Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2018, 5(2):275-280**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Graphs of Pregroups

Faisal H Nesayef

Department of Mathematics, Faculty of Science, University of Kirkuk, Kirkuk, Iraq

Abstract Pregroups was constructed by Stallings in 1971. Subsequently the concept of Pregroups was developed by many other researchers. Stallings originally defined a set with a binary operation satisfying five axioms, namely, P1, P2, P3, P4, and P5. Later it was proved that P3 is a consequence of the other axioms. Stallings has also linked this construction of a Pregroup to Free Product of Groups.
This construction is developed to include a new axiom called P6, which enabled to define a length function on the universal group of Pregroups. Applications of Pregroups with length functions led to direct proofs of many other concepts in combinatorial group theory.
On the other hand the concept of length functions on groups was first introduced by Lyndon [1]. This was used to give direct proofs of many other results in combinatorial group theory. Further work was done by many other researchers such as, Cheswell [2], [3], [4], Hoare [5], Wilkins [6], etc.

Keywords Pregroup, Length Function, Graphs of Groups, Maximal Tree and HNN extension

1. Introduction

1.1. Length functions

Definition 1.1: A length function | | on a group G , is a function giving each element x of G a real number $|x|$, such that for all $x , y , z \in G$, the following axioms are satisfied.

$A1' \ |e| = 0$, e is the identity elements of G. (1)

$A2 \ |x^{-1}| = |x|$ (2)

$A4 \ d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2}\left(|x| + |y| - |xy^{-1}|\right)$ (3)

Lyndon [8] showed that $A4$ is equivalent to $d(x,y) \geq \min\{ d(y,z), d(x,z)\}$ and to

$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m$. (4)

$A1', A2$ and $A4$ imply $|x| \geq d(x,y) = d(y,x) \geq 0$. (5)

Assuming, $A2$ and $A4$ only, it is easy to show that:

i. $d(x,y) \geq |e|$ (6)

ii. $|x| \ \geq |e|$ (7)

iii. $d(x,y) \leq |x| - \frac{1}{2}|e|$, (8)

The Axiom $A3$ states that $d(x,y) \geq 0$ is deductible from $A1', A2$ and $A1'$ is a weaker version of the axiom:

$A1 |x| = 0$ if and only if $x = 1$ in G. (9)

1.2. Pregroups

Definition 1.2. A Pregroup is a set P containing an element called the identity element of P, denoted by 1, a subset D of PXP and a mapping D → P, where (x , y) → x y, together with a map i : P → P where i (x) = x^{-1}, satisfying the following axioms:

We say that x y is defined if (x , y) ∈ D, i.e. x y ∈ P.

P1. For all x ∈ P, 1x and x1 are defined and 1x = x1 = x.

P2. For all x ∈ P, x^{-1} x = x x^{-1} = 1.

P3. For all x ,y ∈P, if x y is defined , then y^{-1} x is defined and $(x\ y)^{-1}$ = y x.
P4. Suppose that x, y, z ∈ P. If x y and y z are defined, then x (y z) is defined, is which case
x (y z) = (x y) z.
P5. If w, x, y, z ∈P, and if w x, x y, y z, are all defined the either w (x y) or (x y) z is defined.

