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# SPECTRAL PROPERTIES OF SELF-ADJOINT LINEAR OPERATORS

**GEZAHEGN ANBERBER TADESSE**

Lecturer, Dilla University,South Region, Ethiopia

## ABSTRACT

The spectral theory mainly deals with a systemic study of inverse operators, their general properties and their relation to the original operators.

Operators on Hilbert space is a basis for a comprehensive study of the spectral theory. In particular, the hermitian or self-adjoint operators are very important in many applications. In this peper, we have generalized this idea by showing that the spectrum of a self-adjoint operator on a Hilbert space also consists entirely of real values.

**KEYWORDS:** Spectral Theory, Self Adjoint, Hilbert Space, Inverse Operator, Projection

## 1. INTRODUCTION

While solving the system of linear algebraic equations, differential equations or integral equations, we come across the problem related to inverse operation. Spectral theory is concerned with such inverse problem.

In chapter one we discuss the preliminary concepts which are frequently used in the project work. In chapter Two we discuss the spectral theory of bounded self-adjoint linear operator and in chapter Three we discuss Projection operators. The definition of projection is suitable for spectral family. Projections are always positive operators.

In Chapter Four we discuss spectral family. The spectral family of a bounded self-adjoint linear operator at points of the resolvent set, at eigen values and at the point of continuous spectrum.

The spectral projection are the given, as in the bounded case, by defining $E_\lambda$ to be the orthogonal projection on the null space of $(T-\lambda I)^+$ for all real $\lambda$. If the two sets are vector spaces, we can introduce the concept of a linear operator, if the sets are normed spaces, we can construct a theory of bounded linear operators on such spaces. Operators that map members of a specified space into the real or complex numbers are called functional. We shall also discuss the Hilbert-adjoint operators as well as the self-adjoint, unitary and normal operators. Finally, we shall look at the spectral family of the Hermitian (self-adjoint) operators which is an important aspect of functional analysis.

## 2. PRELIMINARY CONCEPT

### 2.1 Basic Definition

**Norm**: Let $X$ be a vector space over a field $K$. Then a mapping $\|.\|: X \to [0,\infty)$ is said to be a norm on $X$ if for each $x, y \in X, \lambda \in K$

- $\|x\| \geq 0$ $and$ $\|x\| = 0$ $if\ and\ only\ if$ $x = 0$ $(positive\ definiteness)$
- $\|\lambda x\| = |\lambda|\|x\|$ $(Homogeneity)$
- $\|x+y\| \leq \|x\| + \|y\|$ $(\ triangle\ inequality)$.

If norm is defined on a vector space $X$, then $X$is said to be a **normed space.**

**Linear Functional:** A linear functional on Hilbert space $H$ is a linear map from $H\ to\ \mathbb{C}$. That is $\varphi: H \to \mathbb{C}$.

**Bounded Linear Functional:** A linear functional $\varphi$ is bounded or Continuous, if there exists a constant $k$ such that

$$|\varphi(x)| \leq k\|x\|$$

The norm of a bounded linear functional is

$$\|\varphi\| = sup_{\|x\|=1}|\varphi(x)|$$

If $y \in H, then\ \varphi_y(x) = \langle y, x\rangle$is a bounded linear functional on $H$, with $\|\varphi_y\| = \|y\|$

**Continuous Operator**

An operator $T$ from a normed space $V$ into another normed space $W$ is continuous at a point $x \in D(T)\ if\ for\ any\ \varepsilon > 0\ there\ is\ a\ \delta > 0, such\ that\ \|Tx - Ty\| < \varepsilon\ for\ all\ y \in D(T)\ whenever\ \|x - y\| < \delta$ then $T$ is continuous, if its continuous at all points of $D(T)$.

Let $X$ be a vector space over $K$ and let $\langle .,. \rangle: X \times X \to K$, where for all $x, y, z \in X$

- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle\ (linearity)$
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle\ (Homogeneity)$
- $\langle x, y\rangle = \langle \overline{y, x}\rangle\ (conjugate\ symmetry)$
- $\langle x, x\rangle \geq 0\ and\ \langle x, x\rangle = 0\ \Leftrightarrow x = 0\ (positive\ definiteness)$.

Then the pair $(X, \langle .,. \rangle)$ is said to be an **inner product space. Hilbert space** is a complete inner product space. If $T: X \to X$ is a linear operator on linear space $X$, then $\lambda \in K$ is called **eigen value** of $T$, if there exist an non- zero x$\in X$ then

$$Tx = \lambda x$$

**Resolvent Set**: Let $H$ be a Hilbert space and let $T: D(T) \to H$ be a linear operator with domain $D(T) \subseteq H$. For any $\lambda \in \mathbb{C}$, we define the operator $T_\lambda = \lambda I - T$, then $\lambda$ is said to be a **regular value** if $R_\lambda$ is inverse operator. That is $\lambda$is regular if provided,

- $R_\lambda$exists
- $R_\lambda$ is a bounded linear operator;
- $R_\lambda$ is defined on a dense subspace of $H$

The resolvent set of $T$ is the set of all regular values of $T$

$$\boldsymbol{\rho}(\boldsymbol{T})=\{\lambda \in \mathbb{C}: \lambda\ is\ a\ regular\ value\ of\ T\}$$

**Spectrum**. The set $\sigma(T) = \mathbb{C} - \boldsymbol{\rho}(\boldsymbol{T})$

The spectrum of an operator T is usually divided into three disjoint unions.

- Point spectrum $\sigma_p(T)$.
- Continuous spectrum$\sigma_c(T)$.
- Residual spectrum$\sigma_r(T)$.

Where

$\sigma_p(T) = \{\lambda \in \sigma(T): ker(\lambda I - T) \neq \{0\}\}$

$\sigma_c(T) = \{\lambda \in \sigma(T): (\lambda I - T)^{-1}\ is\ densely\ defined\ but\ not\ bounded\}$

$\sigma_r(T) = \{\lambda \in \sigma(T): R(\lambda I - T)\ is\ not\ dense\ in\ H\}$

Elements of $\sigma_p(T)$ are called eigenvalues.

**Orthogonality**

Two vectors $x\ and\ y$ in inner product space are said to be orthogonal if $\langle x, y\rangle = 0$.

**Orthonormality**

The set containing those vectors $x, y\ such\ that\ \langle x, y\rangle = 0\ and\ \|x\|$=1 is said to be orthonormal set.

**Orthogonal Complement**

For non-empty subset $S$ of Hilbert space we define the orthogonal complement to $S$, denoted by $S^{\perp}$.

$S^{\perp} = \{y \in H: \langle x, y\rangle = 0\ for\ all\ x \in S\}$

**Convergence of Sequence of Operator**

Let $X\ and\ Y$ be normed space. A Sequence $(T_n)\ of\ operators\ T_n \in B(X, Y)$ is said to be:

- **Uniformly Operator Convergent:** If $(T_n)$ converges in the norm of $B(X, Y)$ to$T \in B(X, Y)$

  **i.e** $\|T_n - T\| \to \mathbf{0}$

- **Strongly Operator Convergent:** If $(T_n x)\ converges\ strongly\ in\ Y\ for\ every\ x \in X$

  $\|T_n x - Tx\| \to \mathbf{0}$

**Unitary Operator**: A bounded linear operator $T: H \to H$ on Hilbert space H is said to be unitary if $T$ is bijective and

$T^* = T^{-1}$

**Normal Operator**: A bounded linear operator $T: H \to H$ on a Hilbert space is said to be normal

if $TT^* = T^*T$

**2.2. BASIC THEOREMS**

**Theorem (Cauchy-Schwarz Inequality)**:- If $X$ is inner product space then

$|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}}\langle y. y\rangle^{\frac{1}{2}}$

This equality holds if and only if $x\ and\ y\ are\ linearly\ depedent.$

**Proof. Continuity of Inner Product**-If in an inner product $x_n \rightarrow x$ and $y_n \rightarrow y$**.** Then $\langle x_n, y_n\rangle \rightarrow \langle x, y\rangle$

**Reiszre-Presentation Theorem**: Let $\varphi$ be a continuous linear functional on a Hilbert space$H$. Then there is a unique $z \in H$ such that $\varphi(x) = \langle x, z\rangle\ for\ all\ x \in H$

**Bounded Inverse Theorem**-Let $T: V \rightarrow W$ be a bounded, linear and bijective operator from Banach space $V$ into Banach space $W$ then the inverse operator $T^{-1}: W \rightarrow V$ is also bounded.

**Principle of Uniform Boundness**: Let$(T_n)$ be sequence of bounded, linear operator $T_n: X \rightarrow Y$ from a Banach space $\boldsymbol{X}$ into a normed space $\boldsymbol{Y}$ such that is bounded for every $x \in X, say$

$\|T_n x\| \leq C_N\ n = 1,2$

Where $C_N$ is real number, then the sequence of the norms $\|T_n\|$is bounded, that is there is $C$ a such that$\|T_n\| \leq C$

**Weierstress Approximation Theorem (Polynomial**) - The set of all polynomials W with real coefficient is dense in real space $C[a, b]$. Hence for every $x \in C[a, b]$ and given $\varepsilon > 0$ there exist a polynomial $P$ such that $|x(t) - P(t)| < \varepsilon\ for\ all\ t \in C[a, b]$

Proof in [1], [2] and [3]

# 3. SPECTRAL THEORY OF BOUNDED SELF-ADJOINT LINEAR OPERATORS

## 3.1. Spectral Properties of Bounded Self- Adjoint Linear Operators

### 3.1.1. Definition

Let$H_1 and\ H_2$ be complex Hilbert spaces and T:$H_1 \rightarrow H_2$ bounded linear operator. Then the **Hilbert-adjoint** operator $T^*: H_2 \rightarrow H_1$ defined to be the operator satisfying.

$\langle Tx, y\rangle = \langle x, T^* y\rangle\ for\ all\ x, y \in H$

### 3.1.1. Theorem

If $(H_1 = H_2 = H)$, then $T^*$ exists as bounded linear operator of norm

$\| T^* \| = \| T \|$ and is unique.

**Proof**. Let $H$ be complex Hilbert spaces and $T^*$ be operator we need to show:

- $T^*\ bounded.$
- $T^*\ linear\ operator$
- $T^*\ is\ unique.$

Take $y \epsilon\ H\ and\ T \in B(H)$. We define a bounded linear functional$X' on\ H\ by\ x'(x) = \langle Tx, y\rangle\ \forall x, y \in H \Rightarrow$ X' bounded since $|\ x'(x)\ | = |\ \langle Tx, y\rangle\ | \leq \|\ Tx\ \|\|\ y\ \|$.

By **Riesz-representation theorem** there is unique $z \in H\ such\ that\ x'(x) = \langle x, z\rangle\ for\ all\ x \in H$.here weobserve that $y$ and $z$ must have a certain relation so we can define an operator $T^*y = z$ satisfying$\langle Tx, y\rangle = \langle x, T^*y\rangle\ for\ all\ x \in H$.

