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The Mathematics of Geometric Symmetry in Science and Society

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Abstract Geometric symmetry transformation is a mathematical operation that leaves an object, its position and orientation unaltered. The object is any geometrical object which represents a living or non-living organism which can be microscopic or macroscopic particle. Symmetry is a fundamental attribute of nature and environment. The branch of mathematics that describes and represents geometric symmetry in nature, science and society is group theory and in particular Lie group theory. The aim of this article is to state and highlight various geometric symmetry transformations and their application in various domains of science and society. The specific objectives are to study the role of geometric symmetry transformations in physics, chemistry, biology, mathematics and society especially in the arts and the social sciences.

Keywords Geometric Symmetry, Symmetric Transformations, Science, Society, Group

1. Introduction

Geometric symmetry is a mathematical transformation that produces a figure identical to the original or a mirror image of the original figure. Symmetry operations are defined with respect to a given point (center of symmetry), line (axis of symmetry), and plane (plane of symmetry). In mineralogy and chemistry the laws of symmetry apply to the angular structure of crystals. All classes of crystal are divided into six systems that are based on the length of their axes and other details of symmetry. In physics, a system is said to exhibit symmetry if it remains unchanged in the course of operations such as mirror reversal, reversal in the direction of time, and space-time translation. Many physical systems obey such symmetries, to which the conservation laws of physics are also related. This relationship has come to be of particular importance in particle physics, where certain symmetries called internal symmetries are observed. Such symmetries exist in the mathematical "space" of that realm and underlie the conservation of such quantities as charge, parity, baryon and lepton number, and total strangeness, even as certain particles are substituted for one another. In current theoretical physics, however, such symmetries are now known to be only approximate, except for baryon and lepton number, that is, they are violated in their physical manifestations. When internal symmetries do not operate the same way but instead can be different at each point in space-time, they are called gauge symmetries. Theorists currently hope to reduce all such symmetries to gauge symmetries in their effort to develop a grand unification theory (GUTS) that can incorporate all of the fundamental interactions of matter [1]. In the biological sciences symmetry is seen in most plants and animals. In most applications groups representing geometric symmetry it is matrix representation of the groups that are used for the explanation of symmetries in physical systems, such as application of matrix representation of finite groups to explain symmetries in elementary particle physics [2]. In arts and social sciences symmetry is used in designs as well as socio-political alliances and relationships to produce balance, order, beauty and elegance.

2. Preliminaries and Basic Definitions

[i] Symmetry [3]**:** Symmetry operations or transformations are those mathematical transformations that produce a figure identical to the original or a mirror image of the original figure.

[ii] Geometric symmetry: This is a transformation that leaves a geometric object invariant.

[iii] Symmetry group: The symmetry group of a physical system is the set of all transformations of the system which leaves it dynamically invariant or unchanged. The set S_n of all permutations of {1,2,3…..,n} is called the symmetric group on n letters. The order of S_n is n!.

[iv] Elementary particles: These are particles at the atomic and subatomic levels.

[v] Matrix representation: This is the representation of a group by a matrix which is a rectangular array of numbers enclosed in brackets.

[vi] Special unitary group: A matrix is unitary if we have $A^{*}= A^{-1}$ that is to say the complex conjugate/transposed is equal to the inverse. The trace of a matrix is the sum of its leading diagonal elements. That is tr(A) = $\sum_{i=1}^{n} A_{ii}$, while the defining property of a real orthogonal matrix is $A^{T} = A^{-1}$. The set of complex n x n unitary matrices forms a group which is denoted by U(n). If the determinant of the matrix representation is +1 the group is denoted by SU(n) and is referred as special unitary group.

