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Research Article

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Valuation of the European Options within the Black-Scholes Framework using the Hermite Polynomials

O.L. Babasola, I. Irakoze, A.A. Onoja

Pan African University Institute for Basic Sciences, Technology and Innovation, Kenya

Abstract This paper emphasis on the derivation of the Black-Scholes formula for the valuation of the European options using Hermite polynomial basis. The work categorized the Hermite polynomials into the probabilists' and Physicists' Hermite polynomial and thereby uses the generating function of the probabilists' Hermite polynomials to obtain the Black-Scholes formula for call and put which is generally used in the valuation of the European options.

Keywords Black-Scholes model, European Options, Hermite Polynomials, Option Pricing

1. Introduction

The valuation of option has been of great importance in financial mathematics. An option can either be a call or a put. The main usefulness of an option is to give the holder the right (not obligation) to buy or sell an underlying asset for a predetermined price often called the strike price during a particular period of time. A standard option offers the right to buy (call) or sell (put) an underlying security by a certain date at a set strike price. In comparison to other option structures, standard options are not complicated. Such options are easy to trade and are well-known in the markets. However, the term standard option is a relative measure of complexity, especially when business men and investors are considering various options and structures. Examples of standard options are American options which allow exercise of right at any point during the life of the option and European options that allow exercise to occur only at the maturity date. The first popular mathematical model for pricing European options is the Black-Scholes model which used the assumption of no arbitrage argument to arrive at the fair price of the option. In their work, they considered the stock price process as a Geometric Brownian Motion in order to obtain a closed form expression for the European option price.

2. Literature Review

Different literature explained the applications of orthogonal polynomials and in particular, the Hermite polynomials. [1] discuss how the orthogonal polynomials can be constructed using the basic orthogonality properties and thereby uses the zeros of the constructed orthogonal polynomials as the point of collocation for solving an integral equation. In [2], the contingent claims were priced using the basis of family of Hermite polynomials. However, [3] extend and test the approach of [2] for pricing contingent claim where they use the model to price options on Eurodollar future. [4] wrote a paper on Hermite polynomials based expansion of European options where he seek the closed-form series expansion of option prices to know how the drift, volatility and jump components of underlying risk-neutral dynamics are been translated into option prices. [5] estimate the Neutral Density from option prices and Subjective Density from underlying assets using the Hermite polynomials' expansion.

However the breakthrough made by Fisher Black and Myron Scholes together with Robert Black in the stock option pricing in 1973 is still the most popular model used in the valuation of option especially the European option. According to [6], the stochastic integrals of the Hermite polynomials evaluated in Brownian motion is of great importance in the Black-Scholes option pricing model. The famous Black-Scholes model uses the special martingales which are greatly related to the generating function of Hermite polynomials [7]. In this paper, we present the classic representation of Hermite polynomials and use the generating function of the polynomial to obtain the Black-Scholes formula.

3. Hermite Polynomials

The Hermite polynomials belong to the family of the classical orthogonal polynomial sequence which arise in probability such as the Edgeworth series; in numerical analysis such as Gaussian quadrature, finite methods as shape functions for beams; in combinatorics as an example of an Appell sequence which obeys the umbral calculus; and in physics where they give rise to the eigenstates of the quantum harmonic oscillator [8]. These polynomials were first defined by Laplace in 1810 and then studied in detail by Chebyshev in 1859. However, Chebyshev's work was overlooked and they are later named after Charles Hermite in 1864 who made an improvement on the polynomials.

According to [8], there are two distinct way of standardizing Hermite polynomials and these are known as the probabilists' Hermite polynomials denoted by $H_{e_n}(x)$ and the physicists' Hermite polynomials denoted by $H_n(x)$. These two polynomials are defined mathematically as

$$H_{e_n}(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \tag{3.1}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{3.2}$$

However, it is worth noting that the two definition given in (3.1) and (3.2) are not exactly the same but each is a rescaling of the other. that is,

$$H_n(x) = 2^{\frac{n}{2}} H_{e_n}(\sqrt{2}x)$$

and

$$H_{e_n} = 2^{-\frac{n}{2}} H_n(x)\left(\frac{x}{\sqrt{2}}\right)$$

Without loss of generality, the polynomials H_{e_n} are sometimes denoted by H_n especially in probability theory because the probability density function for the standard normal distribution is $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ which is similar to the weight function of the polynomial.

