many areas of scientific research. Its Bayes factors are constantly developed and modified to suit prevailing situations in hypothesis testing. We transformed the Bayes factor proposed by Wang & Sun (2013) for one random effects model using Natural Log of Gamma Function ($Log_e\ \Gamma(x)$) approximation. The result shows that our modified Bayes factor proved more efficient than the Wang & Sun (2013) Bayes factor in testing the hypothesis of zero between factor variability when the sample size becomes large. *Keywords:* Bayesian approach, Bayes factor, evidence, hypothesis testing, analysis of variance

Introduction

The Bayesian approach to testing a hypothesis about the variance component(s) is computed using the Bayes factor $BF_{01} = \frac{p(y|M_0)}{p(y|M_1)}$, which compares the marginal densities (also known as marginal likelihoods) of the data

$\boldsymbol{y}$under the two models,M_0 (one or more of the variance components is zero) and M_1 (variance unrestricted) suggested by the hypotheses. Analysis of Variance and Regression Analysis are still the most popular tests in many scientific researches; hence developing Bayes factor for these models is a necessary precursor for widespread adoption of the Bayesian method. Several Bayes factors have been proposed over time for student t test, Analysis of Variance(ANOVA) among others. Some of the proposed Bayes factors are Bayesian Information Criterion (BIC)-Based (Wagenmakers, 2007; Masson, 2011; Faulkenberry, 2018), while some other Bayes factors are P-value based (Sellke et al., 2001; Held & Ott, 2018).

Wang & Sun (2013) and Raftery (1995) considered Bayesian hypothesis testing for the balanced one-way random effects model. A special choice of the prior formulation for the ratio of variance components was shown to yield an explicit closed-form Bayes factor without integral representation which can be calculated easily using statistical packages. Furthermore, they studied the consistency issue of the resulting Bayes factor under three asymptotic scenarios: either the number of units goes to infinity, the number of observations per unit goes to infinity, or both go to infinity. Finally, Wang & Sun (2013) illustrated the behavior of the proposed approach using simulation studies.

Egburonu (2018) examined the Wang & Sun (2013) Bayes factor under two cases namely: *Case 1*: factor unit is fixed while observation per unit is increasing (i.e., random). *Case 2*: observation per unit is fixed while number of factor unit is increasing (i.e., random). In all the two cases, the Bayes factors was consistent in increasing the weight of evidence in support of the null hypothesis of zero between factor variability; but as the sample sizes became large, the Wang & Sun (2013) Bayes factor become impracticable. This impracticality situation was as a result of the Gamma function involved in its computational list.

Methodology

Wang & Sun Bayes factor for random effect model

Consider the balanced one-way analysis-of-variance (ANOVA) random effects model,

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad i = 1,2, \ldots, k \text{ and } j = 1,2, \ldots, m \tag{1}$$

where, y_{ij} is the j[th] observation associated with the unit i and μ represents the unknown intercept. Here $k\ (\geq 2)$ is the number of factor/treatment units and $m\ (\geq 2)$ is the number of observations per unit. It is assumed that the random effect (α_i) and the error term (ε_{ij}) are mutually independent, and that $\alpha_i \sim N(0, \sigma_\alpha^2)$ and $\varepsilon_{ij} \sim N(0, \sigma^2)$ for all i and j, The unknown parameters $(\sigma_\alpha^2 \text{ and } \sigma^2)$ are called variance components.

The Wang & Sun (2013) Bayes factor for obtaining the weight of evidence in support of the null hypothesis is given by

$$BF_{01} = \frac{\Gamma\left(\frac{mk-1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\frac{k}{2}+\alpha+\frac{1}{2}\right)\Gamma\left(\frac{mk-k}{2}\right)}\left(\frac{SSE}{SST}\right)^{\frac{(mk-k-2)}{2+\alpha}} \tag{2}$$

where

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{m}\left(y_{ij.} - \bar{y}_{i.}\right)^2 \tag{3}$$

is the sum of error square, and

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{m}\left(y_{ij} - \bar{y}_{..}\right)^2 \tag{4}$$

is the sum total squares.

Wang & Sun (2013) has established through simulation studies that the Bayes factor in Eq. (2) above is robust to a choice of hyper parameter $\alpha \in \left[-\frac{1}{2},\ 0\right]$.