Now let's see $\boldsymbol{T^*}$ **Linear**. Let $y_1, y_2 \in H\ and\ \alpha, \beta \in C, so\ \forall x \in H$

$\langle x, T^*(\alpha y_1 + \beta y_2)\rangle = \langle Tx, \alpha y_1 + \beta y_2\rangle = \langle Tx, \alpha y_1\rangle + \langle Tx, \beta y_2\rangle$

$= \overline{\alpha}\langle Tx, y_1\rangle + \overline{\beta}\langle Tx, y_2\rangle = \langle x, \alpha T^*y_1\rangle + \langle x, \beta T^*y_2\rangle = \langle x, \alpha T^*y_1 + \beta T^*y_2\rangle$

$\Rightarrow T^*(\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2$.

$\therefore\ \boldsymbol{T^*\ Linear\ operator}$.

We want show that the uniqueness of$\|\ \boldsymbol{T^*}\ \| = \|\ \boldsymbol{T}\ \|$**.** Let $y \in H$

$\|\ T^*y\ \|^2 = \langle T^*y, T^*y\rangle = \langle y, TT^*y\rangle \leq \|y\|\|TT^*y\| \leq \|y\|\|T\|\|T^*y\|$

$\Rightarrow \|T^*y\| \leq \|y\|\|T\|$, This shows that $T^*$ is bounded

$\Rightarrow \|T^*\| \leq \|T\|$ (1)

Similarly, let$x \in H$

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|x\|\|T^*Tx\| \leq \|x\|\|T^*\|\|Tx\| \Rightarrow \|Tx\| \leq \|x\|\|T^*\|$$

$\Rightarrow \|\ T\ \| \leq \|\ T^*\ \|$ (2)

Combining(1) and (2) we get

$\|\ T^*\ \| = \|\ T\ \|$. ■

**3.1.2. Definition**

A bounded linear operator $T: H \to H$ on acomplex Hilbert space H is said to be self-adjoint or hermitian, if $T = T^*$. Equivalently, a bounded linear operator $T$ is said to be self-adjoint, if $\langle Tx, y\rangle = \langle x, Ty\rangle\ \forall x, y \in H$.

**3.1.2. Theorem**

Let T:$H \to H$ be a bounded linear operator on a Hilbert space H. Then

- If T is self-adjoint, then $\langle Tx, x\rangle$ is real for all $x \in H$
- If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, then the operator T is self-adjoint

**Proof. 1.** If T is self-adjoint, then for all $x$

$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle$ (1)

By definition $\langle Tx, y\rangle = \langle x, T^*y\rangle$ and since T is self-adjoint, we have:

$\langle Tx, x\rangle = \langle x, Tx\rangle$ (2)

Combining equation (1) and (2) gives

$\overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$

Hence $\langle Tx, x\rangle$is equal to its complex conjugate which implies that it is real. If $\langle Tx, x\rangle$is real for all $x \in H$, then: $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = \langle T^*x, x\rangle$ . Hence $0 = \langle Tx, x\rangle - \langle T^*x, x\rangle = \langle (T - T^*)x, x\rangle$ . Thus, $T - T^* = 0$ . Therefore $T = T^*$

**Remark 1**: If $T$ Self-adjoint or unitary, then $T$ is normal; in general the converse is not true.

**Example 1**: If $I: H \to H$ is identity operator, then $T = 2iI$ is normal, since $T^* = -2iI$, so that $TT^* = T^*T = 4I$ but $T \neq T^*$ and $T^* \neq T^{-1} = \frac{-1}{2}iI$

**3.1.3. Theorem**

Let $T: H \to H$ be a bounded self-adjoint linear operator on complex Hilbert space H. Then:

- All the eigenvalue of $T$ are real.
- Eigenvectors corresponding to distinct eigenvalue of $T$ are mutually orthogonal.

**Proof. a):** Let $\lambda$ be any eigenvalue of $T$ and $x$ a corresponding eigenvectors. Then $x \neq 0$ & $Tx = \lambda x \Longrightarrow \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx,x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle$, using the self-adjointness of T.

$\Rightarrow \lambda\langle x, x\rangle - \overline{\lambda}\langle x, x\rangle = 0$

$\Longrightarrow (\lambda - \overline{\lambda})\langle x, x\rangle = 0$. Here $\langle x, x\rangle = \|x\|^2 \neq 0$

$\therefore \lambda = \overline{\lambda}$

- Let $\lambda$ & $\mu$ be eigenvalues of T and let $x \; and \; y$ be corresponding eigenvectors. Then $Tx = \lambda x$ and $Ty = \mu y$ since T is self-adjiont and $\mu$ real.

$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx,y\rangle = \langle x, T^*y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle$

$\Rightarrow \lambda\langle x, y\rangle - \mu\langle x, y\rangle = 0$

$\Longrightarrow (\lambda - \mu)\langle x, y\rangle = 0, \lambda \neq \mu$

$\therefore \langle x, y\rangle = 0 \; i.e \; x \perp y$

**Example 4:** $T = \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}$

Spectrum $of$ $(\lambda) = \{0,3\}$ distinct real numbers with eigenvectors.

$x_1 = \begin{bmatrix} -1+i \\ 2 \end{bmatrix} \; and \; x_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \; respectively$

Finally $\langle x_1, x_2\rangle = {x_1}^T\overline{x_2} = [-1 + i \;\, 2]\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = 0$. So the eigenvectors are orthogonal.

**3.1.4. Theorem**

Let $T: H \to H$ be a bounded self –adjoint linear operator on a complex Hilbert space$H$. Then a number $\lambda$ belongs to the resolvent set $\rho(T)$ of $T$ if and only if there exists a $c > 0$ such that for ever $x \epsilon H$

$\|T_\lambda x\| \geq c\|x\|\ T_\lambda = T - \lambda I$

**Proof:** $\Longrightarrow$) Suppose that $\lambda\epsilon\ \rho(T)$, $R_\lambda = T^{-1}{}_\lambda : H \to H$ exists and is bounded set $\|R_\lambda\| = k$ where $k > 0$, since $R_\lambda \neq 0$ now I=$R_\lambda T_\lambda$, so that for every $x \in H$

$\Longrightarrow T_\lambda x = y, \Rightarrow x = T^{-1}{}_\lambda$y, since $R_\lambda = T^{-1}{}_\lambda$

$\Longrightarrow x = R_\lambda T_\lambda x$

$\Longrightarrow \|x\| = \|R_\lambda T_\lambda x\| \leq \|R_\lambda\|\|T_\lambda x\| = k\|T_\lambda x\|$

$\Longrightarrow \|x\| \leq k\|T_\lambda x\|$

$\Longrightarrow \|T_\lambda x\| \geq \frac{1}{k}\|x\|$, where $c = \frac{1}{k}$

$\therefore\ \|T_\lambda x\| \geq c\|x\|$

**3.1.5. Theorem**

The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space $H$ is real.

**Proof:** By theorem 2.1.5 we show that a $\lambda = \alpha + i\beta\ (\alpha, \beta\ real)$ with $\beta \neq 0$ which belongs to $\rho(T)$ so that $\sigma(T) \subset \mathbb{R}$.

$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle\ for\ all\ x \in H$ (1)

Since $\langle Tx, x\rangle\ and\ \langle x, x\rangle\ are\ real$

$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \bar{\lambda}\langle x, x\rangle, here\ \bar{\lambda} = \alpha - i\beta$ (2)

Subtract equation (2) from (1)

$\overline{\langle T_\lambda x, x\rangle}$ - $\langle T_\lambda x, x\rangle = (\lambda - \bar{\lambda})\langle x, x\rangle = 2i\beta\|x\|^2$

So that

-2iIm$\langle T_\lambda x, x\rangle = 2i\beta\|x\|^2$

-Im$\langle T_\lambda x, x\rangle = \beta\|x\|^2$

$|\beta|\|x\|^2 = |\mathrm{Im}\langle T_\lambda x, x\rangle| \leq \|T_\lambda x\|\|x\|$

$|\beta|\|x\| \leq \|T_\lambda x\|, for\ \|x\| \neq 0.$

If $\beta \neq 0, then\ \lambda \in\ \rho(T)$. Hence for $\lambda \in \sigma(T)$ we must have $\beta = 0$. i.e $\lambda$ is real

**3.2 Further Spectral Properties of Bounded Self-Adjoint Linear Operators**

### 3.2.1 Theorem

The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H lies in the closed interval [m,M] on the real axis, where

$$\boldsymbol{m = inf_{\|x\|=1}\langle Tx, x\rangle \; M = sup_{\|x\|=1}\langle Tx, x\rangle}$$

**Proof:** $\sigma(T)$ lies on the real axis. We show that any real $\lambda = M + c \; with \; c > 0$ belongs to the resolvent set $\rho(T)$. For every $x \neq 0 \; and \; v = \|x\|^{-1}x \; we \; have \; x = \|x\|v \; and$

$$\langle Tx, x\rangle = \|x\|^2\langle Tv, v\rangle \leq \|x\|^2 sup_{\|\tilde{v}\|=1}\langle T\tilde{v}, \tilde{v}\rangle = \langle x, x\rangle M$$

Hence $-\langle Tx, x\rangle \geq -\langle x, x\rangle M$ and by Cauchy-Schwartz inequality we obtain

$$\|T_\lambda x\|\|x\| \geq -\langle T_\lambda x, x\rangle = -\langle Tx, x\rangle + \lambda\langle x, x\rangle = -\langle x, x\rangle M + \lambda\langle x, x\rangle$$

$$\geq (-M + \lambda)\langle x, x\rangle = C\|x\|^2$$

$$\Longrightarrow \|T_\lambda x\|\|x\| \geq C\|x\|^2, \|x\| \neq 0$$

$$\|T_\lambda x\| \geq C\|x\|$$

$$\therefore \lambda \in \rho(T)$$

### 3.2.2 Theorem

For any bounded self-adjoint linear operator T on a complex Hilbert space H we have:

$$\|T\| = max(|m|, |M|) = sup_{\|x\|=1}|\langle Tx, x\rangle|$$

**Proof.** By the Cauchy-Schwarz inequality

$sup_{\|x\|=1}|\langle Tx, x\rangle| \leq sup_{\|x\|=1}\|Tx\|\|x\| = \|T\|$, where$K = sup_{\|x\|=1}|\langle Tx, x\rangle|$. Hence, $K \leq \|T\|$

We want show $\|T\| \leq K$.If$Tz = 0$, then$0 \leq K\|z\|^2 \Longrightarrow \|Tz\| \leq K \; \forall z \in H, \|z\| = 1$. Otherwise $Tz \neq 0$ for any $z$ of norm 1. $v = \|Tz\|^{\frac{1}{2}}z \; and \; w = \|Tz\|^{-\frac{1}{2}}Tz$. Now set $y_1$=v+w and $y_2 = v - w$. T is self-adjoint

$$\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle = 2(\langle Tv, w\rangle + \langle Tw, v\rangle)$$

$$= 2(\langle Tz, Tz\rangle + \langle T^2z, z\rangle)$$

$$= 2(\langle Tz, Tz\rangle + \langle Tz, Tz\rangle)$$

$$= 2(2\langle Tz, Tz\rangle) = 4\langle Tz, Tz\rangle = 4\|Tz\|^2 \quad (1)$$

Now for every $y \neq 0 \; and \; x = \|y\|^{-1}y \; we \; have \; y = \|y\|x \; and$