Using MATLAB code, the first six probabilists' and physicists' Hermite polynomials are given below:

$$H_{e_0}(x) = 1$$

$$H_{e_1}(x) = x$$

$$H_{e_2}(x) = x^2 - 1$$

$$H_{e_3}(x) = x^3 - 3x$$

$$H_{e_4}(x) = x^4 - 6x^2 + 3$$

$$H_{e_5}(x) = x^5 - 10x^3 + 15x$$

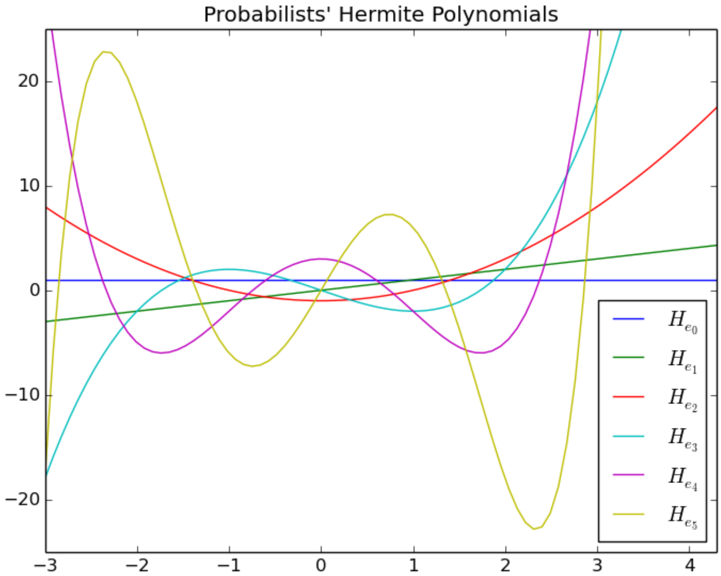

Figure 1: The first six probabilists' Hermite polynomials $H_{e_n}(x)$

for the physicists', we have

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_5(x) = 32x^5 - 160x^3 + 120x$$

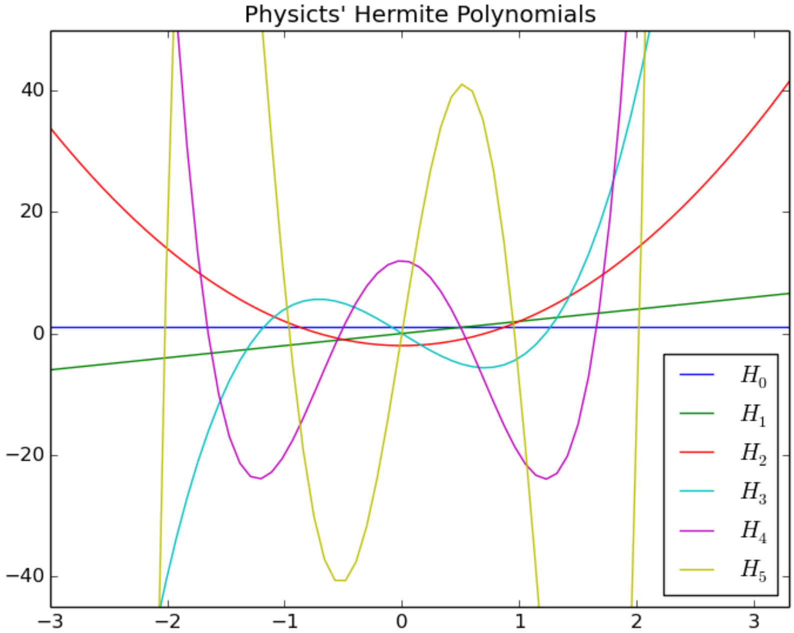

Figure 2: The first six physicist' Hermite polynomials $H_n(x)$

3.1. Probabilists' Hermite Polynomials

The monic Hermite polynomials are often referred to as the probabilists' Hermite polynomials, denoted by He_n, and can be expressed as

$$He_n(x) = n!\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2k}}{2^k(n-2k)!}\frac{(-1)^k}{k!} \tag{3.3}$$

and are orthogonal in $(-\infty,\infty)$ with weight function $w(x) = e^{\frac{-x^2}{2}}$.

They satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2} He_m(x)He_n(x)dx = n!\sqrt{2\pi}\delta_{mn}$$

and their exponential generating function is given by

$$e^{\left(xu-\frac{u^2}{2}\right)} = \sum_{n=0}^{\infty} H_n(x)\frac{u^n}{n!}$$

The three-term recurrence relation is given by

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x).$$

3.2. Normalized Hermite Polynomials

If the Hermite polynomial $He_n(x)$ is normalized and the normalized form is denoted by $\widetilde{He}_n(x)$, then we can express $\widetilde{He}_n(x)$ as [3]:

$$\widetilde{He}_n(x) = \frac{He_n(x)}{\sqrt{n!}}. \tag{3.4}$$

The polynomials are orthogonal in the interval $(-\infty,\infty)$ with respect to the weight function $w(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$, and they satisfy the condition

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2} \widetilde{He}_m(x)\widetilde{He}_n(x)dx = \delta_{mn}.$$

The normalized recurrence relation of the Hermite polynomials $\widetilde{He}_n(x)$ is given by [9]

$$x\widetilde{He}_n(x) = \widetilde{He}_{n+1}(x) + n\widetilde{He}_{n-1}(x).$$

4. Convergence of Hermite Polynomials to the Black-Scholes Model

Consider the Hermite polynomials $H_n(x)$ defined as

$$H_n(x) = (-1)^n \frac{1}{w(x)}\frac{d^n}{dx^n}w(x) \tag{4.1}$$

where $w(x)$ is the reference measure (weight function) which is the standard Gaussian density. The normalized form of the polynomials given as

$$\widetilde{H}_n(x) = \frac{H_n(x)}{\sqrt{n!}} \tag{4.2}$$

Assuming this asset price S_t follows a geometric Brownian motion, then we have

$$S_t = S_0\, e^{\mu t - \frac{\sigma^2 t}{2} + \sigma\sqrt{t} z_t}$$

where μ is the drift rate, σ is the variance rate (volatility) and z_t is the standard normal random variate with mean zero and variance one. That is $z_t \sim N(0,1)$.

Theorem Any arbitrary claim $g(x)$ in an Hilbert space (H) can be written as

$$g(x) = \sum_{n=0}^{\infty} a_n \tilde{H}_n(x)$$

The coefficient a_n is the covariance of the n-th term Hermite polynomial with contingent claim and is given by

$$a_n = \int_{\mathrm{R}} \tilde{H}_n(x) g(x) w(x) dx$$

Proof see [2]

Let $C(x)$ and $P(x)$ denotes the pay off of the European call and put options. where

$$C(x) = (S_t - K)^+$$
$$P(x) = (K - S_t)^+ \tag{4.3}$$

Using the theorem above, we may write the payoff of the call and put option in terms of the basis of Hermite polynomials.

$$C(x) = \sum_{n=0}^{\infty} a_n^{'} \tilde{H}_n(x)$$
$$P(x) = \sum_{n=0}^{\infty} b_n^{'} \tilde{H}_n(x)$$

where the coefficients of the call and put options be given as $a_n^{'}$ and $b_n^{'}$ with

$$a_n^{'} = \int_{-\infty}^{\infty} \tilde{H}_n(x) C(x) w(x) dx$$
$$b_n^{'} = \int_{-\infty}^{\infty} \tilde{H}_n(x) P(x) w(x) dx \tag{4.4}$$