2. Bass-Serre Theory

Definition 2.1: A graph X is a pair $(V(X), E(X))$ of two disjoint sets of elements; a non-empty set $V(X)$, called vertices and a set $E(X)$, called edges, with a function $t: E(X) \to V(X)$ and a function $E(X) \to E(X)$ denoted by $e \to \bar{e}$ such that $e = \bar{\bar{e}}$for all e in $E(X)$, $\bar{e}$ is called the inverse of e, e is not necessarily different from $\bar{e}$.
We define, $o(e) = t(\bar{e})$, so that $o(\bar{e}) = t(e)$. $o(e)$ and $t(e)$ are called the endpoints of the edge e . $o(e)$ is the origin of e, and t (e) is the terminal of e . An edge e with $o(e) = t(e)$ is called a loop. A loop e, for which $\{e = \bar{e}\}$ is called an "inversion loop"
Definition 2.2: A pair of edges $\{e, \bar{e}\}$ is called an unoriented edge.
Definition 2.3: An orientation of a graph X is a set R consisting of exactly one member of each unoriented edge $\{e, \bar{e}\}$ for which $\neq \bar{e}$, together with every edge $e = \bar{e}$.
Definition 2.4: A graph Y is a sub graph of a graph X , if $V(Y) \subseteq V(X)$, $E(Y) \subseteq E(X)$, and if $e \in E(Y)$, then $o(e)$, $t(e)$ and $\bar{e}$ are defined and have the same meaning in Y as they have in X. We write $Y \subseteq X$.
Definition 2.5: A path P of length n in a graph X is a finite sequence of edges:
$P = e_1 \dots e_n$, $n \geq 1$, such that , $t(e_i) = o(e_{i+1})$ for $i = 1, \dots, n-1$.
$o(P) = o(e_1)$ and $t(p) = t(e_n)$ and we say that P is a path from $o(e_1)$ to $t(e_n)$.
The path P is closed if $o(e_1)$ = (e_n) , and reduced if $e_{i+1} \neq \bar{e}_i$ for $i = 1$,…,n-1. For each vertex v of X, we define an empty path 1_v of length zero (i.e. a path without edges) from v to v . The inverse $p^{-1} = \bar{e}_n \dots \bar{e}_1$
Definition 2.6: The product of two path $P_1 = e_1 \dots e_n$ and $P_2 = e_1' \dots e_m'$ in X , such that $t(P_1) = o(P_2)$, is defined by :$P_1 P_2 = e_1 \dots e_n e_1' \dots e_m'$
Definition 2.7: A circuit is a non-empty reduced closed path, such that the terminals of any two different edges are different.
Definition 2.8: A graph X is connected if for each pair of vertices u,v in X, there is a path from u to v.
Definition 2.9: A tree in X is a connected subgraph T of X which contains no circuit.
Definition 2.10: A maximal tree T in a connected graph X, is a subtree in X which is maximal with respect to inclusion.
It can be shown that if T is a tree in a connected graph X, then T is a maximal tree if and only if $V(T) = V(X)$.
Definition 2.11: If X and Y are graphs, then a morphism $f: X \to Y$ is a mapping which takes vertices to vertices, edge to edges, such that $f(o(x)) = o(f(x))$ and $f(\bar{x}) = \overline{(f(x))}$ for all edge x in X.
f is called an isomorphism if it is one-one and onto. An isomorphism $f: X \to X$ is called an automorphim of X. The automorphim of X form a group, denoted by Aut X.
Definition 2.12: A group G acts on a graph X, if there is a homomorphism $\emptyset \to G\ Aut\ X$.
If x is a vertex or an edge in X, $g \in G$; we write gx for $\emptyset(g)(x)$.
If x is an edge, then $(x) = g\ o(gx)$, $g\bar{x} = \overline{(gx)}$.
A group G acts without inversions on a graph X, if $gx \neq \bar{x}$ for any $g \in G$ and any $x \in E(x)$.
G acts with inversions on x, if $gx = \bar{x}$ for some $g \in G$ and any $x \in E(x)$.
Let $V(x)/G$ denote the set of G-orbits in $V(x)$ and $E(x)/G$ denote the set of G-orbits in $E(x)$.
The graph X/G whose vertices and edge are the G-orbits in $V(x)$ and $E(x)$, with induced inverses and origins, is called the quotient graph. The morphism $P: X \to X/G$ is called the projection.
Definition 2.13: Let $v \in V(X)$. The star of v is the set : $star\ (v) = \{x \in E(X): o\ (x) = v\}$.
If $f: X \to Y$ is a morphim of graphs and $v \in V(X)$then f induces a map $f_v : Star(v) \to star(f(v))$ by restriction. We say that f is star injective (surjective) if f_v is injective (surjective) for every $v \in V(X)$.
The following lemmas are proved in [4] and [9] and [8].
Lemma 2.1: Let $P: X \to Y$ be an onto graph morphism which is star surjective. Let T be a tree in Y. Then there is a morphism $q: T \to X$ such that pq is the identity on T.

Lemma 2.2: Let a group G act on a connected graph X with quotient $Y = X/G$. Let $p: X \to Y$ be the projection, and T a maximal tree of Y. Then there is a moephism $q: T \to X$ such that pq is the identity on T.