Transformed Bayes factor

To resolve the problem of impracticability of the Wang & Sun (2013) Bayes factor for large sample sizes, we recommend an transformation of the Wang & Sun (2013) Bayes factor using the Carl Friedrich Gauss Natural Log of Gamma Function ($Log_e\ \Gamma(x)$) approximation.

Carl Friedrich Gauss Natural Log of Gamma ($Log_e\ \Gamma(x)$) approximation is given by:

$$Log_e\Gamma(X) = \left(X - \frac{1}{2}\right)Log_e X - X + \frac{1}{2}Log_e(2\pi) \qquad 0 < X < \infty \tag{5}$$

The transformed Bayes factor is given by:

$$Log_e(BF_{01}) = Log_e\left(\frac{\Gamma\left(\frac{mk-1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\frac{k}{2}+\alpha+\frac{1}{2}\right)\Gamma\left(\frac{mk-k}{2}\right)}\left(\frac{SSE}{SST}\right)^{\frac{(mk-k-2)}{2+\alpha}}\right) \tag{6}$$

Let us make the following transformations:

$$A = \Gamma\left(\frac{mk-1}{2}\right) \tag{7}$$

$$B = \Gamma(\alpha+1) \tag{8}$$

$$C = \Gamma\left(\frac{k}{2} + \alpha + \frac{1}{2}\right) \tag{9}$$

$$D = \Gamma\left(\frac{mk - k}{2}\right) \tag{10}$$

$$E = \left(\frac{SSE}{SST}\right) \tag{11}$$

$$F = \left(\frac{mk - k - 2}{2 + \alpha}\right) \tag{12}$$

Then Eq. (6) becomes,

$$Log_e(BF_{01}) = Log_e\left(\frac{AB}{CD} E^F\right) \tag{13}$$

.

$$\begin{aligned} &= Log_e\left(\frac{AB}{CD}\right) + Log_e(E^F) \\ &= Log_e(AB) - Log_e(CD) + F Log_e(E) \\ &= Log_e(A) + Log_e(B) - Log_e(C) - Log_e(D) + F Log_e(E) \end{aligned} \tag{14}$$

Substituting the values of A, B, C, D, E and F into Eq. (14), then we obtain:

$$Log_e(BF_{01}) = Log_e\left(\Gamma\left(\frac{mk-1}{2}\right)\right) + Log_e(\Gamma(\alpha+1)) - Log_e\left(\Gamma\left(\frac{k}{2} + \alpha + \frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{mk-k}{2}\right)\right) + \left(\frac{(mk-k-2)}{2+\alpha}\right) Log_e\left(\frac{SSE}{SST}\right). \tag{15}$$

where,

$$Log_e\Gamma(X) = \left(X - \frac{1}{2}\right)Log_e X - X + \frac{1}{2}Log_e(2\pi) \qquad 0 < X < \infty \tag{16}$$

Eq. (15) is the modified Bayes factor that will be used to carry out one-way ANOVA with random effects. The Bayes factor in Eq. (15) performs better than the Wang & Sun (2013) in terms of large sample sizes.

The proposed Bayes factor will be interpreted using the Table 1.

Table 1. Bayes factor interpretation

$Log_e(BF_{01})$	**Evidence in support of the null hypothesis**
0 to 1.10	Not worth more a mere mention
1.1 to 2.30	Substantial
2.30 to 4.61	Strong
> 4.61	Decisive

Data analysis

To illustrate the transformed Bayes factor, we used two extreme Cases in which the Wang & Sun (2013) Bayes factors were not practicable in the simulation studies carried out by Egburonu (2018).

Case 1: $\boldsymbol{k = 5 \ and \ m = 100}$

Case 2: $\boldsymbol{k = 35 \ and \ m = 20}$

Simulation study

Data sets were simulated using the native functions implemented in the R software for statistical computing (version 3.4.0 for Windows, R Core Team 2017) from a standard normal population $N(\mu = 0, \sigma = 1)$. Simulation was generated using random seed sets to simplify replication.