Now using the generating function of the probabilists' Hermite polynomials

$$e^{(xu - \frac{u^2}{2})} = \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}$$

$$e^{(\frac{x^2}{2} - \frac{(u-x)^2}{2})} = \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}$$

$$e^{-\frac{(u-x)^2}{2}} = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(u-x)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(u-x)^2}{2}} = \sum_{n=0}^{\infty} H_n(x) w(x) \frac{u^n}{n!}$$

Using the Taylor's series function, we have

$$H_n(x)w(x) = \frac{\partial^n}{\partial u^n}[\frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}}]_{u=0}$$

Then from (4.2)

$$\widetilde{H}_n(x)w(x) = \frac{1}{\sqrt{n!}}\frac{\partial^n}{\partial u^n}[\frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}}]_{u=0}$$

we obtain the generating functions of the European call option by using the coefficient of the call option given in (4.4).

$$a_n^{'} = \frac{1}{\sqrt{n!}}\int_{-\infty}^{\infty} C(x)\frac{\partial^n}{\partial u^n}[\frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}}]_{u=0}dx$$

$$= \frac{1}{\sqrt{n!}}[\int_{-\infty}^{\infty} C(x)\frac{\partial^n}{\partial u^n}\frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}}]_{u=0}dx$$

$$= \frac{1}{\sqrt{n!}}\frac{\partial^n}{\partial u^n}C(x)|_u = 0$$

where

$$C = \int_{-\infty}^{\infty} C(x)\frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}}dx \tag{4.5}$$

is the European call option generating function. From (4.3), we know the payoff of the call option is expressed as

$$\begin{aligned} C(x) &= (S_t - K)^+ \\ &= (S_0 e^{\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t}z_t} - K)^+ \end{aligned} \tag{4.6}$$

(4.6) indicates that $C(x) \geq 0$. Hence

$$S_0 e^{\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t}z_t} - K \geq 0$$

$$S_0 e^{\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t}z_t} \geq K$$

$$e^{\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t}z_t} \geq \frac{K}{S_0}$$

$$\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t}z_t \geq ln\frac{K}{S_0}$$

$$z_t \geq \frac{1}{\sigma\sqrt{t}}[ln\frac{K}{S_0} + \frac{\sigma^2 t}{2} - \mu t] = l \tag{4.7}$$

Therefore,

$$C = \int_l^{\infty}[S_0 e^{\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t}z_t} - K]\frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_l^\infty S_0 e^{\mu t - \frac{\sigma^2}{2} + \sigma\sqrt{t} z_t} e^{-\frac{(u-x)^2}{2}} dx - K \int_l^\infty \frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}} dx$$

$$= S_0 e^{\mu t - \frac{\sigma^2}{2}} \int_l^\infty \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{t} z_t} e^{-\frac{(u-x)^2}{2}} dx - K \int_l^\infty \frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}} dx$$

$$= S_0 e^{\mu t - \frac{\sigma^2}{2}} I_2 - K I_1$$

where

$$I_1 = \int_l^\infty \frac{e^{-\frac{(u-x)^2}{2}}}{\sqrt{2\pi}} dx$$

$$I_2 = \int_l^\infty \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{t} z_t} e^{-\frac{(u-x)^2}{2}} dx$$

Considering I_1, using change of variable, let $m = u - x \Rightarrow dx = -dm$.

Hence

$$I_1 = \int_{-l+u}^{-\infty} \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} (-dm)$$

$$= \int_{-\infty}^{-l+u} \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} dm$$

We note that the formula for a cumulative distribution function of standard normal distribution is given by

$$N(x) = \int_{-\infty}^{x} \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} dm$$

Therefore, $I_1 = N(-l+u)$. Now using (4.7), we compute $-l+u$

$$-l + u = \frac{1}{\sigma\sqrt{t}}[\ln \frac{K}{S_0} + \frac{\sigma^2 t}{2} - \mu t] + u$$

$$= u + \frac{1}{\sigma\sqrt{t}}[\ln \frac{S_0}{K} - \frac{\sigma^2 t}{2} + \mu t]$$

$$= \frac{1}{\sigma\sqrt{t}} \ln \frac{S_0}{K} + \frac{\mu\sqrt{t}}{\sigma} - \frac{\sigma\sqrt{t}}{2} + u$$

$$= \frac{1}{\sigma\sqrt{t}} \ln \frac{S_0}{K} + \frac{\mu\sqrt{t}}{\sigma} - (\frac{\sigma}{2} - \sigma)\sqrt{t} + u$$

$$= \frac{1}{\sigma\sqrt{t}} \ln \frac{S_0}{K} - (\frac{\mu}{\sigma} + \frac{\sigma}{2})\sqrt{t} + u - \sigma\sqrt{t}$$