Lemma 2.3: Let x be a connected graph with sub graphs $X_1 \subseteq X_2 \subseteq X$. let g be a group acting on X and H a subgroup of G. Assume the following:

(1) If v_1 and v_2 are in $v(X_1)$, $g \in G$, and $gv_1 = v_2$, then $g \in H$.

(2) $GX_2 = X$

(3) $V(X_2) \subseteq H.V(X_1)$

Then H = G

The edges E(Y) of a graph Y consists of two disjoint sets. That is $\{e: e \in E(Y) = Y \in E(Y): y \neq \bar{y}\} \dot{U} \{\ell :$ $:\ell$ *is a loop and* $\ell = \ell \in EY$.

Definition 2.14: A graph of groups (Y, G_v, G_e) consists of :

(1) a connected graph Y.

(2) a group G_v associated with each vertex v in Y.

(3) a group G_e associated with each edge e in Y , with $G_e = G_{\bar{e}}$.

(4) a monorphim $\lambda_e : G_e \to G_{t\,(e)}$ for each e in Y, denoted $a \to a^e$.

(5) If $e = \bar{e}$ in Y , then there is an automorphism $\alpha : G_e \to G_e$ of order at most two such that the inner automorphim α^2 is determined by an element $a_0 \in G_e$ fixed by α.

Let G be a group acting on a connected graph X. Let $Y = X/G$. Let T be a maximal tree of $Y, p: X \to Y$ the projection and $q: T \to Y$ as in lemma 1.2. Following Bass-Serre, Chiswell [5] and Khanfar, Y can be made into a graph of groups.

Definition 2.15: Let T be a tree and (T, G_v, G_e) a tree of groups with the monomorphisms, $\lambda_y : G_y \to G_{t\,(y)}$ given by $a \to a^y$ and $\lambda_{\bar{y}} : G_{\bar{y}} \to G_{t(\bar{y})}$ given by $\to a^{\bar{y}}$, where $G_y = G_{\bar{y}}$.

The tree product of (T, G_v, G_e) is defined to be the free product of all G_v with additional relations $a^y = a^{\bar{y}}$, for all $y \in E\,(T)$ and all $a \in G_y$.

The tree product can be presented by $\langle G_v : rel\ G_v, a^y = a^{\bar{y}},\ v \in V(T),\ y \in E(T) \rangle$

Definition 2.16: Let (Y, G_v, G_e) be a graph of groups, and T a maximal tree of Y. The quasi-fundamental group $\prod$ of (Y, G_v, G_e) is given by :

$\prod = < G_v, t_y, t_\ell | rel\ G_v, t_y\, a^y t_y^{-1} = a^{\bar{y}}, t_y t_{\bar{y}} = 1, for\ y \neq \bar{y}, t_y = 1\ if\ y \in T, t_\ell\, c\, t_\ell^{-1} = a_\ell(c), t_\ell^2 = c_0 >$,

where $v \in V\,(Y), y \neq \bar{y}$ in $E\,(Y), a \in G_y, c \in G_\ell$ and ℓ an inversion loop in Y .

Thus II is a quasi H.N.N. extension with base the tree product of (T, G_v, G_ℓ) , stable letters t_y, t_ℓ and associated pairs $(G^y, G^{\bar{y}})$ for each pair of edge $\{y, \bar{y}\}, y \neq \bar{y}$ not in T and $(G_\ell, G_\ell^{\alpha_\ell})$ for each inversion loop.

By a similar situation in Serre [8], $\prod$ is independent (up to isomorphism) of T.

If there are no inversion loops in the graph, then the above definition reduces to:

Let (Y, G_v, G_y) be a graph of graphs, and T a maximal tree of Y. The fundamental group $\prod(Y, G_v, G_y)$ is defined to be the group with presentation $\prod(Y, G_v, G_y) = \langle t_y, G_v | rel\ G_v, t_y G_y^y t_y^{-1} = G_{\bar{y}}^{\bar{y}} t_y t_{\bar{y}} = 1\ all\ y, and\ t_y =$ *1 if y∈T*, where y runs over *EY, v* over *VY, Øy : Gyy→Gyy* is the isomorphism given by *ay→ay, a∈Gy*

If R is an orientation of Y, then $\prod = \langle t_y, G_v | rel\ G_v, t_y a^y t_y^{-1} = a^{\bar{y}}, t_j = 1\ for\ j \in E(T) \rangle$, where y runs through R $v \in V(Y)$.Bass [11] and Serre [8] constructed a graph $\tilde{Y}$on which $\prod$ acts without inversion, and showed that $\tilde{Y}$ is a tree. From this they proved that if the group G acts on tree without inversion, then G is isomorphic to $\prod$ and X is isomorphic to$\tilde{Y}$.