Case 1: $\boldsymbol{k = 5 \text{ and } m = 100}$

Hypothesis for Case 1: we seek to test the hypothesis:

$$H_0 : \sigma_\alpha^2 = 0 \quad \text{against} \quad H_1 : \sigma_\alpha^2 \neq 0 \tag{17}$$

Table 2. Simulated data for **CASE 1 ($\boldsymbol{k = 5 \text{ and } m = 100}$)**

Factors/Treatment Units (k)		Observations per factor (m)														
	A	-0.9	0.18	1.59	-1.13	-0.08	0.13	0.71	-0.24	1.98	-0.14	0.42	0.98	-0.39	-1.04	1.78
	B	1.07	0.26	-0.31	-0.75	-0.86	2.05	0.94	2.01	-0.42	-0.35	-1.03	-0.25	0.47	1.36	0.56
	C	0.3	-1.02	2.87	0.22	-0.97	0.38	-0.12	-0.35	0.6	0.23	1.03	-0.52	1.8	-1.43	0.14
	D	-0.32	-0.32	0.88	-1.89	0.73	0.79	0.19	0.92	1.24	-0.68	0.52	-0.47	-0.5	-1.97	-0.57
	E	-0.21	-2.72	-1.01	-0.83	0.86	-0.24	-0.7	-2.45	1.15	0	0.06	-1.11	0.2	-1.23	-0.13

-2.31	0.88	0.04	1.01	0.43	2.09	-1.2	1.59	1.95	0	-2.45	0.48	-0.6	0.79	0.29	0.74	0.32	1.08
0.46	1.23	1.15	0.11	-0.78	1.24	0.14	1.71	-0.43	-1.04	0.54	-0.67	0.64	-1.72	-1.74	0.69	0.33	0.87
0.45	1.21	-1.32	-1.14	1.68	0.4	0.72	-0.79	1.92	0.07	0.47	-0.09	0.89	-0.37	2.26	1.57	-1.95	-0.62
0.48	1.22	0.12	0.09	-0.2	-0.5	1.09	0.92	0.36	0.3	0.53	0.97	1.91	1.5	-0.31	0.95	-0.09	-0.19
-1.31	1.11	-1.01	1.58	0.29	0.74	-1.26	1.41	0.73	1.28	0.9	0.53	0.78	-1.19	0.35	-1.17	-0.34	-1.02

-0.28	-0.78	-0.6	-1.73	-0.9	-0.56	-0.25	-0.38	-1.96	-0.84	1.9	0.62	1.99	-0.31	-0.09	-0.18	-1.2	-0.84
-2.02	1.21	1.2	1.03	0.79	2.11	-1.45	-0.58	0.41	-0.81	0.09	0.75	-0.65	0.66	0.55	-0.81	-1	0.98
-0.03	-1.72	-1.05	-0.57	0.48	0.1	-0.59	-0.08	2.89	1.17	0.55	0.78	-0.57	0.1	0.28	-0.47	-0.46	0.5
0.73	1.03	-0.31	0.77	-0.02	1.49	0.45	0.17	0.39	-1.29	0.23	0.72	-0.25	-0.32	0.98	0.83	0.38	1.46
1.32	0.04	-0.08	-0.44	0.21	-0.01	-0.45	-2.1	-0.28	-1.57	0.3	0.25	-0.14	-0.71	1.07	-2.02	0.51	-0.61

2.07	-0.56	1.28	-1.05	-1.97	-0.32	0.94	1.14	1.67	-1.79	2.03	-0.7	0.16	0.51	-0.82	-2	-0.48	0.08
-0.17	0.72	-0.84	1.28	-1.34	0.77	0.46	0.27	0.67	0.4	-0.64	-0.27	0.36	-1.31	-0.88	2.08	-2.1	-1.24
-0.41	-0.18	-0.32	0.84	0.48	-0.3	-0.29	0.98	-0.17	1.34	-0.1	-0.32	-1.91	-0.65	-1.4	1.93	0.46	0.53
0.67	-0.37	1.55	2.33	1.4	-0.94	0.83	0.73	0.12	1.56	1.42	-0.36	-0.5	-1.88	-1.14	-0.46	1.17	-0.32
0.03	-0.25	-0.98	1.58	-0.27	-0.51	2.08	0.67	0.2	-0.04	0.12	-0.26	-0.67	0.94	0.49	0.44	-0.07	1.3