3. Geometric Symmetry in the Sciences

In physical science *a complete symmetry arises between space and time*, and this new feature leads to important physical consequences. Action or transformational principles are widely used to express the laws of physics, including those of general relativity. For example, freely falling particles move along geometric geodesics, or curves of extremal path length. Symmetry transformations are changes in the coordinates or variables that leave the action invariant. It is well known that continuous symmetries generate conservation laws in science. Conservation laws are of fundamental importance in physics and so it is valuable to investigate symmetries or symmetric groups produced by symmetric action or transformations. Since the laws of physics are the same at any time (symmetry in time) and any location (symmetry in position) as another, the laws of conservation of energy and momentum are applicable [4].

It is useful to distinguish between two types of symmetries: dynamical symmetries corresponding to some inherent property of the matter or space-time evolution (e.g. the metric components being independent of a coordinate, leading to a conserved momentum one-form component) and non-dynamical symmetries arising because of the way in which we formulate the action. Dynamical symmetries constrain the solutions of the equations of motion while non-dynamical symmetries give rise to mathematical identities. Among the symmetries important in nuclear science are parity P, time reversal invariance T and charge conjugation C (CPT Symmetry) as well as charge independence, charge symmetry and isospin symmetry [5].

Quantum chemistry and mathematical group [6] theory are the modern bases of symmetry considerations in stereochemistry. In quantum chemistry the symmetries of molecular systems are represented by the symmetries of the corresponding molecular Hamiltonian operators. In stereochemistry the structure of molecules is classified by the *symmetry transformations of point groups.* The symmetries of a free molecule can be completely defined by a few types of symmetry transformations. In general, the selection of the three coordinates axes *x*, *y*, *z* is arbitrary. The trivial symmetry transformation is *identity* I which leaves each molecule unchanged. An additional symmetry element is the *axis of rotation* C_n around which a molecule can be rotated by the angle 2π/n without changing its position. Linear molecules, in which all atomic nuclei lie on a straight line (*e.g.* nitrogen N≡N or carbon monoxide C≡O), can be rotated around the connecting axis by arbitrarily small angle and have a continuous axis of rotation with infinite fold symmetry $n \rightarrow \infty$. An additional symmetry element is the *reflection* σ on a plane in which the molecule does not change its position. For example, if the *xy*-plane is the plane of reflection, then replacing all the atomic *z*-coordinates by -*z* does not change the position of the molecule. Depending on the selection of the plane of reflection, a distinction is made between a vertical plane of reflection σ_n and a horizontal plane of reflection σ_h. The next symmetry element is *inversion* in which a molecule remains unchanged during a reflection of all atomic coordinates (*x*, *y*, *z*) at the point of inversion to (-*x*, -*y*, -*z*). An additional symmetry element is *rotary reflection* $S_n = \sigma_h C_n$ in which a molecule is first rotated by an angle $2\pi/n$ around the rotary reflection axis C_n and then reflected on the plane σ_h perpendicular to C_n

through the center of the molecule, without changing its position. The remaining symmetry element is *rotary inversion* in which a molecule does not change its position in spite of rotation followed by inversion. It should also be noted that the compound symmetry transformations of rotary reflection and rotary inversion do not presuppose the partial transformations of rotation, reflection or inversion as symmetry elements of the same molecule. The symmetry transformations of a molecule, when executed one by one, produce symmetry transformations in turn and define as a whole the symmetry structure of the molecule by the mathematical group of these symmetry transformations.

In general, mathematical symmetries are defined by so-called *automorphisms* [7] that means self-mappings of figures or structures whereby the structure remains invariant (example: rotation or reflection of polygons in the plane). The composition of automorphisms satisfies the axioms of a mathematical group. So the symmetry of a molecular structure is defined by its group of automorphisms. There are continuous groups of symmetries (for instance circles and spirals) and discrete groups (for instance, regular polygons, ornaments, Platonic bodies). On account of the finite number of combinations of symmetry elements, it is clear that there can only be a finite number of *point groups.* Thereby many different molecules can belong to the same point group, *i.e.* they can have the same symmetry structure. The classification of point groups also makes it possible to explain the relationship of optical activity and molecular structure in terms of group theory. Symmetric and Lie groups with the corresponding matrix representations have been widely applied in elementary particle and condensed matter physics, hyper elasticity, atomic spectroscopy, crystallography, gauge theories and representation of special functions of mathematical physics. In matter and particle physics group theory is used in classification and identification of elementary particles and antiparticles e.g Bosons, photons, mesons leptons, nucleons, neutrinos etc.