Let $d_1 = \frac{1}{\sigma\sqrt{t}} \ln \frac{S_0}{K} - (\frac{\mu}{\sigma} + \frac{\sigma}{2})\sqrt{t} + u$.

So,

$$-l+u=d_1-\sigma\sqrt{t}. \tag{4.8}$$

If $-l+u=d_2$, then $d_2=d_1-\sigma\sqrt{t}$ and $I_1=N(d_2)$.

Similarly, we consider

$$I_2=\int_l^\infty \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{t}z_t} e^{-\frac{(u-x)^2}{2}} dx$$

Note that

$$\sigma\sqrt{t}x-\frac{(x-u)^2}{2}=\sigma\sqrt{t}z_t-\frac{1}{2}(x^2-2ux+u^2)$$
$$=-\frac{1}{2}(x^2-2x(u+\sigma\sqrt{t})+u^2)$$
$$=-\frac{1}{2}[x-(u+\sigma\sqrt{t})]^2+\frac{1}{2}(2u\sigma\sqrt{t}+\sigma^2 t)$$
$$=-\frac{1}{2}[x-(u+\sigma\sqrt{t})]^2+u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}$$

Therefore,

$$I_2=e^{u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}}\int_l^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[x-(u+\sigma\sqrt{t})]^2} dx$$

let $v=-x+(u+\sigma\sqrt{t}) \Rightarrow dx=-dv$

$$I_2=e^{u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}}\int_{-l+(u+\sigma\sqrt{t})}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}(-dv)$$
$$=e^{u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}}\int_{-\infty}^{-l+(u+\sigma\sqrt{t})} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv$$
$$=e^{u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}} N(-l+u+\sigma\sqrt{t})$$

Using (4.8), $I_2=e^{u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}} N(d_1)$. Therefore, the call option generator C is

$$C=S_0 e^{\mu t-\frac{\sigma^2 t}{2}}.e^{u\sigma\sqrt{t}+\frac{\sigma^2 t}{2}} N(d_1)-KN(d_2)$$
$$=S_0 e^{\mu t+u\sigma\sqrt{t}} N(d_1)-KN(d_2)$$

the value of the put option can be obtain in a similar way as the case of the call option

$$P=KN(-d_2)-S_0 e^{\mu t+u\sigma\sqrt{t}} N(-d_1).$$

Without loss of generality, since u is a dummy variable, we can set it to be equal to zero. Therefore, we have

$$C=S_0 e^{\mu t} N(d_1)-KN(d_2)$$

and

$$P=KN(-d_2)-S_0 e^{\mu t} N(-d_1).$$

However, setting the drift μ to be equal to the risk-free rate r specializes the reference measure to the equivalent martingale measure under Black-Scholes, and the Hermite polynomial pricing model collapses to the Black-Scholes model. Therefore, we have

$$C = S_0 e^{rt} N(d_1) - KN(d_2)$$
$$P = KN(-d_2) - S_0 e^{rt} N(-d_1)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{t}} \ln\frac{S_0}{K} - (\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{t} = \frac{1}{\sigma\sqrt{t}}\left[\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)t\right]$$
$$d_2 = d_1 - \sigma\sqrt{t}$$

5. Conclusion

A separate way for the derivation of the well-known Black-Scholes formula used for the pricing of the European call and put options has been presented. The method make use of the Hermite polynomials and some of the general properties of the polynomial. The derivation shows us that the Black-Scholes formula can be obtained without the use of the Partial differential equation. Moreover, the steps are easy to implement and it involves less computation.

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