$|\langle Ty, y\rangle| = \|y\|^2|\langle Tx, x\rangle| \leq \|y\|^2 sup_{\|\tilde{x}\|=1}|\langle T\tilde{x}, \tilde{x}\rangle| = K\|y\|^2$, so that by the triangle inequality.

$$|\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle| \leq |\langle Ty_1, y_1\rangle| + |\langle Ty_2, y_2\rangle|$$

$$\leq K(\|y_1\|^2 + \|y_2\|^2) \leq 2K(\|v\|^2 + \|w\|^2) = 4K\|Tz\| \quad (2)$$

Combining (1) and (2) we get

$4\|Tz\|^2 \leq 4K\|Tz\|, \|Tz\| \neq 0$

$\|Tz\| \leq K$ *by taking* supremum *over all z of norm* $1. \|T\| \leq K$ *and* $K \leq \|T\|$

$\|T\| = K = max(|m|, |M|) = sup_{\|x\|=1}|\langle Tx, x\rangle|$ ■

**3.2.3 Theorem.**

The residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space $H$ is empty

**Proof.** Suppose not$\sigma_r(T) \neq \emptyset$. Let $\lambda \in \sigma_r(T)$. By definition of $\sigma_r(T)$, the inverse of $T_\lambda$ exists but its domain $\mathfrak{D}(T^{-1}{}_\lambda)$ is not dense in$H$. Hence, by the projection theorem there is a $y \neq 0\epsilon H$ which is orthogonal to$\mathfrak{D}(T^{-1}{}_\lambda)$, but $\mathfrak{D}(T^{-1}{}_\lambda)$is a range of $T_\lambda$, hence, $\langle T_\lambda x, y\rangle = 0, \forall\, x \in H$. since $\lambda$ is real and $T$ self-adjoint, we obtain $\langle x, T_\lambda y\rangle = 0\ \forall\, x$. Taking $x = T_\lambda y$ we get$\|T_\lambda y\|^2 = 0$, so that$T_\lambda y = Ty - \lambda y = 0, since\ y \neq 0$. This shows that $\lambda$ is an eigenvalue of$T$, but contradicts$\lambda \in \sigma_r(T)$. Therefore $(T) = \emptyset$ ■

**3.3 Positive Operators**

In this section we can see the definition of **partial order** is a binary relation "≤" over a set $P$ which is reflexive, antisymmetric, and transitive i.e$\forall a, b, c \in P$. We have that:

- $a \leq a$ ( Reflexive)
- $if\ a \leq b\ and\ b \leq a\ then\ a = b$ (Antisymmetrive)
- $if\ a \leq b\ and\ b \leq c\ then\ a \leq c$ (Transitive)

**3.3.1. Definition**.

Abounded linear operator $T: H \to H$ is called a positive operator if and only if $T$ self-adjoint and$\langle Tx, x\rangle \geqq 0\ for\ all\ x \in H$. A bounded linear operator $T: H \to H$ is said to be **positive,** written

$T \geqq 0\ if\ and\ only\ if\ \langle Tx, x\rangle \geqq 0\ for\ all\ x \in H$

**Remark 2**: The sum of positive operators is positive. Every positive operator on a complex Hilbert space is self-adjoint.

The following theorem is like the familiar statement about real numbers which says that when you multiply two non-negative real numbers the result is a non-negative real numbers.

**3.3.1 Theorem**

Let $S$ and $T$ be a positive and self-adjoint such that $ST = TS$. Then their product $ST$ is also self-adjoint and positive.

**Proof.** Since $(ST)^* = T^*S^* = TS$ we see that $ST$ is self-adjoint if and only if $ST = TS$. We show that $ST$ is positive. This is true for $n = 0$. Assume it is true for any $n$. We consider

$S_1 = \frac{1}{\|S\|}S, S_{n+1} = S_n - {S_n}^2\ (n = 1,2,3 \ldots \ldots..)$ (*)

and prove by induction such that $0 \leqq S_n \leqq I$ (**)

- For $n = 1$ the inequality $(**)$ holds. Indeed, the assumption $0 \leqq S$ *implies* $S_1 \leqq I$is obtained by application of the Schwarz inequality $\|Sx\| \leqq \|s\|\|x\|$

  $\langle S_1 x, x\rangle = \frac{1}{\|S\|}\langle Sx, x\rangle \leqq \frac{1}{\|S\|}\|sx\|\|x\| \leqq \|x\|^2 = \langle Ix, x\rangle$

  $\Rightarrow S_1 \leqq I.$

  Suppose $(**)$ holds for any $= k$ *that is* $0 \leqq S_k \leqq I. Thus, 0 \leqq I - S_k \leqq I.$

  Since $S_k$ is self-adjoint, for every$x \in H \;\&\; y = S_k x$, we obtain

  $\langle {S_k}^2 (I - S_k)x, x\rangle = \langle (I - S_k)S_k x, S_k x\rangle = \langle (I - S_k)y, y\rangle \geqq 0.$

  By definition this proves ${S_k}^2 (I - S_k) \geqq 0$. Similarly, $S_k (I - S_k)^2 \geqq 0$.

  Since $S_k$ is self-adjiont. It is clear that from remark (2) sum of positive operator is positive.

  $0 \leqq {S_k}^2 (I - S_k) + S_k (I - S_k)^2 = S_k - {S_k}^2 = S_{k+1}.$

  Hence $0 \leqq S_{k+1}$ *and* $S_{k+1} \leqq I$ *follows from* ${S_k}^2 \geqq 0$ *and* $I - S_k \geqq 0$

- We now show that $\langle STx, x\rangle \geqq 0$ *for all* $x \in H$. Form $(**)$ we obtain successively

  $S_1 = {S_1}^2 + S_2, S_2 = {S_2}^2 + S_3, S_3 = {S_3}^2 + S_4, S_4 = {S_1}^2 + {S_2}^2 + {S_3}^2 + S_4, \cdots,$

  $S_n = {S_1}^2 + {S_2}^2 + {S_3}^2 + {S_4}^2 + \cdots + {S_n}^2 + S_{n+1}$

  Since $S_{n+1} \geqq 0$ *this implies*

  ${S_1}^2 + {S_2}^2 + {S_3}^2 + {S_4}^2 + \cdots + {S_n}^2 = S_1 - S_{n+1} \leqq S_1$

  By the definition of $\leqq$ and self-adjiontness of $S_j{}'s$

  $\sum_{j=1}^{n} \|S_j x\|^2 = \sum_{j=1}^{n} \langle S_j x, S_j x\rangle = \sum_{j=1}^{n} \langle {S_j}^2 x, x\rangle \leqq \langle S_1 x, x\rangle$

  Since $n$ is arbitrary, the infinite series $\|S_1 x\|^2 + \|S_2 x\|^2 + \cdots$

  $\lim_{n\to\infty} \sum_{j=1}^{n} \|S_j x\|^2 = \|S_1 x\|^2$. Hence $\lim_{n\to\infty} S_n x = Sx.\left(\sum_{j=1}^{n} {S_j}^2\right)x = (S_1 - S_{n+1})x \to S_1 x$

  $\lim_{n\to\infty}\left(\sum_{j=1}^{n} {S_j}^2\right) x = \lim_{n\to\infty}(S_1 - S_{n+1})x \to S_1 x.$

  All the $S_j{}'s$ commute with $T$ since they are sums and product of $S_1 = \|S\|^{-1}S$, $S$ and $T$ commute. Using $S = \|S\|S_1, \left(\sum_{j=1}^{n} {S_j}^2\right)x = (S_1 - S_{n+1})x \to S_1 x, T \geqq O$ and the continutity of inner product, we thus obtain for every $x \in H$ *and* $y_j = S_j x$ $\langle STx, x\rangle = \|S\|\langle TS_1 x, x\rangle = \|S\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle T{S_j}^2 x, x\rangle = \|S\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle Ty_j, y_j\rangle \geqq 0$

  $\therefore \langle STx, x\rangle \geqq 0$, then $ST$ are positive. ■

**3.3.2. Definition**

A monotone sequence $(T_n)$ of self-adjoint linear operators $T_n$on Hilbert space H is a sequence $(T_n)$ which is either monotone increasing, that is,

$T_1 \leqq T_2 \leqq T_3 \leqq \cdots$

Or monotone decreasing, that is,

$T_1 \geqq T_2 \geqq T_3 \geqq \cdots$

The following theorem is a generalization of the familiar fact that if you have an increasing sequence of real numbers which is bounded above, then the sequence converges

**3.3.2. Theorem**

Let$(T_n)$ be a sequence of bounded self-adjoint linear operators on a complex Hilbert space $H$ such that

$T_1 \leqq T_2 \leqq T_3 \leqq \cdots T_n \leqq \cdots \leqq K\ (*)$.

Where $K$ is bounded self-adjointlinear operator on $H$. Suppose that any $T_j$ commutes with $K$ and with every $T_m$. Then $(T_n)$ is strongly operator convergent $(T_n x \to Tx\ for\ all\ x \in H)$ and limit operator $T$ is linear, bounded and self-adjoint and satisfies$T \leqq K$.

**Proof.** We consider $S_n = K - T_n$. The sequence $\left(\langle {S_n}^2 x, x\rangle\right)$ converges $\forall x \in H$;$T_n x \to Tx$, where $T$ is linear, self-adjoint and bounded by uniform boundnees theorem $\langle {S_n}^2 x, x\rangle = \langle S_n x, S_n x\rangle = \langle x, {S_n}^2 x\rangle$**.** Therefore, $S_n$ is self-adjoint? ${S_m}^2 - S_n S_m = (S_m - S_n)S_m = (T_n - T_m)(k - T_m)$. Let $m < n$. Then $T_n - T_m$ and $K - T_m$ are positive by $(*)$, since these operators commute, their product is positive. Hence on the left ${S_m}^2 - S_n S_m \geqq 0, that\ is\ {S_m}^2 \geqq S_n S_m\ for\ m < n.$

Similarly $S_n S_m - {S_n}^2 = S_n(S_m - S_n) = (k - T_n)(T_n - T_m) \geqq 0$. So that $S_n S_m \geqq {S_n}^2$, together ${S_m}^2 \geqq S_n S_m \geqq {S_n}^2, m < n$. By definition, using the self-adjointness of $S_n$

$\langle {S_m}^2 x, x\rangle \geqq \langle S_n S_m x, x\rangle \geqq \langle {S_n}^2 x, x\rangle = \langle S_n x, S_n x\rangle = \|S_n x\|^2 \geqq 0.$ (**)

This show that $\left(\langle {S_n}^2 x, x\rangle\right)$ with fixed x is a monotone decreasing sequence of non-negative numbers. Hence, $\left(\langle {S_n}^2 x, x\rangle\right)$ converges $\forall x \in H$.