-0.9	-0.92	0.33	-0.14	0.43	-0.05	-0.91	1.3	0.77	1.05	-1.41	1	-1.7	-0.53	-1.37	-2.21	1.82	-0.65
0.99	1.09	0.84	0.06	0.32	-0.9	-0.65	-0.26	-0.93	0.82	-1.62	-1.03	-1.26	0.39	-1.13	0.54	1.18	0.03
-0.04	-2.72	-0.05	0	-1.23	0.25	1.61	-0.04	1.03	-0.4	0.07	2.04	0.65	1.28	-0.63	1.35	1.67	1.17
0.04	-0.18	2.28	-0.8	1.23	-0.03	-0.66	-0.25	2.53	0.16	0.75	-0.43	-0.74	0.14	-0.72	1.32	-1.57	0.05
-1.36	1.55	1.45	0.93	-1.31	0.28	-0.8	-0.38	-0.61	-2.43	-0.51	-1.07	0.14	-0.89	-0.08	1.6	-1.2	-1.66

-0.28	-0.39	0.39	1.6	1.68	-1.18	-1.36	-1.51	-1.25	1.96	0.01	-0.84	-0.6					
0.52	-0.65	0.5	-1.27	-0.08	-1.35	-0.27	1.09	0.7	-0.44	-0.79	-0.86	-0.75					
0.01	1.31	-0.09	-1.13	0.59	0.09	-0.23	1.49	-0.35	0.42	-2.1	-1.37	-0.68					
-0.03	1.41	0.83	0.18	1.34	1.2	0.87	-0.12	0.34	-0.99	1.13	0.23	0.92					
0.07	-0.95	-0.95	0.87	0.16	1.3	-1.08	0.39	-0.03	-1.26	-0.94	-0.2	0.14					

Source: Simulation result

$$SST = \sum_{i=1}^{5}\sum_{j=1}^{100}(y_{ij} - \bar{y}_{..})^2 = 531.7194$$

$$SSB = m\sum_{i=1}^{5}(\bar{y}_{i.} - \bar{y}_{..})^2 = 12.3304$$

$$SSE = SST - SSB = 519.3890$$

Wang & Sun Bayes factor for case 1 (k = 5 and m = 100)

The Wang and Sun (2013) Bayes factor for obtaining the weight of evidence in support of the null hypothesis for $(\boldsymbol{k = 5\ and\ m = 100})$at $\alpha = -\frac{1}{2}$ is computed as follows:

$$BF_{01} = \frac{\Gamma\left(\frac{mk-1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\frac{k}{2}+\alpha+\frac{1}{2}\right)\Gamma\left(\frac{mk-k}{2}\right)}\left(\frac{SSE}{SST}\right)^{\frac{(mk-k-2)}{2+\alpha}}$$

$$= \frac{\Gamma\left(\frac{500-1}{2}\right)\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(\frac{5}{2}+\left(-\frac{1}{2}\right)+\frac{1}{2}\right)\Gamma\left(\frac{500-5}{2}\right)}\left(\frac{519.3890}{531.7194}\right)^{\frac{(500-5-2)}{2+\left(-\frac{1}{2}\right)}} \qquad (18)$$

$=$ value out of range in Gamma function

The Bayes factor BF_{01}, for testing the null hypothesis under Case 1($\boldsymbol{k = 5\ and\ m = 100}$ signifies that the data is "out of the range of the Gamma function". By implication, its inverse BF_{10} is also "out of the range of the Gamma function".

Transformed Bayes factor for case 1 (k = 5 and m = 100)

The transformed Bayes factor for obtaining the weight of evidence in support of the null hypothesis for **($\boldsymbol{k = 5\ and\ m = 100}$)**at $\alpha = -\frac{1}{2}$ is computed as follows:

$$Log_e(BF_{01}) = Log_e\left(\Gamma\left(\frac{mk-1}{2}\right)\right) + Log_e(\Gamma(\alpha+1)) - Log_e\left(\Gamma\left(\frac{k}{2}+\alpha+\frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{mk-k}{2}\right)\right) + \left(\frac{(mk-k-2)}{2+\alpha}\right)Log_e\left(\frac{SSE}{SST}\right)$$

$$Log_e(BF_{01}) = Log_e\left(\Gamma\left(\frac{500-1}{2}\right)\right) + Log_e\left(\Gamma\left(-\frac{1}{2}+1\right)\right) - Log_e\left(\Gamma\left(\frac{5}{2}+(-\frac{1}{2})+\frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{500-5}{2}\right)\right) + \left(\frac{(500-5-2)}{2+(-\frac{1}{2})}\right)Log_e\left(\frac{519.3890}{531.7194}\right) \qquad (19)$$

$$= Log_e\left(\Gamma\left(\frac{499}{2}\right)\right) + Log_e\left(\Gamma\left(\frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{5}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{495}{2}\right)\right) + (-7.7114)$$