The unitary and special unitary groups of order n given respectively by U(n) and SU(n) are symmetric groups that have far reaching implications for particle and condensed matter physics. The group U(n) is of great importance in gauge theories for the electroweak interactions as well as for the Grand Unification Theories (GUTS). In particular the U(1) symmetric group is an invariance principle in quantum electrodynamics. The SU(2) is called an isotopic or isospin or strong interaction symmetry group for the nucleon. As a transformation group it can be used to switch the electric charge of the proton or nucleus.

In quantum mechanics the assumption that space is homogenous or possesses translational symmetry leads to the conclusion that the linear momentum of a closed isolated system does not change as the system moves. This makes it possible to study separately the motion of the centre of mass and the internal motion of the system. The assumption that space is isotropic or possesses rotational symmetry means that the total angular momentum of such a system is constant. In the chemistry of dynamical symmetries we get unexpected degeneracies of energy levels of the hydrogen atom and the isotropic harmonic oscillator. If two or more energy states or levels in an atom have the same numerical value of energy they are said to be degenerate. A free electron possesses space displacement symmetry while the electron in hydrogen atom does not. Important symmetries in physics include continuous symmetries and discrete symmetries of space-time; internal symmetries of particles as well as super symmetry of physical theories. Symmetry in physics is generally used to separate the analysis of a physical phenomenon into two parts: the initial conditions and the laws of nature that governs the phenomenon. The applications of symmetry in physics leads to the conclusion that certain physical laws, particularly conservation laws governing the behavior of objects and particles are not affected when their geometric coordinates including time are transformed by means of symmetric transformations. In particle physics symmetry can be used to derive conservation laws and analyze particle interactions. Symmetry feature prominently in the physics of relativity and quantum mechanics [8]. A very important example of symmetry in relativistic physics is that the speed of light has the same value in all frames of reference which is known as Poincare group, the symmetry group of special relativity. The property of symmetry describes certain physical phenomena, geometric shapes and mathematical equations that remain unchanged despite changes in orientation or other properties. Symmetry also is responsible for the orderly, mutually corresponding arrangement of various parts of a body, whether in animals or plants resulting in a proportionate and balanced form.

The principle of symmetry is of great importance in the fields of biology, mathematics, physics, chemistry and mineralogy. In biology, the regular distribution of various parts of an animal's body on two opposite sides of a linear axis, or a median plane, is known as bilateral symmetry. The proportional arrangement of similar parts of a body around a central axis, as in the case of jellyfish or starfish, is known as radial symmetry. The bodies of protozoans, such as those of the order Radiolaria, which have a round form about a central point or nucleus, are said to have a spherical symmetry. In plants and flowers symmetry is quite evident. In geometry, symmetry is a feature of certain plane and solid shapes.

4. Groups Defining Geometric Symmetry

Definition 4.1: A permutation of a set S is a bijection on S, that is, a function f: S →S that is one to-one and onto. For example, if S = {1,2,3,4,5}, then

$\begin{bmatrix} 1,2,3, & 4,5 \\ 3,5,4,1,2 \end{bmatrix}$ is the permutation such that f(1)=3, f(2)=5, f(3)=4, f(4)=1, f(5)=2. We express the same permutation as f: 1→ 3→ 4→ 1, 2→ 5→ 2 or f = (1 3 4)(2 5).