We show that $T_n x \to Tx$ is converges. By assumption, every $T_n$ commute withevery $T_m$ and with K. Hence the $S_j's$ all commute. These operators are self-adjoint, since $-2\langle S_n S_m x, x\rangle \leqq -2\langle {S_n}^2 x, x\rangle\ by\ (**)\ where\ m < n$

$\|S_m x - S_n x\|^2 = \langle (S_m - S_n)x, (S_m - S_n)x\rangle$

$= \langle (S_m - S_n)^2 x, x\rangle$

$= \langle {S_m}^2 x, x\rangle - 2\langle S_n S_m x, x\rangle + \langle {S_n}^2 x, x\rangle$

$\leqq \langle {S_m}^2 x, x\rangle - \langle {S_n}^2 x, x\rangle$.

Since $(S_n x)$ is Cauchy sequence in $H$ is completes. We show that $T$ is self-adjoint because $T_n$ is self-adjoint and the inner product is continuous.

$\langle Tx, x\rangle = \lim_{n\to\infty}\langle T_n x, x\rangle = \lim_{n\to\infty}\langle x, T_n x\rangle = \langle x, Tx\rangle$.

We show that $T$ is bounded. Now we must show that $T$ is bounded and $(T_n x)$ converges to it $sup_n \|T_n x\|$ must be finite for every $x$, there is $M$ such that $M > 0$.

$\Rightarrow \|T_n\| \leq M, \forall n$

$\Rightarrow \|T_n x\| \leq M\|x\|$

$\Rightarrow \lim_{n\to\infty} \|T_n x\| \leq M\|x\|$ *this implies* $\|Tx\| \leq M\|x\|$.

Hence $T$ is bounded, by uniform boundness theorem. Finally $\langle Tx, x\rangle = \lim_{n\to\infty}\langle T_n x, x\rangle \leqq \langle Kx, x\rangle$. Therefore $T \leqq K$ ■

### 3.4. Square Roots of Positive Operators

#### 3.4.1. Definition

Let $T: H \to H$ be positive bounded self-adjoint linear operator on complex Hilbert space H. Then a bounded self-adjoint operator $A$ is called a square root of $T$ if, $A^2 = T$. In addition, $A \geqq 0$, then $A$ is called a positive square root of $T$ and is denoted by $A = T^{\frac{1}{2}}$ exists and unique.

#### 3.4.1. Theorem

Every positive bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space $H$ has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on $H$ which commutes with$T$.

**Proof. a)** We show that if the theorem holds under the additional assumption $T \leqq I$ it also holds without that assumption. If $T = 0$, we can take $A = T^{\frac{1}{2}} = 0$. Let $T \neq 0$ by Cauchy-Schwarz inequality.

$\langle Tx, x\rangle \leqq \|Tx\|\|x\| \leqq \|T\|\|x\|^2 = \langle Ix, x\rangle$

$\langle \frac{T}{\|T\|} x, x\rangle = \langle Ix, x\rangle, since\ \|T\| \neq 0\ set\ Q = \left(\frac{1}{\|T\|}\right)$, We obtain

$\langle Qx, x\rangle \leqq \|x\|^2 = \langle Ix, x\rangle\ that\ is\ Q \leqq I.$

Assuming that $Q$ has a unique positive square root. $B = Q^{\frac{1}{2}}$ we have $B^2 = Q$ and $T = \|T\|Q \Rightarrow T^{\frac{1}{2}} = \|T\|^{\frac{1}{2}} Q^{\frac{1}{2}} = T^{\frac{1}{2}} B = \|T\|B^2 = \|T\|Q = T$. Hence if we prove the theorem under the additional assumption $T \leqq I$.

- We obtain the existence of the operator $A = T^{\frac{1}{2}}\ from\ A_n x \to Ax\ where\ A_0 = 0$ &

  $A_{n+1} = A_n + \frac{1}{2}\left(T - {A_n}^2\right)\ n = 0,1,2, \dots\ (i)$

  We consider$(i)$, since $A_0 = 0, we\ have\ A_1 = \frac{1}{2}T, A_2 = T - \frac{1}{8}T^2 \dots$.,

  $A_{n+1} = A_n + \frac{1}{2}\left(T - {A_n}^2\right)\ n = 0,1,2, \dots$

  Each $A_n$is polynomial in$T$, and they also commutes with $T$. We now prove

  $A_n \leqq I\ n = 0,1,2, \dots$ *(ii)*

$A_n \leqq A_{n+1}\; n = 0,1,2, \dots$ *(iii)*

$A_n x \to Ax, A = T^{\frac{1}{2}}$ *(iv)*

$ST = TS, \Rightarrow AS = SA$ *(v)*

Where $S$ is a bounded linear operator on $H$.

**Proof** $(\boldsymbol{ii})$**.** This is true for $n = 0$. Assume true for $n$. Since $I - A_{n-1}$is self-adjoint

$(I - A_{n-1})^2 \geqq 0$. Also $T \leqq I$. This implies $I - T \geqq 0$. From this $(i)$and we obtain$(ii)$.

$0 \leqq \frac{1}{2}(I - A_{n-1})^2 + \frac{1}{2}(I - T) = I - A_{n-1} - \frac{1}{2}\left(T - {A_{n-1}}^2\right) = I - A_n.$

$\Rightarrow\; 0 \leqq I - A_n.$

**Proof** $(\boldsymbol{iii})$**.** We use induction $(i)$ gives $0 = A_0 \leqq A_1 = \frac{1}{2}T$

We show that $A_{n-1} \leqq A_n$ for fixed $n$ implies $A_n \leqq A_{n+1}$ from $(i)$ we have;

$A_{n+1} - A_n = A_n + \frac{1}{2}\left(T - {A_n}^2\right) - A_{n-1} - \frac{1}{2}\left(T - {A_{n-1}}^2\right)$

$= A_n - \frac{1}{2}{A_n}^2 - A_{n-1} + \frac{1}{2}{A_{n-1}}^2$

$= A_n - A_{n-1} - \frac{1}{2}{A_n}^2 + \frac{1}{2}{A_{n-1}}^2$

$\geqq (A_n - A_{n-1})[\mathrm{I}-\frac{1}{2}(A_n + A_{n-1})]$

$= (A_n - A_{n-1})I - \frac{1}{2}(2I)] = 0.$

Hence $A_n - A_{n-1} \geqq 0$ and $(I - \frac{1}{2}(A_n + A_{n-1}) \geqq 0.$

Therefore $A_n \leqq A_{n+1}\; n = 0,1,2, \dots.$

**Proof** $(\boldsymbol{iv})$**.**$(A_n)$ Monotone increasing by $(iii)$ and $A_n \leqq I$ by$(ii)$. Hence (by the theorem monotone sequence) implies the existence of a bounded self-adjoint linear operator $A$. Such that $A_n x \to Ax\; for\; all\; x \in H$, since $(A_n x)$ converges $(i)$ gives

$A_{n+1}x - A_n x = \frac{1}{2}\left(Tx - {A_n}^2 x\right) \to 0\; as\; n \to \infty$

$Tx - A^2 x = 0\; \forall x.$

$Tx = A^2 x, x \neq 0$. Hence $T = A^2$. Also $A \geqq 0$. Because $0 = A_0 \leqq A_n$ by $(iii)$

$\Rightarrow \lim_{n\to\infty}\langle A_n x, x\rangle = \lim_{n\to\infty}\langle x, A_n x\rangle = \langle x, Ax\rangle = \langle Ax, x\rangle \geqq 0$, by continuity of inner product

**Proof**$(\boldsymbol{v})$**.** Let $S \in L(H,H)$ be any linear operator that commute with $A. SA_n = A_n S, \forall n$

Since$A_n \to A,\; A_n Sx \to ASx$. Using continuity of inner product of$S$,

$\lim_{n\to\infty} SA_n x = S\lim_{n\to\infty} A_n x = SAx = \lim_{n\to\infty} A_n Sx = ASx$. Hence $ST = TS \Rightarrow AS = SA$

- **Uniqueness:** Let both $A$ and $B$ be positive square root of $T$. Then$A^2 = B^2 = T$. Also $BT = B^2B = BB^2 = TB$. So that $AB = BA$ by $(v)$. Let $x \in H$ be arbitrary and $y = (A - B)x$. Then $\langle Ay, y\rangle \geqq 0$ & $\langle By, y\rangle \geqq 0$, because $A \geqq 0$ & $B \geqq 0$. Using $AB = BA$ & $A^2 = B^2$ we obtain $\langle Ay, y\rangle + \langle By, y\rangle = \langle (A + B)y, y\rangle = \langle (A^2 - B^2)x, y\rangle = 0$.

Hence $\langle Ay, y\rangle = \langle By, y\rangle = 0$, since $A \geqq 0$ and $A$ is self-adjoint. It has itself a positive square root $C$, that is $C^2 = A$ and $C$ is self-adjoint. We obtain: $0 = \langle Ay, y\rangle = \langle C^2y, y\rangle = \langle Cy, Cy\rangle = \|Cy\|^2$ $and$ $Cy = 0, also$ $Ay = C^2y = C(Cy) = 0$.

Similarly, $B \geqq 0$ and $B$ is self-adjoint, it has itself a positive square root $D$. That is $D^2 = B$ and $D$ is self-adjiont.$0 = \langle By, y\rangle = \langle D^2y, y\rangle = \langle Dy, Dy\rangle = \|Dy\|^2$ & $Dy = 0$, also $By = D^2y = D(Dy) = 0$. Hence $By = 0$, since$B \geqq 0$. Hence $(A - B)y = 0$, using $y = (A - B)x, \forall\, x \in H$. $\|Ax - Bx\|^2 = \langle (A - B)^2x, x\rangle = \langle (A - B)y, x\rangle = 0$. Hence $A = B$. Then $A$ is unique. ■

**Example 5**

Let $T: L^2(0,1) \to L^2(0,1)$be a linear operator defined by

$Tf(x) = xf(x)\; for\; all\; f \in L^2(0,1), for\; all\; x \in (0,1)$

- Show that T is a positive operator.
- Find the lower and upper bounds of T.
- Find norm of T.

**Solution. (a)** For all $f \in L^2(0,1)$ we have that. $T$ is self-adjoint

$$\langle Tf, g\rangle = \int_0^1 xf\left(\overline{xg(x)}\right) dx = \int_0^1 f(x)\overline{xg(x)}dx = \langle f, Tg\rangle\ \forall\, f, g \in L^2(0,1)$$

$$\langle Tf, f\rangle = \int_0^1 xf\left(\overline{xf(x)}\right) = \int_0^1 x|f(x)|^2\, dx \geqq 0.$$

Therefore $T$ is positive operator.

- **First we Notice that**

$$M = sup_{\|f\|=1}\langle Tf, f\rangle = sup_{\|f\|=1} \int_0^1 x|f(x)|^2dx \leqq sup_{\|f\|=1} \int_0^1 |f(x)|^2dx = 1$$

We prove that $M = 1$, consider

$$f_\varepsilon(x) = \begin{cases} 0, if\ x \in [0,1-\varepsilon) \\ \varepsilon^{\frac{-1}{2}}, if\ x \in [1-\varepsilon, 1] \end{cases}.$$

$$\Rightarrow \|f\|^2 = \int_{1-\varepsilon}^1 \varepsilon^{-1}dx = 1\ \&$$

$$M = \langle Tf_\varepsilon, f_\varepsilon\rangle = \int_0^1 x|f_\varepsilon x|^2dx = \int_{1-\varepsilon}^1 \frac{x}{\varepsilon}dx = \frac{x^2}{2\varepsilon}|_{1-\varepsilon}^1 = \frac{1-(1-\varepsilon)^2}{2\varepsilon} = 1$$

$M = 1$.