But,

$$Log_e\Gamma(X) = \left(X - \frac{1}{2}\right)Log_e X - X + \frac{1}{2}Log_e(2\pi) \qquad 0 < X < \infty \qquad (20)$$

Hence,

$$Log_e\,\Gamma\left(\frac{499}{2}\right) = \left(\frac{499}{2} - \frac{1}{2}\right)Log_e\left(\frac{499}{2}\right) - \frac{499}{2} + \frac{1}{2}Log_e(2\pi) = 1125.764$$

$$Log_e \Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - \frac{1}{2}\right) Log_e\left(\frac{1}{2}\right) - \frac{1}{2} + \frac{1}{2} Log_e(2\pi) = 0.4189$$

$$Log_e \Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2} - \frac{1}{2}\right) Log_e\left(\frac{5}{2}\right) - \frac{5}{2} + \frac{1}{2} Log_e(2\pi) = 0.2515$$

$$Log_e \Gamma\left(\frac{495}{2}\right) = \left(\frac{495}{2} - \frac{1}{2}\right) Log_e\left(\frac{495}{2}\right) - \frac{495}{2} + \frac{1}{2} Log_e(2\pi) = 1114.737 \quad (21)$$

Then

$$Log_e(BF_{01}) = 1125.764 + 0.4189 - 0.2515 - 1114.737 + (-7.7114)$$

$$= 3.483 \quad (22)$$

Table 3. Decision rule for proposed Bayes factor interpretation

$Log_e(BF_{01})$	**Evidence in support of the null hypothesis**
0 to 1.10	Not worth more a mere mention
1.1 to 2.30	Substantial
2.30 to 4.61	Strong
> 4.61	Decisive

The $Log_e(BF_{01}) = 3.483$ indicate strong evidence in support of the null hypothesis of no variability between the five treatments. ($H_0 : \sigma_\alpha^2 = 0 \; against H_1 : \sigma_\alpha^2 \neq 0$) stated in equation (3.1) under **Case 1** ($\boldsymbol{k = 5 \; and \; m = 100}$). This can be seen in **Table 3.** This is a more informative technique compared to the Wang & Sun (2013) Bayes factor.

Case 2 (k = 35 and m = 10)

Hypothesis for case 2: we seek to test the hypothesis:

$$H_0 : \sigma_\alpha^2 = 0 \quad \text{against} \quad H_1 : \sigma_\alpha^2 \neq 0 \tag{23}$$

Table 4. Simulated data for **CASE 2B (** $\boldsymbol{k = 35\ and\ m = 10}$ **)**

Factors/Treatment Units (k)		**Observations per factor (m)**									
	A	-0.9	0.18	1.59	-1.13	-0.08	0.13	0.71	-0.24	1.98	-0.14
	B	0.42	0.98	-0.39	-1.04	1.78	-2.31	0.88	0.04	1.01	0.43
	C	-0.38	-1.96	-0.84	1.9	0.62	1.99	-0.31	-0.09	-0.18	-1.2
	D	2.09	-1.2	1.59	1.95	0	-2.45	0.48	-0.6	0.79	0.29
	E	0.74	0.32	1.08	-0.28	-0.78	-0.6	-1.73	-0.9	-0.56	-0.25
	F	-0.84	2.07	-0.56	1.28	-1.05	-1.97	-0.32	0.94	1.14	1.67
	G	-1.79	2.03	-0.7	0.16	0.51	-0.82	-2	-0.48	0.08	-0.9
	H	-0.92	0.33	-0.14	0.43	-0.05	-0.91	1.3	0.77	1.05	-1.41
	I	1	-1.7	-0.53	-1.37	-2.21	1.82	-0.65	-0.28	-0.39	0.39
	J	1.6	1.68	-1.18	-1.36	-1.51	-1.25	1.96	0.01	-0.84	-0.6
	K	0.4	0.72	-0.79	1.92	0.07	0.47	-0.09	0.89	-0.37	2.26
	L	1.07	0.26	-0.31	-0.75	-0.86	2.05	0.94	2.01	-0.42	-0.35
	M	-1.03	-0.25	0.47	1.36	0.56	0.46	1.23	1.15	0.11	-0.78
	N	1.24	0.14	1.71	-0.43	-1.04	0.54	-0.67	0.64	-1.72	-1.74
	O	2.04	0.65	1.28	-0.63	1.35	1.67	1.17	0.01	1.31	-0.09
	P	-2.72	-0.05	0	-1.23	0.25	1.61	-0.04	1.03	-0.4	0.07
	Q	0.69	0.33	0.87	-2.02	1.21	1.2	1.03	0.79	2.11	-1.45
	R	-0.32	-0.32	0.88	-1.89	0.73	0.79	0.19	0.92	1.24	-0.68
	S	0.5	-0.41	-0.18	-0.32	0.84	0.48	-0.3	-0.29	0.98	-0.17
	T	-0.58	0.41	-0.81	0.09	0.75	-0.65	0.66	0.55	-0.81	-1
	U	0.98	-0.17	0.72	-0.84	1.28	-1.34	0.77	0.46	0.27	0.67