Definition 4.2: the set S_n of all permutations of {1,2,3…..,n} is called the symmetric group on n letters. The order of S_n is n!. As an example we discuss the dihedral group D_{2n} which is group of symmetries of a regular polygon of n-sides under a combination of rotations and reflections. For n=5, the group D_{2n} = D_{10} consists of the symmetries of the pentagon which are the permutations that can be realized via a rigid motion (a combination of rotations and reflections).

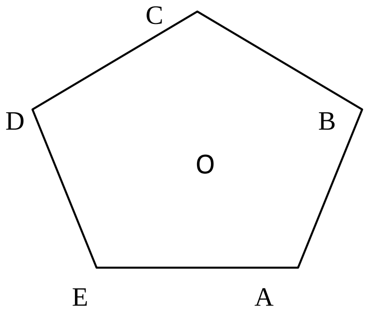

Figure 1

All the symmetries can be generated by two basic operations R and F, where R is anti-clockwise rotation by $\frac{360}{n}$ = $\frac{360}{5}$ = 72 degrees and F ("flip") is reflection about the line joining the centre of the pentagon O to the vertex considered as the first vertex (which is OA in fig 1). The group D_{10} contains 10 elements which consists of 5 rotations I, R, R^2, R^3, R^4 and n reflections F, RF, R^2F, R^3F. R^4F.

5. Groups and their Representations

A group is a set of objects or operations equipped with a binary operation that satisfy closure, associativity, existence of identity and existence of inverse elements. A group that is commutative is called an Abelian group. For example consider the set {E, A, B, C} that combine according to the following group multiplication table.

.	E	A	B	C
E	E	A	B	C
A	A	B	C	E
B	B	C	E	A
C	C	E	A	B

The elements of the set {E, A, B, C} form an Abelian group where the elements are abstract mathematical entities. A representation of the group is a set of particular objects {E, A, B, C} that satisfy the multiplication table. For example consider the following representation:

C_4 = {E= 1, A= i, B= -1, C= -i} where the combination rule is ordinary multiplication. The group is both Abelian and cyclic where i the imaginary unit $\sqrt{-1}$ is the generator. Another representation is given by the successive rotation of 90^0 in a plane. Remember the matrix that represents a rotation through angle φ in 2-dimensions is given by:

$$\begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}$$

With $\varphi = 0, \frac{\pi}{2}, \pi, 3\pi/2$, we have

E = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Thus we have seen two representations of the group with the multiplication table above i.e. the group elements and matrix elements.

The representation of group elements by matrices is a very powerful technique and has been almost universally adopted among scientists, especially physicists. It can be shown that all the elements of finite groups and continuous groups of the type important in physics can be represented by unitary matrices. A matrix is unitary if we have $A^* = A^{-1}$ that is to say the complex conjugate/transposed is equal to the inverse. The trace of a matrix is the sum of its leading diagonal elements. That is tr(A) = $\sum_{i=1}^{n} A_{ii}$, while the defining property of a real orthogonal matrix is $A^T = A^{-1}$. The set of complex n x n unitary matrices forms a group which is denoted by U(n). If the determinant of the matrix representation is +1 the group is denoted by SU(n) and is referred as special unitary group.

6. Faces of Geometric Symmetry

The basic symmetry operations used to describe or transform geometric objects or particles in the physical sciences include the following:

6.1: Reflection in a mirror plane denoted by σ.

6.2: Rotation about an axis denoted by C.

6.3: Rotary-reflection which consists of a rotation about an axis followed by or preceded by a reflection in a plane perpendicular to the axis which is denoted by S.

6.4: Inversion through a central point, denoted by i.

6.5: The identity corresponding to the return of all the points to their original positions and denoted by E.