We proceed similarly in order to prove that $m = inf_{\|f\|=1}\langle Tf, f\rangle = 0$ using the function

$$g_\varepsilon(x) = \begin{cases} \varepsilon^{\frac{-1}{2}}, if x \in [0,\varepsilon] \\ 0, if\ x \in [\varepsilon, 1] \end{cases}$$

Hence, $m = 0$

**(c)** $\|T\| = \max(|m|, |M|) = sup_{\|f\|=1}|\langle Tf, f\rangle| = 1$

## 4. PROJECTIONS

### 4.1.1 Definition

Let $H$ be a Hilbert space over Complex number. A bounded linear operator $P$ on is $H$ called:

- a projection, if $P^2$ =P
- An orthogonal projection, if $P^2$ = P and P* = P.

**Note**

The range $Ran(P) = P(H)$ of a projection on a Hilbert space $H$ always is a closed linear subspace of $H$ on which $P$ acts like the identity. If in addition $P$ is orthogonal, then $P$ acts like zero operator on$(R(P))^{\perp}$.

If $x = y + z$ with $y \in R(P)$ and $z \in (R(P))^{\perp} = \mathcal{N}(P)$ is the decomposition guaranteed by the projection theorem, then $Px = y$. Thus the projection theorem sets up a one –to –one correspondence between orthogonal projection and closed linear subspaces.

$x = y + z = Px + (I - P)x$

This show that the projection of $H$ onto $Y^{\perp}\ is\ I - P$

### 4.1.1. Theorem

A bounded linear operator $P: H \to H$ on a Hilbert space $H$ is a projection if and only if $P$ is self-adjoint and idempotent.

**Proof.** $(\Rightarrow)$Suppose that $P$ is a projection on $H$ and denote $P(H)\ by\ Y$. Then $P^2 = P$ because for every $x \in H\ and\ Px = y \in Y$ we have; $P^2x = Py = y$, hence $P^2x = Px \Rightarrow P^2 = P$ is idempotent. Now consider any two vectors $x_1, x_2 \epsilon\ H$, from decomposition we can write$x_1 = y_1 + z_1\ and\ x_2 = y_2 + z_2$, where $y_1, y_2 \in Y = R(P)\ and\ z_1, z_2 \in Y^{\perp} = N(P)$. Then, $\langle y_1, z_2\rangle = \langle y_2, z_1\rangle = 0$, because $Y \perp Y^{\perp}$. We show that $P$ is self-adjoint; $\langle Px_1, x_2\rangle = \langle y_1, y_2 + z_2\rangle = \langle y_1, y_2\rangle = \langle y_1 + z_1, y_2\rangle = \langle x_1, Px_2\rangle$. Hence, $P$ is self-adjoint.

$(\Leftarrow)$ Suppose that $P$ is self-adjoint and idempotent, denoted $P(H)$ by $Y$. Then for every $x \in H$

$x = Px + (I - P)x$

Orthogonality; $Y = P(H) \perp (I - P)(H)$, follows from:

$\langle Px, (I - P)v\rangle = \langle x, P(I - P)v\rangle = \langle x, Pv - P^2v\rangle = \langle x, 0\rangle = 0.$

Let, $Q = (I - P)$, $y \subset \ker Q$ can see from $QPx = Px - P^2x = QPx = 0$. Next, $y \supset \ker Q$, $Qx = x - Px \Rightarrow Qx = x$. Hence $Y = \{0\}$, since $\{0\}$ is closed, so inverse image $\{x: QPx = 0\} = kerQ$. Hence $Y$ is closed subspace of $H$.

Therefore, $Y$ is projection on $H$.

**4.1.2. Theorem**

For any projection $Y$ on a Hilbert space $H$.

- $\langle Px, x\rangle = \|Px\|^2$
- $0 \leq P \leq I$
- $\|P\| \leqq 1;\ \|P\| = 1\ if\ P(H) \neq \{0\}$

**Proof.**

- $\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, Px\rangle = \|Px\|^2$
- $0 \leq \|Px\|^2 = \langle Px, Px\rangle = \langle Px, x\rangle \leq \|x\|^2 = \langle x, x\rangle = I \Rightarrow 0 \leq P \leq I$
- By using Schwarz inequality;

  $\langle Px, x\rangle \leqq \|x\|^2$

  $\|Px\|^2 \leqq \|x\|^2, since\ \|x\| \neq 0$

  $\|P\| \leqq 1, \forall\ x \in H$ (1)

  $\|Px\| = \|P(Px)\| \leqq \|P\|\|Px\|$

  $\|Px\| \leqq \|P\|\|Px\|, since\ \|P\| \neq 0$

  $1 \leqq \|P\| \ldots$ (2)

  By combining (1) and (2) we get ; $\|P\| = 1$ ■

**4.1.3. Theorem**

Products projections on Hilbert space $H$ are satisfying the following two conditions:

- $P = P_1P_2$ is projection on H if and only if the projections $P_1\ and\ P_2$ commute. Then $P$ projects $H$ onto $Y = Y_1 \cap Y_2\ where\ Y_j = P_j(H)$.
- Two closed subspaces $Y$ and $V$ of $H$ are orthogonal if and only if the corresponding projections satisfy $P_YP_V = 0$.

**Proof.** I(⇐). Suppose that $P_1\ and\ P_2$ commute, then show that $P$ is self-adjiont and idempotent $P^* = (P_1P_2)^* = {P_2}^*{P_1}^* = P_2P_1 = P_1P_2 = P$. Hence $P^* = P$, then $P$ is self-adjoint. $P^2 = (P_1P_2)^2 = (P_1P_2)(P_1P_2) = {P_1}^2{P_2}^2 = P_1P_2 = P_2P_1 = P$. Then $P$ is idempotent. Hence $P$ is projection. Then $Px = P_1(P_2x) = P_2(P_1x) = x, \forall\ x \in H$. Since $P_1$ projects $H$ onto $Y_1$, we must have :

$Px = x = P_1(P_2x) \in Y_1$

$Px = x = P_2(P_1x) \in Y_2$

$Px \in Y_1 \cap Y_2\ \ since\ x \in H$ was arbitrary. This show $P$ projects $H\ into\ Y = Y_1 \cap Y_2$ . Actually $P$ projects $H$

onto $Y$. Indeed, if $y \in Y$, then $y \in Y_1$ and $y \in Y_2$. $Py = P_1P_2y = P_1y = y$. Then $y \in Y_1 \cap Y_2$ , $y \in Y$ $Hence$ $Y = Y_1 \cap Y_2$ .

($\Rightarrow$) . Suppose $P = P_1P_2$ is projection defined on $H$ . It must be self-adjoint $\langle Py_1, y_2\rangle = \langle P_1P_2y_1, y_2\rangle = \langle P_2y_1, P_2y_1\rangle = \langle y_1, P_2P_1y_2\rangle = \langle y_{1,}Py_2\rangle$. Then $P_1P_2 = P_2P_1$.

II ($\Rightarrow$). If $Y \perp V, then\ Y \cap V = \{0\}\ and\ P_YP_Vx = \{0\}\ for\ all\ x \in H$ by part (a), so that $P_YP_V = 0$.

($\Leftarrow$). If $P_YP_V = 0$, then for every $y \in Y$ and $v \in V$. We obtain $\langle y, v\rangle = \langle P_Yy, P_Vv\rangle = \langle y, P_YP_Vv\rangle = \langle y, 0\rangle = 0$. Hence $Y \perp V$. ■

**3.1.5 Theorem**

Let $P_1\ and\ P_2$ be projections on Hilbert space $H$. Then:

- The sum $P = P_1 + P_2$ is a projection on $H$ if and only if $Y_1 = P_1(H)\ and\ Y_2 = P_2(H)$ are orthogonal.
- If $P = P_1 + P_2$ is a projection, $P$ projects $H$ onto$Y = Y_1 \oplus Y_2$.

**Proof.** I($\Rightarrow$). Suppose that $P = P_1 + P_2$ is a projection.

Let $x \in Y_1 \Rightarrow \|x\|^2 \geq \|(P_1 + P_2)x\|^2 = \langle (P_1 + P_2)x, (P_1 + P_2)x\rangle$

$= \langle (P_1 + P_2)^2x, x\rangle = \langle (P_1 + P_2)x, x\rangle$

$= \langle P_1x, x\rangle + \langle P_2x, x\rangle$

$= \|x\|^2 + \langle {P_2}^2x, x\rangle$

$= \|x\|^2 + \|P_2x\|^2 \Rightarrow \|P_2x\| = 0.$

For any $y \in Y_2$ and $\in Y_1$ $\langle x, y\rangle = \langle x, P_2y\rangle = \langle P_2x, y\rangle = \langle 0, y\rangle = 0 \Rightarrow \langle x, y,\rangle = 0$. Therefore $Y_1 \perp Y_2$.

($\Leftarrow$) If $Y_1 \perp Y_2$, then $P_2P_1 = P_1P_2 = 0$ which implies $P^2 = P$. Since $P_2$ and $P_1$ are self-adjoint, so is $P_1 + P_2$. Hence $P$ is projection.

II. We determine the closed subspace $Y \subset H$ onto which $P$ projects. Since $P_1 + P_2, \forall\ x \in H$. $y = Px = P_1x + P_2x$. Here $P_1x \in Y_1$ and $P_2x \ \in Y_2$ hence, $y \in Y_1 \oplus Y_2$.

So that;

$$Y \subset Y_1 \oplus Y_2. \tag{1}$$

We show that $Y \supset Y_1 \oplus Y_2$. Let $v \in Y_1 \oplus Y_2$ be arbitrary, then $v = y_1 + y_2$, here $y_1 \ \in Y_1$ and $y_2 \in Y_2$ applying in $P$.Using $Y_1 \perp Y_2$, thus we obtain: $pv = p_1(y_1 + y_2) + p_2(y_1 + y_2) = p_1y_1$. Hence,

$$v \in Y\ and\ Y \supset Y_1 \oplus Y_2 \tag{2}$$

Combining (1) and (2 ) we have;

$$Y = Y_1 \oplus Y_2$$

**4.2 Further Properties of Projections**

**4.2.1 Theorem**

Let $P_1$ and $P_2$ be projections defined on a Hilbert Space $H$. Denote by $Y_1 = P_1(H)$and $Y_2 = P_2(H)$ the subspace onto which $H$ is projected by $P_1$and $P_2$, and let $\mathcal{N}(P_1)$ and $\mathcal{N}(P_2)$ be the Null space of these projection. Then the following conditions are equivalent.