	V	-1.03	-1.26	0.39	-1.13	0.54	1.18	0.03	0.52	-0.65	0.5
	W	0.4	-0.64	-0.27	0.36	-1.31	-0.88	2.08	-2.1	-1.24	0.99
	X	1.09	0.84	0.06	0.32	-0.9	-0.65	-0.26	-0.93	0.82	-1.62
	Y	-1.27	-0.08	-1.35	-0.27	1.09	0.7	-0.44	-0.79	-0.86	-0.75
	Z	0.3	-1.02	2.87	0.22	-0.97	0.38	-0.12	-0.35	0.6	0.23
	AA	1.03	-0.52	1.8	-1.43	0.14	0.45	1.21	-1.32	-1.14	1.68
	BB	1.57	-1.95	-0.62	-0.03	-1.72	-1.05	-0.57	0.48	0.1	-0.59
	CC	-0.08	2.89	1.17	0.55	0.78	-0.57	0.1	0.28	-0.47	-0.46
	DD	1.34	-0.1	-0.32	-1.91	-0.65	-1.4	1.93	0.46	0.53	-0.04
	EE	-1.13	0.59	0.09	-0.23	1.49	-0.35	0.42	-2.1	-1.37	-0.68
	FF	0.52	-0.47	-0.5	-1.97	-0.57	0.48	1.22	0.12	0.09	-0.2
	GG	-0.5	1.09	0.92	0.36	0.3	0.53	0.97	1.91	1.5	-0.31
	II	0.17	0.39	-1.29	0.23	0.72	-0.25	-0.32	0.98	0.83	0.38
	JJ	0.95	-0.09	-0.19	0.73	1.03	-0.31	0.77	-0.02	1.49	0.45

$$SST = \sum_{i=1}^{35}\sum_{j=1}^{10}(y_{ij} - \bar{y}_{..})^2 = 374.8342$$

$$SSB = m\sum_{i=1}^{35}(\bar{y}_{i.} - \bar{y}_{..})^2 = 36.7433$$

$$SSE = SST - SSB = 338.0909$$

Wang & Sun Bayes factor for case 2E (k = 35 and m = 10)

The Wang and Sun (2013) Bayes factor for obtaining the weight of evidence in support of the null hypothesis for **($k = 35$ and $m = 10$)**at $\alpha = -\frac{1}{2}$ is computed as follows:

$$BF_{01} = \frac{\Gamma\left(\frac{mk-1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\frac{k}{2}+\alpha+\frac{1}{2}\right)\Gamma\left(\frac{mk-k}{2}\right)}\left(\frac{SSE}{SST}\right)^{\frac{(mk-k-2)}{2+\alpha}}$$

$$= \frac{\Gamma\left(\frac{350-1}{2}\right)\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(\frac{35}{2}+\left(-\frac{1}{2}\right)+\frac{1}{2}\right)\Gamma\left(\frac{350-35}{2}\right)}\left(\frac{338.0909}{374.8342}\right)^{\frac{(350-35-2)}{2+\left(-\frac{1}{2}\right)}} \quad (24)$$

$$= \text{undefined}$$

The Bayes factor BF_{01}, for testing the null hypothesis of no between treatment variability under **Case 2($k = 35$ and $m = 10$** signifies "**undefined**". Indicating that its inverse $BF_{10} = 0$.It means that there is entirely no evidence in support of the alternative hypothesis at this point.