7. Functions Representing Geometric Symmetry

7.1: symmetry about the y-axis

F(x,y) = F(-x,y) e.g $y=x^2$

7.2: symmetry about the x-axis

F(x,y)=F(x,-y) e.g $x=y^2$

7.3: symmetry about the origin

F(x,y)=F(-x,-y) e.g $y=x^3$

7.4: symmetry about the 45^0 line F(x,y) = F(y,x) e.g $\sqrt{x} + \sqrt{y} = 1$

7.5: multiple symmetries

e.g (i) $x^2 + y^2 = 1$ is symmetric about x-axis, y-axis, origin and the line y = x (ii) $x^2 - y^2 = 1$ is symmetric about x-axis, y-axis and the origin. (iii) xy = 1 is symmetric about the origin and the line y = x

8. Geometric Symmetry and Society

Geometric symmetry reflects beauty, order, balance and elegance which are found in both natural and artificial systems and objects. In artificial objects especially in designs of fabrics and physical structures symmetry feature prominently. In particular symmetry is used in art, decorative design, buildings and architecture. symmetry often appears as in the design of bridges, buildings, and auto mobiles and in the artistic patterns of quilts, wallpaper, and even sidewalks. The development of Magnetic Resonance Imaging (MRI) in medicine illustrates the process of movement from symmetry to asymmetry. In 1946, two scientists, independently of each other, found that certain nuclei in the periodic system when placed in a magnetic field absorbed energy in the radiofrequency range and re-emitted this energy as the nuclei relaxed to their original orientation. Because the strength of the magnetic field and the radiofrequency must match each other, the phenomenon was called *nuclear magnetic resonance*: *nuclear* because it is only the nuclei of the atoms that react; *magnetic* because it happens in a magnetic field; and *resonance* because of the direct dependence of field strength and frequency.

The symmetry of stability of the spin of the nuclei of atoms was perturbed by a large magnetic field tuned to a radiofrequency range moving the nuclei into instability asymmetric to their normal movement). P. C. Lauterbur [9] suggested the use of this phenomenon for medical imaging by adding a second, weaker magnetic field, the gradient field, to pick up the re-emitted signal [9]. In the development of MRI we see the interaction of symmetry and asymmetry in science and technology on two levels: (1) the perturbation of symmetry into asymmetry in the nucleus; and (2) the employment of biotechnology to produce an MRI machine that can now be used to image the symmetries and asymmetries of the human body. If one looks at the resting state positron emission tomography (PET) scan or a resting state MRI scan of the brain, at almost every level, the brain seems to possess a left-right dihedral reflective symmetry. Organs on the left are matched by the same organs on the right. And some regions like the hippocampus exist in a horseshoe-like ring symmetrically divided in the middle with half on the left and the other half on the right. From this structural symmetry one might infer that the symmetry which we see in the external physical world arises from the symmetrical functions occurring in the symmetrical structure of the brain. We have already seen that in nonlinear systems creativity often takes place in the transitions from symmetry to asymmetry and vice versa. Humans continue to evolve in both biological evolution through minor mutations and cultural evolution through new adaptations to society [10].

9. Conclusion

We have studied how significant geometric symmetric transformations are in science and society. We have discovered an interaction of symmetry and asymmetry between science, technology and society on three levels: (1) the mental world in which science and technology exist in theories and designs; (2) the physical and social world; and (3) the mind and brain where creativity arises. We also know that mathematical instruments like chaos theory can best represent the emergence of creativity from the interaction between symmetry and asymmetry. In this paper an attempt has been made to present geometric symmetry and the corresponding groups as useful tools in science and society. The paper also highlighted the description and significance of geometric symmetry and symmetric groups in biology chemistry, physics and mathematics.

10. Recommendations

Based on the study we recommend the following:

(1) Deeper studies should be done on the mathematics and applications of geometric symmetry and symmetric transformations.

(2) Multidisciplinary research especially among scientists should be carried out to analyze the role of symmetry in understanding natural systems and behavior.

(3) Studies should be carried out on the significance of symmetry in crafts (fabrics, pottery etc), arts (languages, history, philosophy), culture (dances, music, food etc) and society (relationships, politics, values etc).

(4) Research centers and academic institutions especially the National Mathematical Centre, Abuja should mount courses and conferences on the theory and applications of symmetry in science and society.

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