- $P_2P_1 = P_1P_2 = P_1$
- $Y_1 \subset Y_2 = P_1(H) \subset P_2(H)$
- $\mathcal{N}(P_1) \supset \mathcal{N}(P_2)$
- $\|P_1x\| \leqq \|P_2x\|$
- $P_1 \leqq P_2$

**Proof.** (ii ⇒ i). Suppose $Y_1 \subset Y_2 = P_1(H) \subset P_2(H)$. We want show that $P_2P_1 = P_1P_2 = P_1$. For every $x \in H$, we have $P_1x \in Y_1$. Hence, $P_1x \in Y_2$ by (ii.) $P_2(P_1x) = P_1x \Rightarrow P_2P_1 = P_1 \;\Rightarrow (P_2P_1)^* = P_1{}^*P_2{}^* = P_1P_2 = P_2P_1 = P_1$. Since, $P_1$ is self-adjoint. Therefore, $P_1 = P_1P_2 = P_1$.

(i ⇒ iv). Suppose $P_2P_1 = P_1P_2 = P_1$. We want show that $\|P_1x\| \leqq \|P_2x\|$ for all $x \in H$

$\Rightarrow \langle P_1x, x\rangle \leq \|x\|^2$, since $P_1x = x$

$\Rightarrow \|P_1x\|^2 \leq \|x\|^2$, for all $x \in H$

$\Rightarrow \|P_1\|^2\|x\|^2 \leq \|x\|^2, \|x\| \neq 0$. We have $\|P_1\| \leq 1$ by (1) $\langle Px, x\rangle = \|Px\|^2$

$\Rightarrow \|P_1x\| = \|P_1P_2x\| \leqq \|P_1\|\|P_2x\| \leqq \|P_2x\|$.

Therefore, $\|P_1x\| \leqq \|P_2x\|$

(iv ⇒ v). Suppose $\|P_1x\| \leqq \|P_2x\|$. We want show that $P_1 \leqq P_2$. From $\langle Px, x\rangle = \|Px\|^2$ and (iv.) in present theorem we have for all $x \in H$. $\langle P_1x, x\rangle = \|P_1x\|^2 \leqq \|P_2x\|^2 = \langle P_2x, x\rangle$

Therefore, $P_1 \leqq P_2$by definition of positive operators.

(v.⇒ iii.)Suppose $P_1 \leqq P_2$. We want show that $\mathcal{N}(P_1) \supset \mathcal{N}(P_2)$. Let $x \in \mathcal{N}(P_2) \Longrightarrow P_2x = 0$ by (iii.) sect. 2 and (v.) in the present theorem, $\|P_1x\|^2 = \langle P_1x, x\rangle \leq \langle P_2x, x\rangle = \langle 0, x\rangle = 0$. Hence, $P_1x = 0, x \in \mathcal{N}(P_1)$. Therefore, $\mathcal{N}(P_1)$ $\supset\mathcal{N}(P_2)$. ■

**4.2.2. Theorem**

Let$P_1$ & $P_2$ be Projections on a Hilbert space $H$. Then

- The difference $P = P_2 - P_1$ is a projection if and only if $Y_1 \subset Y_2$ where $Y_j = P_j(H)$.
- If $P = P_2 - P_1$ is a projection, $P$ projects H onto $Y$, where $Y$ is the orthogonal complement of $Y_1$ *in* $Y_2$.

**Proof.**(⇒) Suppose that $P = P_2 - P_1$ is a projection. We want to show that $Y_1 \subset Y_2$. If $P = P_2 - P_1$ is a projection. $P = P^2 \Rightarrow P_2 - P_1 = (P_2 - P_1)^2 = P_2{}^2 - P_2P_1 - P_1P_2 + P_1{}^2$.

$P_2 + P_2P_1 = 2P_1,*$

by theorem (2.2) (i.). Multiply both sides by $P_2$ we have:

$P_2P_1P_2 + P_2P_1 = 2P_2P_1$

$P_1P_2 + P_2P_1P_2 = 2\ P_1P_2$

Hence $P_2P_1P_2 = P_2P_1, P_2P_1P_2 = P_1P_2$ and by $(*)$

$P_1 = \ P_1P_2 = \ P_1 \ \ **$

Therefore, $Y_1 \subseteq Y_2$.

**(⇐)** Suppose $Y_1 \subset Y_2$, where $Y_j = P_j(H) \Rightarrow P^2 = (P_2 - P_1)^2 = {P_2}^2 - P_2P_1 - P_1P_2 + {P_1}^2 = P_2 - P_1$. Therefore, $P$ is idempotent. $\Rightarrow P^* = (P_2 - P_1)^* = {P_2}^* - {P_1}^* = P_2 - P_1$. Therefore $P$ is self-adjoint.

$\therefore P$ is projection.

**(b)** $Y = P(H)$consists of all vectors of the form

(8) $Y = Px = P_2x - P_1x \ for \ all \ x \in H$

Since $P_2P_1 = P_1P_2 = P_1 \ implies \ Y_1 \subset Y_2$

$P_2y = {P_2}^2x - P_2P_1x = P_2x - P_1x = y$

This show that y$\in Y_2$, also from (8) and (1)

$P_1y = P_1P_2x - {P_1}^2x = P_1x - P_1x = 0$

$\Rightarrow P_1y = 0$

$y \in \mathcal{N}(P_1) = Y_1^{\perp}$

Together $y \in V \ where \ V = Y_2 \cap Y_1^{\perp}$, since the projection of H onto $Y_1^{\perp}$ is $I - P_1$, every $v \in V$

is the form

(9) $v = (\ I - P_1) + \ y_2, (y_2 \in Y_2)$

Using again,$P_2P_1 = P_1$, we obtain from (9), since $P_2y_2 = y_2$

$Pv = (P_2 - P_1)(I - P_1)y_2$

$= \left(P_2 - P_2P_1 - P_1 + {P_1}^2\right)y_2$

$= (P_2 - P_2P_1 - P_1 + P_1)y_2 = (P_2 - P_1)y_2$

$= P_2y_2 - P_1y_2 = y_2 - P_1y_2 = v$

$\therefore Pv = v$ so that $v \in Y$, since $v \in V$ was arbitrary

$\therefore Y \supseteq V$

$\therefore Y = P(H) = V = Y_2 \cap Y_1^{\perp}$

**4.2.3. Theorem**

Let $P_n$ is a monotone increasing sequence of projection $P_n$ defined on a Hilbert Space $H$. Then

- $(P_n)$ is strongly operator convergent, say $P_n x \to Px$ for every $x \in H$, and the limit operator $P$ is a projection defined on $H$.
- $P$ projects $H$ onto
- $P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}$
- $P$ has the null space

$\mathcal{N}(P) = \bigcap_{n=1}^{\infty} \mathcal{N}(P_n)$

**Proof a)** Let $m < n$, by assumption,$P_m \leqq P_n$, so that we have $P_m(H) \subset P_n(H)$ and $P_n - P_m$ is projection. Hence for every fixed $x \in H$, we obtain by 2.1.2

$\|P_n x - P_m x\|^2 = \|(P_n - P_m)x\|^2 = \langle (P_n - P_m)x, x\rangle = \|P_n x\|^2 - \|P_m x\|^2$.

Now $\|P_n\| \leq 1$ by 3.1.2, so that $\|P_n x\| \leq \|x\|$ for all $n$. Hence, $(\|P_n x\|)$ is bounded a sequence of numbers.$(\|P_n x\|)$is also monotone by 3.2.1. Since$(P_n)$is monotone. Hence $\lim_{n\to\infty} P_n x \longrightarrow Px$ is converges. Since $(P_n x)$ is Cauchy.

$\|P_n x - P_m x\| = \|(P_n - P_m)x\| \leq \|P_n - P_m\|\|x\|$

$\Longrightarrow \lim_{n\to\infty} P_n x = Px$, since $H$ is complete.

The linearity of $P$.

$P(\alpha x + \beta y) = \lim_{n\to\infty} P_n(\alpha x + \beta y)$

$= \lim_{n\to\infty}(P_n \alpha x + P_n \beta y)$

$= \alpha \lim_{n\to\infty} P_n x + \beta \lim_{n\to\infty} P_n\, y$

$= \alpha Px + \beta Py$

$\Longrightarrow \|P_n x\| \leq \|P_n\|\|x\| \Longrightarrow \|P_n x\| \leq \|P_n\|\|x\|$

$\Longrightarrow \|P_n x\| \leq \|x\|$

$\Longrightarrow \|p_n\| \leq \|x\|$. Therefor $P_n$is bounded,

$\langle P_n x, x\rangle = \langle x, P_n^* x\rangle = \langle x, P_n x\rangle$ , therefore $P_n$ is self-ad joint, and $\langle P_n^2 x, x\rangle = \langle P_n x, P_n x\rangle = \langle x, P_n x\rangle = \langle P_n x, x\rangle$, hence $P_n$ is projection**.**

b). We determine $P(H)$. Let $m < n$. Then $P_m \leqq P_n$, that is $P_n - P_m \geqq 0$ and $\langle (P_n - P_m)x, x\rangle \geqq 0$. The continuity of inner product

$\lim_{n\to\infty}\langle (P_n - P_m)x, x\rangle = \lim_{n\to\infty}\langle x, (P_n - P_m)x\rangle$

$= \lim_{n\to\infty}\langle x, (P - P_m)x\rangle$

$\therefore\ \langle (P - P_m)x, x\rangle \geqq 0.$

That is $P_m \leqq P$ and 3.2.1 yields $P_m(H) \subset P(H)$ for every $m$. Hence $\cup P_m(H) \subset P(H)$. Furthermore, for every $m$ and for every $x \in H$, we have, $P_m x \in P_m(H) \subset \cup P_m(H)$. Since $P_m x \longrightarrow Px, Px \in \overline{\cup P_m(H)}$. Hence $P(H) \subset \overline{\cup p_m(H)}$

$\cup P_m(H) \subset P(H) \subset \overline{\cup P_m(H)}$

$\Rightarrow P(H) = \mathcal{N}(I - P) = \{0\}$

$\Rightarrow P(H) = \{0\}$is closed

$\therefore\ P(H) = \overline{\cup_{m=1}^{\infty} P_m(H)}$

c). We determine $\mathcal{N}(P)$

$\mathcal{N}(P) = P(H)^{\perp} \subset P_n(H)^{\perp}$for all $n$. Since $P(H) \supset P_n(H)$ by part (b). Hence $\mathcal{N}(P) \subset \cap P_n(H)^{\perp} = \cap \mathcal{N}(P_n)$

$\Rightarrow \mathcal{N}(P) \subset \cap \mathcal{N}(P_n)$ (1)

On other hand, if $x \in \cap \mathcal{N}(P_n)$, then $x \in \mathcal{N}(P_n)$ for all $n$. So that $P_n x = 0$ and $P_n x \longrightarrow Px \implies Px = 0$. That is $x \in \mathcal{N}(P)$. Since $x \in \mathcal{N}(P_n)$ was arbitrary.