The above argument can be stated as follows:

Let G be a group acting on a connected graph X. Let $Y = X/G$, (Y, G_v, G_y) be the graph of group associated with the action of G on x, let II be the quasi –fundamental group of (Y, G_v, G_y) relative to a maximal tree T of Y. Let R be an orientation of Y and $\tilde{Y}$ be the tree constructed as above on which $\prod$ acts. If X is a tree, then $\tilde{Y} \cong x$ and $\prod \cong G$.

3. Some Constructions

A " tree of groups " , which should not be confused with tree product, consists of :

(a) A set I partially ordered by < which contains no infinite descending chain, has a least element, and such that for all $i , j , k \in I , if\ i \leq k , j \leq k$ then either $i \leq j$ or $j \leq i$.

(b) A class of groups$\{G_i\} , i \in I$.

(c) For all $i , j \in I , if\ i < j$, a monomorphism $\emptyset_{ij} = G_i \to G_j$, with the condition that for all $i , j , k \in I , if\ i < k , j < k$, then $\emptyset_{jk} . \emptyset_{ij} = \emptyset_{ik} G_i \to G_k$.

Identify $x \in G_i$, with $\emptyset_{ij}(x) \in G_j$, i.e. put an equivalence relation ~ on elements of $G = \dot{U}_{i \in I} G_i$, by :

$x \sim y \Leftrightarrow$ either for some $i \leq j , \emptyset_{ij}(x) = y$ or for some $i \leq j , \emptyset_{ij}(y) = x$.

Denote the equivalence classes containing x by [x] for all elements of $i \in \dot{U} G_i$.

Let $p = \{[x_i]: i \in I\}$. The product of two elements $[x_i] , [x_j] , x_i \in G_i , x_j \in G_i$is defined in P by

$[x_i][x_j] = [\emptyset_{ij}(x_i)x_j]$, where there exists a monomorphism $\emptyset_{ij} : G_i \to G_j$

$= [x_{i\,ji}(x_j)]$, where there exists a monomorphism $\emptyset_{ji} : G_j \to G_i$

Assuming that $\emptyset_{ii} = id_{G_i}$

This multiplication between elements of P is well defined. It is easy to show that P is a pregraoup.

Since the map $x \to [x]$ is a monomorphism of G_i into P, then each G_i is embedded in P.

Definition 3.1: The product $g_i g_j$ of two elements of G_i , G_j is not defined in neither of g_i , g_j is the image of an element in a factor comparable with both G_i , G_j .

Any element $g \in U(P)$ can be written uniquely as:

$g = g_{i_1} g_{i_2} \dots g_{i_n}$, where (1)

(i) No g_{i_j} is an image.

(ii) No $g_{i_j} , g_{i_{j+1}}$ is defined, i.e. $G_{i_j} , G_{i_{j+1}}$are not comparable where $g_{i_j} \in G_{i_j}$ and $g_{i_{j+1}} \in G_{i_{j+1}}$.

Now we make a further assumption that for the " tree of groups " given by Stallings , there is a unique tree T in which every edge is not the composite of any two monomorphism , $\emptyset_{ij} , \emptyset_{ik} , i < j < k$.

For a reduced from $g = g_{i_1}, g_{i_2}, \dots, g_{i_n}$ take the shortest path $u_1 u_2 \dots u_n$ in T starting in G_0, involving all vertices G_{i_j} in the correct order , and ending in G_0 where $g_{i_j} \in G_{i_j}$ in (1).