Transformed Bayes factor for case 2 (k = 35 and m = 10)

The transformed Bayes factor for obtaining the weight of evidence in support of the null hypothesis for **case2 ($k = 35$ and $m = 10$)**at $\alpha = -\frac{1}{2}$ is computed as follows:

$$Log_e(BF_{01}) = Log_e\left(\Gamma\left(\frac{mk-1}{2}\right)\right) + Log_e(\Gamma(\alpha+1)) - Log_e\left(\Gamma\left(\frac{k}{2}+\alpha+\frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{mk-k}{2}\right)\right) + \left(\frac{(mk-k-2)}{2+\alpha}\right)Log_e\left(\frac{SSE}{SST}\right)$$

$$Log_e(BF_{01}) = Log_e\left(\Gamma\left(\frac{350-1}{2}\right)\right) + Log_e\left(\Gamma\left(-\frac{1}{2}+1\right)\right) - Log_e\left(\Gamma\left(\frac{35}{2}+(-\frac{1}{2})+\frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{350-35}{2}\right)\right) + \left(\frac{(350-35-2)}{2+(-\frac{1}{2})}\right)Log_e\left(\frac{338.0909}{374.8342}\right)$$

$$= Log_e\left(\Gamma\left(\frac{349}{2}\right)\right) + Log_e\left(\Gamma\left(\frac{1}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{35}{2}\right)\right) - Log_e\left(\Gamma\left(\frac{315}{2}\right)\right) + (-21.5279) \quad (25)$$

But,

$$Log_e\,\Gamma(X) = \left(X - \frac{1}{2}\right) Log_e X - X + \frac{1}{2} Log_e(2\pi) \qquad 0 < X < \infty \tag{26}$$

Hence,

$$Log_e\,\Gamma\left(\frac{349}{2}\right) = \left(\frac{349}{2} - \frac{1}{2}\right) Log_e\left(\frac{349}{2}\right) - \frac{349}{2} + \frac{1}{2} Log_e(2\pi) = 724.5938$$

$$Log_e\,\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - \frac{1}{2}\right) Log_e\left(\frac{1}{2}\right) - \frac{1}{2} + \frac{1}{2} Log_e(2\pi) = 0.4189 \tag{27}$$

$$Log_e\,\Gamma\left(\frac{35}{2}\right) = \left(\frac{35}{2} - \frac{1}{2}\right) Log_e\left(\frac{35}{2}\right) - \frac{35}{2} + \frac{1}{2} Log_e(2\pi) = 32.0764$$

$$Log_e\,\Gamma\left(\frac{315}{2}\right) = \left(\frac{315}{2} - \frac{1}{2}\right) Log_e\left(\frac{315}{2}\right) - \frac{315}{2} + \frac{1}{2} Log_e(2\pi) = 637.7487$$

Then

$$Log_e(BF_{01}) = 724.5938 + 0.4189 - 32.0764 - 637.7487 + (-21.5279)$$

$$= 33.6597 \tag{28}$$

The $Log_e(BF_{01}) = 33.6597$ indicates a decisive evidence in support of the null hypothesis of no between factor variability in the thirty-five factors. ($H_0 : \sigma_\alpha^2 = 0\ against H_1: \sigma_\alpha^2 \neq 0$) stated in equation (3.2) under **Case 2($k = 35$ and $m = 10$.** This can be seen in **Table 5.** This is a more informa-tivetechnique compared to the Wang & Sun (2013) especially for handling One Way ANOVA with random effects for large sample sizes.

Table 5. Decision rule for proposed Bayes factor interpretation

$Log_e(BF_{01})$	**Evidence in support of the null hypothesis**		
0 to 1.10	Not worth more a mere mention		
1.1 to 2.30	Substantial		
2.30 to 4.61	Strong		
> 4.61	Decisive		

Conclusion

The transformed Bayes factor has proved to be more efficient in testing the null hypothesis of zero between treatment variability for the factors/treatments been compared. In all the two cases (1 and 2), the transformed Bayes factor showed strong and decisive evidence in favour of the null hypothesis respectively. This result is contrary to the Wang and Sun (2013) that reported an "Out of Range of the Gamma function"; thereby leading to a loss of information with respect to the two cases. The modified Bayes factor provides smaller values than the Wang & Sun (2013) and Faulkenberry (2018) Bayes factors. Although computationally intricate, the proposed Bayes factor will give researchers an insight into the One Way ANOVA with random effects from a Bayesian perspective.

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