$\cap \mathcal{N}(P_n) \subset \mathcal{N}(P)$ (2)

By combining (1) and (2) we gets

$\mathcal{N}(P) = \cap_{n=1}^{\infty} \mathcal{N}(P_n)$

**Example 6**

Let $P: C^3 \to C^3$ be the linear operator defined by $P(x, y, z) = (x, y, 0), \forall\, x, y, z \in C^3$. Then $P$ is orthogonal projection.

**Solution**

Since $C^3$ is finite dimensional $P \in B(C^3)\ and\ clealy\ P^2 = P.$

Follows from $\langle P(x, y, z), (u, v, w)\rangle = x\overline{u} + y\overline{v} = \langle (x, y, z), P(u, v, w)\rangle$. Then $P$ is self-adjoint. Therefore $P$ is projection. Orthogonal projection of $P$ has:$Imp = \{(x, y, 0): x, y \in \mathbb{C}\}$ orthogonal projection $P$ matrix representation:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In generally a $n \times n$ diagonal matrix whose diagonal element either $0\ or\ 1$ is the matrix of an orthogonal projection $B(C^3)$.

## 5. SPECTRAL FAMILY

**5.1.1 Definition**

A one-parameter family of projection is called spectral family. Spectral family can be obtained from the finite dimension case as follows. Let $T: H \longrightarrow H$ be a self-adjoint linear operator on a unitary space $\boldsymbol{H} = \mathbb{C}^{\boldsymbol{n}}$**.** Then $T$ is bounded and we may choose a basis for $H$ and represent $T$ by a Hermitian matrix which we denote simply by $T$. The spectrum of the operator consists of the eigenvalues of that matrix which are real. For simplicity let us assume that the matrix $T$ has $n$ different eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n$. Then theorem 1.1.1(6) implies that $T$ has an orthonormal set of $n$ eigenvectors $x_1, x_2 \ldots x_n$ where $x_j$ corresponding to $\lambda_j$ and we write these vectors as column vectors. This a basis for $H$, so that every $x \in H$ has a unique representation

$$x = \sum_{j=1}^{n} \gamma_j x_j \,, \gamma_j = \langle x, x_j \rangle = x^T \overline{x_j} \tag{1}$$

In (1) we obtain the second formula from the first one by taking the inner product$\langle x, x_k \rangle$, where $x_k$is fixed and using the orthonormality. The essential fact in (1) is that $x_j$ is an eigen vectors of $T$, so that we have$\boldsymbol{T x_j = \lambda_j x_j}$, consequently, if we apply $T$ to (1) we simply obtain

$$Tx = \sum_{j=1}^{n} \lambda_j \gamma_j x_j \tag{2}$$

Looking at (1) more closely, we see that we can define an operator

$$P_j : H \longrightarrow H \tag{3}$$

$x \mapsto \gamma_j x_j \Rightarrow p_j x = \gamma_j x_j = \langle x, x_j \rangle x_j . P_j$ is the projection (orthogonal projection) of $H$ onto the eigenspace of $T$ corresponding to $\lambda_j$. Formula (1) can now be written,

$$x = \sum_{j=1}^{n} P_j x \tag{4}$$

Hence, $I = \sum_{j=1}^{n} P_j$. Where $I$ is identity operator on $H$. Formula (2) becomes

$$Tx = \sum_{j=1}^{n} \lambda_j P_j x \tag{5}$$

Hence $T = \sum_{j=1}^{n} \lambda_j P_j$. This is representation of $T$ in terms of projection. Instead of the projection $P_1, P_2, \ldots, P_n$ themselves we take sums of such projection. More precisely, for any real $\lambda$ we define,

$$E_\lambda = \sum_{\lambda_j \leqq \lambda} P_j \quad (\lambda \in \mathbb{R}) \tag{6}$$

Hence $E_\lambda{}^2 = E_\lambda$, moreover $E_\lambda$ is symmetric operator. This is one-operator family of projection, $\lambda$ being the parameter. From (6) we see that for any $\lambda$ the operator $E_\lambda$ is the projection of $H$ onto the subspace $V_\lambda$ spanned by all those $x_j$ for which $\lambda_j \leqq \lambda$ it follows that $V_\lambda \subset V_\mu$ $(\lambda \leqq \mu)$. As $\lambda$ traverse $\mathcal{R}$ in the positive sence$E_\lambda$ grows from 0 to $I$. The growth occurring at the eigenvalues of $T$ and $E_\lambda$ remaining unchanged for $\lambda$ in any interval that is free of eigenvalues.

**Properties of $E_\lambda$**

- $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$, if $\lambda < \mu$
- $E_\lambda = 0$, if $\lambda < \lambda_1$
- $E_\lambda = I$, if $\lambda \geqq \lambda_n$

- $E_{\lambda+o} = \lim_{\mu\to\lambda+o} E_\mu = E_\lambda$

**Definition**

A real spectral family (or real decomposition unity) is a one parameter family $\mathcal{E} = (E_\lambda), \lambda \in \mathbb{R}$ of projection $E_\lambda$ defined on Hilbert space $H$ (of any dimension) which depends on a real parameter $\lambda$ and such that,

$$E_\lambda \leqq E_\mu \tag{7}$$

Hence, $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda, \lambda < \mu$

$$\lim_{\lambda\to-\infty} E_\lambda x = 0 \tag{8a}$$

$$\lim_{\lambda\to+\infty} E_\lambda x = x \tag{8b}$$

$$E_{\lambda+o}x = \lim_{\mu\to\lambda+o} E_\mu x = E_\lambda x \; (\forall\, x \in H) \tag{9}$$

We see from the definition that a real spectral family can be regarded as a mapping $\boldsymbol{R} \longrightarrow \boldsymbol{\mathcal{B}}(\boldsymbol{H}, \boldsymbol{H}), \lambda \longrightarrow E_\lambda$; Strongly operator continuous from right. To each $\lambda \in R$ there is corresponds a projection $\boldsymbol{E_\lambda} \in \boldsymbol{\mathcal{B}}(\boldsymbol{H}, \boldsymbol{H})$**,** where $\mathcal{B}(H, H)$ is the space of all bounded linear operators from $H$ into $H$. We assume, for simplicity, that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $T$ are all different and $\lambda_1 < \lambda_2, < \cdots < \lambda_n$. Then we have,

$E_{\lambda_1} = P_1$

$E_{\lambda_2} = P_1 + P_2$

$E_{\lambda_3} = P_1 + P_2 + P_3$

$\vdots$

$E\lambda_n = P_1 + P_2 + \cdots + P_n$.

Hence, conversely ;

$P_1 = E_{\lambda_1}$

$P_j = E\lambda_j - E\lambda_{j-1}$ j=2, ...,n

Since $E_\lambda$ remains the same for $\lambda$ in the interval$[\lambda_{j-1}, \lambda_j)$, this can be written$P_j = E\lambda_j - E\lambda_{j-0}$. Now equation (4) becomes $x = \sum_{j=1}^{n} P_j x = \sum_{j=1}^{n} (E\lambda_j - E\lambda_{j-0})x$ and equation (5) becomes

$$Tx = \sum_{j=1}^{n} \lambda_j P_j x = \sum_{j=1}^{n} \lambda_j (E\lambda_j - E\lambda_{j-0})x.$$

If we drop the $x$ and write $\delta E_\lambda = E_\lambda - E_{\lambda-0}$. Since $E_\lambda = \sum_{\lambda_j \leqq \lambda} P_j$. We arrive at

$$T = \sum_{j=1}^{n} \lambda_j \delta E\lambda_j \tag{10}$$

This is the spectral representation of the self-adjoint linear operator $T$ with eigenvalues $\lambda_1 < \lambda_2, < \cdots < \lambda_n$ on that n-dimensional Hilbert space $H$. The representation shows that for

any $x, y \in H$,

$\langle Tx, y\rangle = \sum_{j=1}^{n} \lambda_j \langle \delta E_{\lambda_j} x, y\rangle$ (11)

We note that this may be written as a Riemann-stieltjes integral

$\langle Tx, y\rangle = \int_{-\infty}^{+\infty} \lambda dw(\lambda)$ … … … … … … … … (12).

Where $w(\lambda) = \langle E_\lambda x, y\rangle$.

### 5.2 Spectral Family of a Bounded Self-Adjiont Linear Operator

To define Spectral family$(\varepsilon)$ we need the operator $T_\lambda = T - \lambda I$ from resolvent theorem $\|T_\lambda x\| \geqq c\|x\|$. Then $\mathcal{B}_\lambda{}^2 = T_\lambda{}^2$, the positive square root of $T_\lambda{}^2$ denote by $\mathcal{B}_\lambda{}^2$. $\mathcal{B}_\lambda = \left(T_\lambda{}^2\right)^{\frac{1}{2}} = |T_\lambda|$ and operator $T_\lambda{}^+ = \frac{1}{2}(\mathcal{B}_\lambda + T_\lambda)$which is called positive part of $T_\lambda$. The spectral family $\varepsilon$ of $T$ is then defined by $\mathcal{E} = (E_\lambda), \lambda \in \mathbb{R}$, where $E_\lambda$ is the projection of $H$ onto the null space $\mathcal{N}(T_\lambda^+)$of $T_\lambda^+$. We proceed step wise and consider at first the operator:

$\mathcal{B} = (T^2)^{\frac{1}{2}}$(Positive square root of $T^2$),

$T^+ = \frac{1}{2}(\mathcal{B}+T)$ (Positive part of $T$),

$T^- = \frac{1}{2}(\mathcal{B}-T)$(Negative part of $T$),

and the projection of $H$ onto the null space of $T^+$ which we denote by $E$.

$E: H \longrightarrow V = \mathcal{N}(T^+) = \ker\,(\lambda I - T)^+$

$T = T^+ - T^- \Longrightarrow \frac{1}{2}(\mathcal{B} + T) - \frac{1}{2}(\mathcal{B}-T) = T^+ - T^-$

$\Longrightarrow \frac{1}{2}\mathcal{B} + \frac{1}{2}T - \frac{1}{2}\mathcal{B} + \frac{1}{2}T = T^+ - T^-$

$\Longrightarrow \boldsymbol{T} = \boldsymbol{T}^+ - \boldsymbol{T}^-$ by subtraction

$\mathcal{B} = T^+ + T^- \Longrightarrow T^+ + T^- = \frac{1}{2}(\mathcal{B} + T) + \frac{1}{2}(\mathcal{B}-T)$.

$= \frac{1}{2}\mathcal{B} + \frac{1}{2}T + \frac{1}{2}\mathcal{B} - \frac{1}{2}T$.

$= \mathcal{B}$

$\Longrightarrow \boldsymbol{\mathcal{B}} = \boldsymbol{T}^+ + \boldsymbol{T}^-$ by addition.