Define $P(g) = g_0 u_1 g_1 u_2 g_2 \dots . u_{i_1} g_{i_1} u_{i_2} \dots u_{i_n} g_{i_n} u_{i_{n+1}} \dots u_{i_{2m}} g_{2n}$ (2)

Where u_i is an edge directed away from $G_{i-1} , g_r \in G_r\ and\ g_r = 1$ except that $g_r = g_{i_j}$in the appropriate position on the path.

Definition 3.2: $g_0 u_1 g_1 u_2 g_2 \dots . g_{2_{m-1}} u_{2m} g_{2m}$, is reduced if $u_1 \dots . u_{2m}$, is a path and

(i) $g_j = 1 \to u_j \neq u_{j+1}$

(ii) g_r , g_{r+t} are successive non-identity terms in $g \to g_r g_{r+t}$is not defined $t \geq 2$.

(iii) No non-identity term is an image.

The following properties can be deduced from (i), (ii) and (iii):

(iv) Two successive terms g_r , g_{r+t} cannot be non-identities.

(v) For two successive non-identities g_r , g_{r+t} for some $t \geq 2$, in the subsequence $u_r g_r u_{r+1} \dots u_{r+t} g_{r+t}$, we have $g_r g_{r+t}$ is in reduced from

(vi) m is the minimum , i.e. no shorter path has these properties.

(vii) If g_j is any non- identity terms in (2) , then $u_j = u_{j+1}^{-1}$

(viii) The length of the path is even.

(ix) g_0 and g_{2n} are the identity of G_0 for $m \neq 0$.

Definition 3.3: Let U(P) be the universal group of Stallings tree for $g \in U(P)$ in reduced form , define $|g| = {}^1/_2$ the length of the path P(g) given in (2).

$|\ \ | : U(P) \to R,$ satisfies

$A1'$, $|1| = 0$,1 identity element of $U(P)$.
$A2$ $|g| = |g^{-1}|$
For A4, suppose
$P(x_1) = g_0u_1g_1u_2g_2 \dots g_{2n-1}u_{2n}g_{2n}$
$P(x_2) = h_0v_1h_1v_2h_2 \dots h_{2m-1}v_{2n}h_{2m}$ and
$P(x_3) = k_0w_1k_1w_2k_2 \dots k_{2\ell-1}w_{2\ell}k_{2\ell}$, are all reduced,
Then $|x_1| = n$, $|x_2| = m$ and $|x_3| = \ell$.
Then consider the following

$$g_0u_1g_1u_2g_2 \dots g_{2n-1}u_{2n}g_{2n}h_{2m}^{-1}v_{2m}^{-1}h_{2m-1}^{-1}v_{2m-1}^{-1} \dots v_1^{-1}h_0^{-1} \quad (1)$$

Note that if $u_{2n} \neq v_{2m}$, then (1) is reduced and $|x_1x_2^{-1}| = n + m$.
Now (1) is reduced unless $u_{2n} = v_{2m}$ in which case $g_{2n-1}h_{2m-1}^{-1}$ is also defined.
If so, then put $g_{2n-1}h_{2m-1}^{-1} = a_1$
Then $g_0u_1g_1u_2g_2 \dots g_{2n-2}u_{2n-1}a_1v_{2m-1}^{-1} \dots v_1^{-1}h_0^{-1}$ is reduced unless a_1 is an image and $u_{2n-1} = v_{2m-1}$, in which case a_1 is after identification in the same factor as g_{2n-2} , h_{2m-2}^{-1} and so $g_{2n-2}a_1h_{2m-2}^{-1}$ is also defined and equal to a_2 say.
Suppose $d(x_1, x_2)$ and $(x_2, x_3) \geq r$, then by induction on i, there exists a_r such that a_{r-i} are images.
$u_{2n-i-1} = v_{2m-i-1}$, $a_i = g_{2n-i}\, a_{i-1}h_{2m-i}^{-1}$ is after identification defined , for all $0 \leq i \leq r$.
Similarly, there exists b_r such that b_{r-i} are images,
$v_{2m-i-1} = w_{2\ell-i-1}$, $b_i = h_{2m-i}b_{i-1}k_{2\ell-i}^{-1}$ is defined , for all $0 \leq i \leq r$.
Thus $u_{2n-i-1} = w_{2\ell-i-1}$, $0 \leq i \leq r$.
Moreover, since for any $0 \leq i \leq r - 1$, both a_i , b_i are in the same factor and are images , therefore a_ib_i is defined and is an image.
Let $c_i = a_ib_i = g_{2n-i}\, a_{i-1}h_{2m-i}^{-1}h_{2m-i}b_{i-1}k_{2\ell-i}^{-1} = g_{2n-i}\, c_{i-1}k_{2\ell-i}^{-1}$
Hence $d(x_1, x_3) \geq r$
Therefore A4 is satisfied.
Another example of a pregroup can be constructed by using pregroups instead of groups in the tree with some extra condition follows:

(a) A set I, partially ordered by<, which contains no infinite descending chain , has a least element , and such that for all $i, j, k \in I$, $if\ i \leq k, j \leq k$ then either $i \leq j\ or\ j \leq i$.
(b) A class of pregroups $\{p_i\}$, $i \in I$ such that each $i < j$ some j , P_i satisfies the following:
If xy and yz are both defined, then x (yz) is defined in P_i for $x, y, z \in P_i$.
(c) For all $i, j \in I$ if $i < j$, a monomorphism $\emptyset_{ij} : P_i \to P_j$, such that $\{\emptyset_{ij}\}$ satisfies the following :
If $i < j < k$, then $\emptyset_{jk}\ \emptyset_{ij} = \emptyset_{ik} : P_i \to P_j$
(d) If $x_1, x_2 \in p_i$, and $\emptyset_{ij}(x_1)\, \emptyset_{ij}(x_2)$ is defined in P_j , then x_1x_2 is defined in P_i.

Put a relation ~ on elements of $\dot{\bigcup}_{i \in I} P_i$ by :
$x{\sim}y \Leftrightarrow$ either for some $i \leq j$, $\emptyset_{ij}(x) = y$, or for some $i \leq j$, $\emptyset_{ij}(y) = x$
Let R be the equivalence relation on $\dot{\bigcup}_{i \in I} P_i$ generated by ~ .
Denote the equivalence class containing x by [x] for all elements $x \in \dot{\bigcup}_{i \in I} P_i$.
Let $P = \{[x_i] : x \in UP_i, i \in I\}$, then P is a pregroup .
The product of two elements $[x_i][x_j]$, $x_i \in P_i$, $x_j \in P_j$ is defined by:
$[x_i][x_j] = [\emptyset_{ij}(x_i)x_j]$ or $= [x_i\, \emptyset_{ji}(x_j)]$

When there exists monomrphism $\emptyset_{ij} : P_i \to P_j$ and $\emptyset_{ij}(x_i)x_j$ is defined in P_j, or there exists monomrphism $\emptyset_{ji} : P_j \to P_i$ and $x_i\emptyset_{ji}(x_j)$ is defined in P_i.This multiplication is well defined.

References

[1]. Lyndon, R.C.; Length Function in Groups, Math. Scand, 12, 1963, 209-234.
[2]. Chiswell, I.M.; Abstract Length Function in groups, Math. Proc. Camb. PhilSoc., 80, 1976, 451-463.
[3]. Chiswell, I.M.; Length Function and Free products with amalgamation of groups´, Math. Proc. Camb. Phil. Soc, (3), 1981, 42-58.
[4]. Chiswell, I.M.; On Groups acting on Trees, Ph.D Thesis, University of Michigan, 1973.
[5]. Hoare, A. H. M.; On Length Functions and Nielson Methods in Free Groups, J. London Mathematical Society, (2), 14, 1976, 188-192.
[6]. Wlikens, D.L.; On Non Archimedian length in Groups, Mathematika, 23, 1976, (57-61).
[7]. Stallings, J.R.,; Group Theory and Three Dimensional Manifolds, Yale University Press, New Haven and London, 1971.
[8]. Serre, J. P., Trees, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[9]. Khanfar, M. M.I.; Combinatorial Properties of Groups with Length Function Ph.D. Thesis, University of Birmingham. U.K, 1978.
[10]. Nesayef, F.H.; Presentation of groups generated by elements of length zero and one, International journal of Mathematics and Computer Applications Research, Vol.4, Issue 6, 2014, (45-52.
[11]. Bass, H., Some Remarks on Group Actions on Trees, Comm. in Algebra, 4 (12), 1976, 1091-1126.