#### 5.2.1 Lemma

The operators just defined have the following properties

- $\mathcal{B}, T^+$and $T^-$are bounded and self-adjoint
- $\mathcal{B}, T^+$and $T^-$ commute with every bounded linear operator that $T$ commute with; in particular
  - $\mathcal{B}T = T\mathcal{B}\; T^+T = TT^+T^-T = TT^-T^+T^- = T^-T^+$
- $E$ commutes with every bounded self-adjoint linear operator that $T$ commute with; in particular

- $ET = TE\ E\mathcal{B} = \mathcal{B}E$
- Furthermore
  - $T^+T^- = 0\ T^-T^+ = 0$
  - $T^+E = ET^+ = 0\ T^-E = ET^- = T^-$
  - $TE = -T^-\ T(I - E) = T^+$
  - $T^+ \geqq 0\ T^- \geqq 0$

**Proof. a**

- ***Claim 1*** $\mathcal{B}$ is bounded

  **Proof of Claim 1**

  $||\mathcal{B}||^2 = \langle \mathcal{B}, \mathcal{B} \rangle \leq \langle T^+ + T^-, T^+ + T^- \rangle$

  $= T^+T^+ + T^+T^- + T^-T^+ + T^-T^-$

  $\leq ||T^+||^2 + T^+T^- + T^-T^+ + ||T^-||^2$

  $= ||T^+||^2 + 2Re\langle T^+, T^- \rangle + ||T^-||^2$

  $\leq ||T^+||^2 + 2|\langle T^+, T^- \rangle| + ||T^-||^2$

  $\leq (||T^+|| + ||T^-||)^2$

  $||\mathcal{B}|| \leq ||T^+|| + ||T^-||$

  $\therefore \mathcal{B}$ is bounded $\Leftrightarrow$ continuous. Since $\lambda = 1$

**Claim 2**$\mathcal{B}$ is self-adjoint

**Proof of Claim 2** Since $\mathcal{B} = T^+ + T^-$

$\langle \mathcal{B}x, y \rangle = \langle (T^+ + T^-)x, y \rangle = \langle x, (T^+ + T^-)^* y \rangle$

$= \langle x, \mathcal{B}^* y \rangle$

$\Rightarrow \mathcal{B} = \mathcal{B}^*$

$\therefore \mathcal{B}$ is self-adjoint

- Suppose $TS = ST$. Then $T^2S = TST = ST^2$ and $\mathcal{B}S = S\mathcal{B}$ follows from theorem (positive square root) $T^+S = \frac{1}{2}(\mathcal{B}S + TS) = \frac{1}{2}(S\mathcal{B} + ST) = ST^+$ . Therefore, $T^+S = ST^+ \Rightarrow T^-S = ST^- \Rightarrow T^-S = \frac{1}{2}(\mathcal{B}S - TS) = \frac{1}{2}(S\mathcal{B} - ST) = ST^-$.

  Then show that $\mathrm{T^+T^- = T^-T^+}$.

  $T^+T^- = \frac{1}{2}(\mathcal{B}S + TS).\frac{1}{2}(\mathcal{B}S - TS) = \frac{1}{2}(S\mathcal{B} - ST).\frac{1}{2}(S\mathcal{B} + ST) = T^-T^+ \Rightarrow T^+T^- = T^-T^+$.

- For every $x \in H$ *we have* $y = Ex \in Y = \mathcal{N}(T^+) = Ker(T^+)$. Hence $T^+y = 0$ *and* $ST^+Y = S0 = 0$. From TS=ST and (b) we have S $T^+ = T^+S$ *and* $T^+SEx = T^+Sy = ST^+y = S0 = 0 \Rightarrow T^+SEx = 0$, *hence* $SEx \in Y$. *since* $E$ *projects* $H$ *onto* $Y$. We thus have $ESEx = SEx$. For every $x \in H$.that is $ESE = SE$. A projection is self adjoint. $ES = E^*S^* = (SE)^* = (ESE)^* = E^*S^*E^* = ESE = SE$. Therefore, $ES = SE$.

  We prove $(3) - (6)$

**Proof (3)**

From $B = (T^2)^{\frac{1}{2}} \Rightarrow B^2 = T^2$, *and also* $BT = TB$ *by* $(6)$ Hence $T^+T^- = T^-T^+ = \frac{1}{2}(B - T).\frac{1}{2}(B + T) = \frac{1}{4}(B^2 + BT - TB - T^2) = 0 \Rightarrow T^+T^- = T^-T^+ = 0$

**Proof (4)**

Let $T^+Ex = 0\ \forall\ x \in H$, since $T^+$ is self adjoint. We have $ET^+x = T^+Ex = 0$ *by* $(3)$ *and* $(c)$. Therefore, $ET^+ = T^+E = 0$. Furthermore $T^+T^-x = 0$ *by* $(8)$, *so that* $T^-x \in \mathcal{N}(T^+) = ker(T^+)$. Hence $ET^-x = T^-Ex = T^-x\ \forall x \in H$. Therefore, $ET^- = T^-E = T^-$

**Proof (5)**

From a, b and (4), since $T = T^+ - T^-$we have $TE = (T^+ - T^-)E = T^+E - T^-E = -T^-E$. $TE = -T^-$, since $T^-$is self adjoint. Again by (4) $T(I - E) = T - TE$=$T + T^- = T^+$. Therefore, $T(I - E) = T^+$.

**Proof (11)** By (4) and (b) $T^- = ET^- + ET^+ = E(T^- + T^+) = EB \geqq 0$, since $E$ and $B$ are self-adjoint and $E \geqq 0$ and $B \geqq 0$ by definition of positive operators.

$T^+ = B - T^- = B - EB = (I - E)B \geqq 0$

$\therefore T^+ \geqq 0$, *since* $I - E \geqq 0$ and B$\geqq$ 0

**In second step instead of $T$ we consider $T_\lambda = T - \lambda I$**

Instead of $B, T^+, T^-$ and $E$ we now have to take $B_\lambda = \left(T_\lambda{}^2\right)^{\frac{1}{2}}$, $T_\lambda{}^+ = \frac{1}{2}(B_\lambda + T_\lambda)$. $T_\lambda{}^- = \frac{1}{2}(B_\lambda - T_\lambda)$ and projection $E_\lambda : H \to Y_\lambda = \mathcal{N}(T_\lambda{}^+)$.

**5.2.2. Lemma**

The previous lemma remains true if we replace. $T, B, T^+, T^-, E$ *by* $T_\lambda, B_\lambda, T_\lambda{}^+, T_\lambda{}^-, E_\lambda$ Respectively, where $\lambda$ *is real*, moreover, for any real $\kappa.\lambda, \mu, \upsilon, \tau$ *following operator all commute*: $T_\kappa, B_\lambda, T_\mu{}^+, T_\upsilon{}^-, E_\tau$

**5.2.1. Theorem**

Let $T: H \to H$ is a bounded self-adjoint linear operator on a complex Hilbert space H. Furthermore, let $E_\lambda(\lambda$ *is real*$)$ be the projection of H onto the null space$Y_\lambda = \mathcal{N}(T_\lambda{}^+)$ of the positive part $T_\lambda{}^+$ of $T_\lambda = T - \lambda I$.then $\varepsilon = (E_\lambda)_{\lambda \in \mathbb{R}}$ is spectral family on the interval $[m, M] \subset \mathbb{R}$, where $m$ and $M$ are given by(1) see section 3.2.

**Proof.** We shall prove $\lambda < \mu \Rightarrow E_\lambda < E_\mu$

$\lambda < m \Rightarrow E_\lambda = 0.$

$\lambda \geq M \Rightarrow E_\lambda = I.$

$\mu \to \lambda + 0 \Rightarrow \mathrm{E}_\mu x \to E_\lambda x.$

In the proof we use part of **Lemma 5.2.2 formulated for $\boldsymbol{T_\lambda, T_\mu, T_\lambda}^+$** $etc$ instead of $T, T^+$ $etc$

$T_\mu{}^+ T_\mu{}^- = 0$

$T_\lambda E_\lambda = -T_\lambda{}^- T_\lambda (I - E_\lambda) = T_\lambda{}^+ T_\mu \mathrm{E}_\mu = -T_\mu{}^-$

$T_\lambda{}^+ \geqq 0 \; T_\lambda{}^- \geqq 0 \; T_\mu{}^+ \geqq 0 \; T_\mu{}^- \geqq 0$

**Proof (7)**

Let $\lambda < \mu$ we have $T_\lambda = T_\lambda{}^+ - T_\lambda{}^- \leqq T_\lambda{}^+ \; becuase - T^- \leqq 0$. Hence $T_\lambda{}^+ - T_\mu \geqq T_\lambda - T_\mu = (\mu - \lambda)I \geqq 0$. $T_\lambda{}^+ - T_\mu$ is self-adjiont and commutes with $T_\mu{}^+$ by Lemma of 5.2.2 and $T_\mu{}^+ \geqq 0$. Theorem 3.3.1 $T_\mu{}^+(T_\lambda{}^+ - T_\mu) = T_\mu{}^+(T_\lambda{}^+ - T_\mu{}^+ + T_\mu{}^-) \geqq 0$. Here $T_\mu{}^+ T_\mu{}^- = 0$. Hence $T_\mu{}^+ T_\lambda{}^+ \geqq T_\mu{}^{+2} \; \forall \; x \in H$. $\langle T_\mu{}^+ T_\lambda{}^+ x, x \rangle \geqq \langle T_\mu{}^{+2} x, x \rangle = \|T_\mu{}^+ x\|^2 \geqq 0$. Since $T_\mu{}^+$is self-adjoint. This show that $T_\lambda{}^+ x = 0 \Rightarrow T_\mu{}^+ x = 0$. Hence $\mathcal{N}(T_\lambda{}^+) \subset \mathcal{N}(T_\mu{}^+)$.

$E_\lambda < E_\mu \; for \; \lambda < \mu$

**Proof (8)**

Suppose not $E_\lambda \neq 0 \Rightarrow E_\lambda z \neq 0 \; for \; \exists \; z$, we set $x = E_\lambda z$. Then $E_\lambda x = E_\lambda{}^2 z = E_\lambda z = x$, we may assume $\|x\| = 1$.

$\langle T_\lambda E_\lambda x, x \rangle = \langle T_\lambda x, x \rangle = \langle Tx, x \rangle - \lambda \geqq inf_{\|\tilde{x}\|=1} \langle T\tilde{x}, \tilde{x} \rangle - \lambda = m - \lambda > 0$, contradiction the fact that $\lambda > m$. $T_\lambda E_\lambda = -T_\lambda{}^- \leqq 0$.

$T_\lambda E_\lambda = 0 \Rightarrow E_\lambda$=0

$\therefore \; \lambda < m \Rightarrow E_\lambda = 0$

**Proof (9)**

Suppose $\lambda > M$ but $E_\lambda \neq I$ so that $I - E_\lambda \neq 0$. Then $(I - E_\lambda)x = x$ for some $x$ of norm 1. Hence $\langle T_\lambda (I - E_\lambda) x, x \rangle = \langle T_\lambda x, x \rangle = \langle Tx, x \rangle - \lambda \leq sup_{\|\tilde{x}\|} \langle \langle T\tilde{x}, \tilde{x} \rangle - \lambda \rangle = M - \lambda < 0$. Then there is contradiction.

$T_\lambda (I - E_\lambda) = (T_\lambda{}^+) \geqq 0$

$\therefore \; \lambda \geqq M \Rightarrow E_\lambda = I$

## 6. REFERENCES


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